

An analytic study of a pseudo-complex-structure.

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0. Introduction

In this paper we give an analytic study of a pseudo-complex-structure, which is roughly speaking, the abstract substitute of an imbedding of a real manifold into a complex manifold. Until now the main tool of this analysis was the theory of partial differential equations developed by Kohn [4] and Hörmander [2]. But the results obtained so far seem to be applied only to the 2-regular case. (See Definition 2 for the concept of the regularity.)

On the other hand, the geometric study of pseudo-complex-structures was already made by N. Tanaka [8], [9], [10]. And this formulation, when combined with a recent work of Hörmander [3], yields a powerfull theorem (Theorem 1.5) which is a partial generalization of Kohn-Hörmander theory to the general μ -regular case.

This theorem is of particular importance in the study of a complex manifold. In fact it follows from this the finite-dimensionality of the space of gloval sections of every analytic vector bundle over a complex manifold having some nice real submanifold, which is in general neither compact, nor pseudo-concave.

I am greatly indebted to Professor N. Tanaka not only for the formulation of basic geometric materials, but also for a number of stimulating conversations out of which especially Theorems 1 - 2 grew. Therefore this paper might be regarded as a collaboration with him. I thank also Professor S. Matsuura for his serious interest and encouragement.

1. The concept of a pseudo-complex-structure and many other concepts related to it were already introduced in [9] [10]. But we reproduce these here. We assume the differentiability

of class C^∞ . Let E be a (real or complex) vector bundle over a manifold M . We shall denote by $\underline{E}$ the sheaf of germs of local C^∞ sections of E . For $p \in M$, E_p denotes the fiber of E over p . Similarly $\mathcal{S}_p$ is a stalk of $\mathcal{S}$ over p if $\mathcal{S}$ is a sheaf over M . $\Gamma_\Omega(\mathcal{S})$ is the set of sections over Ω and $\Gamma(\mathcal{S})$ denotes the set of global sections. For simplicity we use $\Gamma_\Omega(E)$, $\Gamma(E)$ instead of $\Gamma_\Omega(\underline{E})$, $\Gamma(\underline{E})$ respectively.

Let M be a real submanifold of a complex manifold $\tilde{M}$ and let $T^C(M)$, $T^C(\tilde{M})$ denote the complexified tangent bundles of M , $\tilde{M}$ respectively. As usual, $T_p^C(M)$ is regarded as a subspace of $T_p^C(\tilde{M})$ for any p of M . Now we set

$$(1.1) \quad S_p = T_p^C(M) \cap T_p^{(1,0)}(\tilde{M}) \quad p \in M$$

where $T_p^{(1,0)}(\tilde{M})$ is the holomorphic part of $T_p^C(\tilde{M})$. Then there uniquely exists a (complex) vector bundle S over M whose fiber over p is just S_p provided that

$$(1.2) \quad \dim S_q = \dim S_{q'}, \quad q, q' \in M.$$

The pair (M, S) has the following properties:

(i) S is a subbundle of $T^C(M)$.

(ii) $S_p \cap \bar{S}_p = (0)$

(iii) S is completely integrable, i.e. $[S, S] \subset S$.

Thus a real submanifold M of a complex manifold $\tilde{M}$ gives rise to a pair (M, S) with the properties (i), (ii), (iii) if (1.2) is satisfied.

Conversely, if M is an arbitrary real analytic manifold and if moreover S is a real analytic subbundle of $T^{\mathbb{C}}(M)$ with properties (ii), (iii), then there exists a complex manifold $\tilde{M}$ of which M is a real submanifold and for which (1.1) holds.

Definition 1. A pair (M, S) with the properties (i), (ii), (iii) is called a pseudo-complex-structure over M .

In general the concept of a pseudo-complex-structure will provably be much wider than that of a real submanifold of a complex manifold satisfying (1.2). Especially a pseudo-complex-structure (M, S) constructed as above from an imbedding of M into a complex manifold will be called the pseudo-complex-structure induced by the imbedding under consideration.

Now let (M, S) be a pseudo-complex-structure and put

$$D_p = \{ \operatorname{Re} x ; x \in S_p \}$$

where $\operatorname{Re} x$ denotes the real part of x . We call the Pfaffian system D whose fiber over p is D_p the first Pfaffian system of (M, S) .

Definition 2. A pseudo-complex-structure (M, S) is called

μ -regular if there exists a series of Pfaffian systems

$$D = D^1 \subsetneq D^2 \subsetneq \dots \subsetneq D^\mu = T(M) \quad \text{such that} \quad \underline{D}^i = [\underline{D}^{i-1}, \underline{D}] + \underline{D}^{i-1} \\ (2 \leq i \leq \mu) .$$

The above $\underline{D}^i$ is uniquely determined for a given μ -regular pseudo-complex-structure (M, S) . So we call $\underline{D}^i$ the i -th Pfaffian system of (M, S) . By successive use of the Jacobi identity

$$[\underline{D}^i, \underline{D}^j] \subseteq \underline{D}^{i+j} ,$$

where $\underline{D}^i = T(M)$ for $i > \mu$. Thus the usual bracket operation for vector fields gives rise to a Lie algebra structure of $\Gamma(\underline{\quad})$ ($\underline{\quad} = \sum_{i=1}^{\mu} \underline{\mathcal{G}}^i$, $\underline{\mathcal{G}}^i = D^i/D^{i-1}$) and the bracket operation of this shall be denoted by $[\quad, \quad]^*$. But we have then

$$[fX, gY]^* = fg[X, Y]^* \quad X, Y \in \Gamma(\underline{\quad})$$

where f, g are C^∞ functions. Therefore $[\quad, \quad]^*$ induces also a Lie algebra structure of $\mathfrak{m}_p = \sum_{i=1}^{\mu} \mathcal{G}_p^i$ for any point p of M , whose bracket operation we shall denote by $[\quad, \quad]_p^*$. This Lie algebra $\mathfrak{m}_p$ has the properties:

- (i) $\mathfrak{m}_p$ is finite-dimensional.
- (ii) $[\mathcal{G}_p^i, \mathcal{G}_p^j]^* \subseteq \mathcal{G}_p^{i+j} \quad (\mathcal{G}_p^i = 0 \text{ for } i > \mu)$
- (iii) $\mathfrak{m}_p$ is generated by $\mathcal{G}_p^1$.

$$(iv) \quad [\operatorname{Re} ix, \operatorname{Re} iy]_p^* = [\operatorname{Re} x, \operatorname{Re} y]_p^* \quad i = \sqrt{-1} \quad x, y \in S_p.$$

We call this nilpotent Lie algebra $\mathfrak{m}_p$ the *Levi-Tanaka form* of (M, S) at p (which will be short referred to *L-T form* at p). Put

$$L_p(x, y) = [\operatorname{Re} x, \operatorname{Re} iy]_p^* \quad x, y \in S_p.$$

Then L_p is a $\mathcal{G}_p^2$ -valued symmetric bilinear form by (iv).

Definition 3. Let (M, S) be a regular pseudo-complex-structure and let $\mathfrak{m}_p = \sum_{i=1}^n \mathcal{G}_p^i$, L_p be as above. We say that (M, S) is *totally indefinite* at p if, for any non-zero linear form α of $\mathcal{G}_p^2$, the (real-valued) symmetric bilinear form

$$\langle \alpha, L_p(x, y) \rangle$$

is indefinite (not semi-definite). When (M, S) is totally indefinite at every point of M , we simply call (M, S) *totally indefinite*.

Total indefiniteness was suggested by the condition of sub-ellipticity in Hörmander [1].

In order to state our main theorem we need still the concept of an analytic vector bundle over a pseudo-complex-structure. First of all we define the analyticity of a function.

Definition 4. Let (M, S) be a pseudo-complex-structure and let f be a C^∞ function on an open set Ω of M . We say that f is (M, S) -analytic in Ω if $Xf = 0$ for any $X \in \Gamma_\Omega(\bar{S})$.

The (M, S) -analyticity is a local property, that is, f is (M, S) -analytic in $\Omega = \bigcup_\lambda \Omega_\lambda$ if and only if f is (M, S) -analytic in each Ω_λ .

Definition 5. Let (M, S) is a pseudo-complex-structure and let E be a vector bundle over M and $\mathcal{S}$ be a subsheaf of $\underline{E}$.

The pair $(E, \mathcal{S})$ is called an analytic vector bundle over (M, S) if it satisfies the following conditions:

(i), if the values in E_p of sections $s_1, \dots, s_m$ of $\mathcal{S}$ over an open set Ω are linearly independent at every p of Ω , then $\sum_{j=1}^m f_j s_j$ is a section of $\mathcal{S}$ if and only if $f_1, \dots, f_m$ are all (M, S) -analytic in Ω .

(ii), for any $p \in M$, there exist a series of local sections $s_1, \dots, s_e$ ($e = \text{fiber dim. of } E$) of $\mathcal{S}$ whose values in E_p are linearly independent.

Remark 1. Let (M, S) be the induced pseudo-complex-structure by an imbedding of M into a complex manifold $\tilde{M}$ and let $\tilde{E}$ be an analytic vector bundle over $\tilde{M}$ and $\tilde{\mathcal{S}}$ denote the sheaf of germs of local analytic sections of $\tilde{E}$. Denote by $\mathcal{S}$ the smallest subsheaf of $\underline{E}$ ($E = \tilde{E}|_M$) containing $\tilde{\mathcal{S}}|_M$ with property (i) of definition 5.

Then $(E, \mathcal{J})$ is an analytic vector bundle over M and we call $(E, \mathcal{J})$ the restriction of $\tilde{E}$.

Our main theorem is now stated as follows:

Theorem 1. Let (M, S) be a totally indefinite regular pseudo-complex-structure over a compact manifold M and let $(E, \mathcal{J})$ be an analytic vector bundle over (M, S) . Then $\Gamma(\mathcal{J})$ is a finite-dimensional vector space (over $\mathbb{C}$).

Now we proceed to formulate another important theorem. Let (M, S) be a pseudo-complex-structure. Then $\bar{S}_p$ is a Lie subalgebra of $T^{\mathbb{C}}(M)_p$ since S is completely integrable. Then there exists a unique Lie algebra sheaf $\mathcal{A}'$ whose stalk $\mathcal{A}'_p$ over p is the normalizer of $\bar{S}_p$ in $T^{\mathbb{C}}(M)_p$. The $\bar{S}$ is a subsheaf of $\mathcal{A}'$ such that $\bar{S}_p$ is an ideal of $\mathcal{A}'_p$. So $\mathcal{A} = \mathcal{A}' / \bar{S}$ is again a Lie algebra sheaf of (M, S) . We call this sheaf $\mathcal{A}$ the tangential sheaf of (M, S) . Our second theorem is the following:

Theorem 2. Let (M, S) be a totally indefinite regular pseudo-complex-structure over a compact manifold M and let $\mathcal{A}$ denote its tangential sheaf. Then $\Gamma(\mathcal{A})$ is finite-dimensional.

Remark 2. $\mathcal{A}$ is, of course, a subsheaf of $T^{\mathbb{C}}(M)/S$. If (M, S) is induced by an imbedding of M into a complex manifold $\tilde{M}$, $E = T^{\mathbb{C}}(M)/\bar{S}$ can be naturally identified with

the restriction to M of the (real) tangent bundle $T(\tilde{M})$ of $\tilde{M}$ and moreover $(E, \mathcal{A})$ is the restriction of $T(\tilde{M})$ in the sense of Remark 1. Therefore, in this case, Theorem 2 is a consequence of Theorem 1.

The key of the proofs of Theorems 1-2 is a consequence of Hörmander [3] stated as below.

Lemma 1. Let (M, S) be a μ -regular pseudo-complex-structure on M and suppose that $X_p^1, \dots, X_p^m$ ($X^i \in \Gamma(S)$) span S_p for every p of M . Then, for any compact set K of M and for any $0 < \epsilon < \frac{1}{\mu}$, there exists a positive constant such that

$$\|u\|_{(\epsilon)}^2 \leq C \left(\sum_{j=1}^m (\|X^j u\|_{(0)}^2 + \|\bar{X}^j u\|_{(0)}^2) + \|u\|_{(0)}^2 \right) \quad u \in C_0^\infty(K)$$

But what we really need is the following refinement of Lemma 1 for a totally indefinite regular pseudo-complex-structure.

Lemma 2. In addition to the hypothesis of lemma 1, assume that (M, S) is totally indefinite. Then, for any compact subset K of M and for any $0 < \epsilon < \frac{1}{\mu}$, there exists a positive constant C such that

$$\|u\|_{(\epsilon)}^2 \leq C \left(\sum_{j=1}^m \|\bar{X}^j u\|_{(0)}^2 + \|u\|_{(0)}^2 \right) \quad u \in C_0^\infty(K)$$

This lemma follows from Lemma 1 by generalizing a technique of Kohn [4].

Now, in view of Lemma 2,

the proof of Theorem 1 is almost evident. The proof of Theorem 2 still needs a minor differential geometric trick. We give these proofs in paragraph 3.

2. Now we are in a position to apply Theorems 1-2 to the study of the automorphism group of a pseudo-complex-structure and to the study of an analytic vector bundle over a complex manifold.

Definition 6. Let (M, S) , (M', S') be two pseudo-complex-structures. A diffeomorphism f of M onto M' is called an isomorphism of (M, S) onto (M', S') if $(df)_p$ maps S_p isomorphically onto $S'_{f(p)}$ for any $p \in M$. An isomorphism of (M, S) onto itself is called an automorphism of (M, S) .

As an application of Theorem 2 we have

Theorem 3. Let (M, S) be a totally indefinite regular pseudo-complex-structure over a compact manifold M . Then the automorphism group of (M, S) is a Lie transformation group over M with respect to some natural topology.

Proof. Let $\mathcal{A}$ denote the infinitesimal automorphism group of (M, S) (i.e. the Lie algebra of generators of 1-parameter subgroups of the automorphism group of (M, S)). Let $\mathcal{A}$ be the tangential sheaf of (M, S) and ρ denote the natural projection of $\Gamma(T^C(M))$ to $\Gamma(T^C(M)/S)$. Then

$$\rho(\mathcal{W}) = \Gamma(\mathcal{A}) \cap \rho(\Gamma(T(M)))$$

But ρ is one-to-one on $\Gamma(T(M))$ since $S_p \cap \bar{S}_p = (0)$. Thus $\mathcal{W}$ is finite-dimensional. Now a theorem of Palais [6] implies the conclusion of Theorem 3.

Remark 3. This theorem was proved by Naruki [4] when (M, S) is 2-regular. It was shown by N. Tanaka that the finite-dimensionality of the automorphism group of a pseudo-complex-structure follows (without compactness assumption) under the assumptions of strong-regularity and non-degeneracy of (M, S) . For all of these, we refer to [9].

Now let (M, S) be the pseudo-complex-structure induced by an imbedding of M into a complex manifold $\tilde{M}$ and let $(E, \mathcal{J})$ be the restriction to M of an analytic vector bundle $(\tilde{E}, \tilde{\mathcal{J}})$ over $\tilde{M}$. Assume

$$(2.1) \quad \text{fiber dim. of } S + \text{complex dim. of } \tilde{M} = \text{real dim. of } M.$$

Then the restriction mapping of $\Gamma(\tilde{\mathcal{J}})$ into $\Gamma(\mathcal{J})$ is one-to-one. In fact the condition (2.1) implies that $T_p(\tilde{M})$ is the unique complex subspace of $T_p(\tilde{M})$ which contains $T_p(M)$ for every p of M , and vice versa. We say that M is *generally imbedded* if the condition (2.1) is satisfied. As an application of Theorem 1 we have

Theorem 4. Suppose that a complex manifold $\tilde{M}$ has a compact generally imbedded submanifold M and that the pseudo-complex-structure induced by the inclusion map of M into $\tilde{M}$ satisfies the hypotheses of Theorem 1. Then the space of global sections of an analytic vector bundle over $\tilde{M}$ is always finite-dimensional. In particular, the holomorphic automorphism group of $\tilde{M}$ is a Lie transformation group over $\tilde{M}$ with respect to some natural topology.

This theorem was suggested by N. Tanaka. Note that for any neighbourhood of M in $\tilde{M}$ the conclusion of this theorem holds. So one can easily construct a complex manifold M which is neither compact, nor pseudo-concave and for which the conclusion of Theorem 4 holds. (See Example 2).

Example 1. We give some examples which clarify what the validity of the conclusion would be in Theorems 1-3 without the total indefiniteness. Set

$$M_r: |z_0|^2 + \sum_{j=1}^r |z_j|^2 - \sum_{j=r+1}^{n-1} |z_j|^2 - |z_u|^2 = 0$$

where $(z_0, \dots, z_n)$ is the homogeneous coordinate of $P^n(C)$.

The pseudo-complex-structure (M_r, S_r) induced by the inclusion $M_r \hookrightarrow P^n(C)$ is regular, but not totally indefinite when $r = 0$, or $r = n-1$. Since M_0 (or M_{n-1}) can be imbedded into C^n and since C^n is a Stein manifold, global sections of an analytic vector bundle over C^n , hence also over (M_0, S_0) (or (M_{n-1}, S_{n-1})) form an infinite-dimensional vector space.

But the automorphism group of (M_0, S_0) (or of (M_{n-1}, S_{n-1})) is still Lie transformation group. This follows from the result of N. Tanaka remarked after Theorem 3. However this is not true for $M_0 \times P^m(C)$ in $P^n(C) \times P^m(C)$ ($m \geq 1$) because of the infinite-dimensionality of the space of (M_0, S_0) -analytic functions. But, when $1 \leq r < n-1$, the hypothesis of Theorems 1-3 holds for $M_r \times P^m(C)$ in $P^n(C) \times P^m(C)$, although the pseudo complex structure is degenerate in the sense of [9].

Example 2. (due to N. Tanaka) Put

$$G = GL(n, C), \quad K = U(n)$$

$$H = \{(a_{ij}) \in G; a_{ij} = 0 \text{ if } |i-j| \geq 2 \text{ or if } i: \text{even}\}$$

Then $M = K/K \cap H$ is imbedded generally into the complex manifold $\tilde{M} = G/H$. M is obviously compact and the pseudo-complex-structure induced by this imbedding is $(n-1)$ -regular and totally indefinite.

3. In this paragraph we shall prove Lemma 2 and Theorems 1-2. Before proceeding we need an algebraic lemma which makes the meaning of total indefiniteness more clear. Let V^* be the dual space of a n -dimensional complex vector space V . The (real) vector space of Hermitian forms on V (resp. V^*) shall be denoted by F (resp. F^*). The notation F^* may be justified by the fact that F^* can be naturally identified with the dual space of F . In fact, we can define the bilinear form on $F \times F^*$ by setting

$$(f, g^*) = \sum_{j,k=1}^n f(e_j, e_k) g^*(e_k^*, e_j^*) \quad f \in F, g^* \in F^*$$

where $\{e_1^*, \dots, e_n^*\}$ is the dual base of a base $\{e_1, \dots, e_n\}$ of V . Note that $(\cdot, \cdot)$ is independent of the choice of $\{e_1, \dots, e_n\}$ and that $(f, g^*) = 0$ for any $g^* \in F^*$ implies $f = 0$. These facts gives us the desired identification.

Lemma 3. Notations being as above, for a subspace of L , the following statements are equivalent.

- (i) L contains no semi-definite element except 0.
- (ii) $L^\perp$ contains a (positive) definite element, where $L^\perp = \{g^* \in F^* : (f, g^*) = 0 \text{ for any } f \in L\}$.

This lemma follows from a corresponding theorem for quadratic forms due to L. L. Dines [1], but we prefer to give a direct proof in the Appendix.

Now let (M, S) be a totally indefinite regular pseudo-complex-structure and let F_p (resp. F_p^*) denote the vector

space of Hermitian forms on S_p (resp. S_p^*). Put

$$\begin{aligned} \eta^\circ(x, y) &= \langle \eta, [\operatorname{Re}(x), \operatorname{Re}(iy)]_p^* \rangle + i \langle \eta, [\operatorname{Re}(x), \operatorname{Re}(y)]_p^* \rangle \\ &= -\frac{i}{2} \langle \eta, [x, \bar{y}] \rangle \quad x, y \in S_p \quad \eta \in (\mathcal{G}_p^2)^* \end{aligned}$$

where $[\ , \]_p^*$ is the bracket operation of the Levi-Tanaka form $m_p = \sum_{i=1}^u \mathcal{G}_p^i$ at p . Then η° is a usual Hermitian form on S_p . Set

$$L(p) = \{ \eta^\circ : \eta \in (\mathcal{G}_p^2)^* \}.$$

The subspace $L(p)^\perp$ of F_p^* being as in Lemma 3 consider the vector bundle $L^\perp$ with its fiber $L(p)^\perp$ over p . Since $L(p)^\perp$ contains certainly a positive definite element by Lemma 3, and since the set of positive definite elements in $L(p)^\perp$ is convex, there exists $g \in \Gamma(L)$ such that the value $g(p)$ at p of g is positive definite for any $p \in M$. Thus if $\{X^k\}_{k=1}^m \in \Gamma(\bar{S})$ is a frame of $\bar{S}$ (, that is, if $\{X_p^k\}_{k=1}^m$ is a base of $\bar{S}_p$ for $p \in M$), we have

$$(3.1) \quad \sum_{j,k=1}^m g_{jk} [X^j, \bar{X}^k] \in \Gamma(S \oplus \bar{S})$$

where $g_{jk}(p) = g(p)(Y_p^j, Y_p^k)$ ($\{Y_p^k\}_{k=1}^m$ is the dual base of $\{X_p^k\}_{k=1}^m$).

Proof of Lemma 2. Since the validity of (2.1) is entirely a local property, we may assume that $X_p^1, \dots, X_p^o$

are linearly independent for any $p \in M$ replacing M by some suitable open subset of M . g_{jk} being as above, we shall define three norms $\| \cdot \|$, $\| \cdot \|_1$, $\| \cdot \|_2$ on $\underbrace{C_0^\infty \times \dots \times C_0^\infty}_\rho$ by setting

$$\|U\|^2 = \sum_{j=1}^{\rho} \|u_j\|_{(0)}^2$$

$$\|U\|_1^2 = \sum_{j,k=1}^{\rho} (g_{jk} u_j, u_k)_{(0)} \quad U = (u_1, \dots, u_\rho)$$

$$\|U\|_2^2 = \sum_{j,k=1}^{\rho} (g_{jk} u_k, u_j)_{(0)}$$

where $(\cdot, \cdot)_{(0)}$ is the polar form of $\| \cdot \|_{(0)}^2$.

$\| \cdot \|$, $\| \cdot \|_1$, $\| \cdot \|_2$ are equivalent on $C_0^\infty(K)^\rho$ for any compact subset K of M , since $(g_{jk}(p))$ is positive definite. Note that, for any $X \in \Gamma(T^{\mathbb{C}}(M))$, there exists $c \in C_0^\infty(M)$ such that

$$(Xu, v)_{(0)} + (u, \bar{X}v)_{(0)} = (cu, v)_{(0)} \quad u, v \in C_0^\infty(M).$$

Therefore it follows from (3.1) that there exists $X_0 \in \Gamma(S \oplus \bar{S})$ such that

$$\begin{aligned} \|Xu\|_1^2 - \|\bar{X}u\|_2^2 &= \sum_{j,k} ((g_{jk} X^j u, X^k u) - (g_{jk} \bar{X}^k u, \bar{X}^j u)) \\ &= (u, X_0 u), \end{aligned}$$

where we have put $Xu = (X^1 u, \dots, X^\rho u)$, $\bar{X}u = (\bar{X}^1 u, \dots, \bar{X}^\rho u)$.

This implies that, for any compact subset K there

5.2

exists a positive constant C such that

$$\|\bar{\mathcal{X}}u\|_2^2 \leq \|\mathcal{X}u\|_1^2 + C\|u\|_{(0)} (\|\mathcal{X}u\| + \|\bar{\mathcal{X}}u\|) \quad u \in C_0^\infty(K).$$

Since $\|\cdot\|$, $\|\cdot\|_1$, $\|\cdot\|_2$ are equivalent on $C_0^\infty(K)^\rho$, we obtain

$$\|\bar{\mathcal{X}}u\|^2 \leq C(\|\mathcal{X}u\|^2 + \|u\|_{(0)} (\|\mathcal{X}u\| + \|\bar{\mathcal{X}}u\|)) \quad u \in C_0^\infty(K)$$

for some other $C > 0$.

Using the inequality $|ab| \leq \delta|a|^2 + \frac{1}{\delta}|b|^2$ for sufficiently small δ , we obtain

$$(2.3) \quad \|\bar{\mathcal{X}}u\|^2 \leq C(\|\mathcal{X}u\|^2 + \|u\|_{(0)}^2)$$

for another $C > 0$. On the other hand, Lemma 1 implies

$$(2.4) \quad \|u\|_{(\varepsilon)}^2 \leq C(\|\mathcal{X}u\|^2 + \|\bar{\mathcal{X}}u\|^2 + \|u\|_{(0)}^2) \quad u \in C_0^\infty(K).$$

Combining (2.3) and (2.4) we conclude that there exists a positive constant C such that

$$\|u\|_{(\varepsilon)}^2 \leq C(\|\mathcal{X}u\|^2 + \|u\|^2) \quad u \in C_0^\infty(K).$$

Q.E.D.

Proof of Theorem 1. Let (M, S) be a regular compact pseudo-complex-structure and let $(E, \mathcal{S})$ is an analytic vector bundle over (M, S) . First we shall introduce the Sobolev norms $\| \cdot \|_{(\sigma)}$ on $\Gamma(E)$ suitable for our purpose. Let $\{\Omega_\alpha\}$ be a finite covering of M such that there exist sections of $\mathcal{S}$ $s_\alpha^1, \dots, s_\alpha^e$ satisfying (ii) in Definition 5 for any $p \in \Omega_\alpha$ and let $\{\varphi_\alpha\}$ is a partition of unity subordinate to $\{\Omega_\alpha\}$. Define

$$\|U\|_{(\sigma)}^2 = \sum_\alpha \sum_{j=1}^e \|\varphi_\alpha u_\alpha^j\|_{(\sigma)}^2 \quad U \in \Gamma(E)$$

where $U = \sum_{j=1}^e u_\alpha^j s_\alpha^j$.

Recall that U is an element of $\Gamma(\mathcal{S})$ if and only if u_α^j are all analytic. Applying (2.2) to $\varphi_\alpha u_\alpha^j$, we obtain

$$\|\varphi_\alpha u_\alpha^j\|_{(\varepsilon)}^2 \leq C \left(\sum_{k=1}^0 \|(X^k \varphi_\alpha) u_\alpha^j\|^2 + \|\varphi_\alpha u_\alpha^j\|^2 \right) \quad U \in \Gamma(\mathcal{S})$$

since $X^k u_\alpha^j = 0$ by the (M, S) -analyticity of u_α^j .

Therefore there exists a positive constant C such that

$$\|U\|_{(\varepsilon)}^2 = \sum_{\alpha, j} \|\varphi_\alpha u_\alpha^j\|_{(\varepsilon)}^2 \leq C \|U\|_{(0)}^2 \quad U \in \Gamma(\mathcal{S}).$$

By the generalized Rellich lemma, $\Gamma(\mathcal{S})$ is finite-dimensional.

Q.E.D.

Proof of Theorem 1.6. Choosing the (complex-valued) 1-forms $\zeta^1, \dots, \zeta^\pi$ such that $\zeta_p^1, \dots, \zeta_p^\pi$ span $\bar{S}_p^\perp$ where $\bar{S}^\perp$ is the bundle of annihilators in $T^*\mathbb{C}(M)$ of $\bar{S}$, we introduce Sobolev norms $\| \cdot \|_{(\sigma)}$ on $\Gamma(T^*\mathbb{C}(M)/\bar{S})$ by setting

$$\|s\|_{(\sigma)}^2 = \sum_{j=1}^{\pi} \|\zeta^j(X)\|_{(\sigma)}^2 \quad s \in \Gamma(T^*\mathbb{C}(M)/\bar{S})$$

where we have chosen a vector field X such that $\rho(X) = s$ (ρ : the canonical projection of $\Gamma(T^*\mathbb{C})$ onto $\Gamma(T^*\mathbb{C}/S)$). The right hand side is independent of the choice of such a X , so $\| \cdot \|_{(\sigma)}$ is well defined.

Suppose that $\rho(X) \in \Gamma(\mathcal{A})$ and that $\chi_p^1, \dots, \chi_p^\rho$ where $\chi^j \in \Gamma(\bar{S})$ ($j = 1, 2, \dots, \rho$) span $\bar{S}_p$ for any $p \in M$. Then

$$\zeta^j(X^k) = 0.$$

Taking the Lie derivative of this with respect to X , we obtain

$$\begin{aligned} 0 &= \langle L_X(\zeta^j) | X^k \rangle + \langle \zeta^j | [X^k, X] \rangle \\ &= \langle L_X(\zeta^j) | X^k \rangle \end{aligned}$$

since $[X^k, X] \in \Gamma(\bar{S})$ by $\rho(X) \in \Gamma(\mathcal{A})$.

This can be rewritten in the form

$$(2.5) \quad x^k(\zeta^j(X)) = \langle x^k \rfloor d\zeta^j, X \rangle.$$

From the complete integrability of $\bar{S}$, it follows

$$x^k \rfloor d\zeta^j \in \Gamma(\bar{S}^\perp).$$

This together with (2.5) implies that there exists a positive constant C such that

$$\sum_{k=1}^p \|x^k(\zeta^j(X))\|_{(0)}^2 \leq C \|s\|_{(0)}^2$$

where $s = \rho(X)$.

Applying Theorem 2.2 to this we obtain

$$\|\zeta^j(X)\|_{(\varepsilon)}^2 \leq C \|s\|_{(0)}^2$$

for some $C > 0$, and hence

$$\|s\|_{(\varepsilon)}^2 = \sum_{j=1}^p \|\zeta^j(X)\|_{(\varepsilon)}^2 \leq C \|s\|_{(0)}^2 \quad s \in \Gamma(\mathcal{A})$$

for another $C > 0$. By the generalized Rellich lemma,

$\Gamma(\mathcal{A})$ is finite-dimensional.

Q.E.D.

Appendix

In this appendix we shall prove Lemma 3. Let $(\cdot, \cdot)$ be a (fixed) positive definite hermitian form on an n -dimensional complex vector space V . For $A \in \text{Hom}_{\mathbb{C}}(V, V)$ we define $A^* \in \text{Hom}_{\mathbb{C}}(V, V)$ by the following identity:

$$(Au, v) = (u, A^*v) .$$

A is called a hermitian endomorphism (with respect to $(\cdot, \cdot)$) if $A = A^*$. Given a Hermitian form f on V , there exists one and only one Hermitian endomorphism A_f such that

$$f(u, v) = (A_f(u), v) .$$

We denote by F_e the vector space of Hermitian endomorphisms. F_e can be then identified with F by the mapping $f \mapsto A_f$. We also introduce an inner product $(\cdot, \cdot)$ of F_e by putting

$$(A, B) = \text{Sp}(A, B) .$$

Then Lemma 3 is equivalent to the following.

Lemma 3'. For a subspace L of F_e the following conditions are equivalent.

- (i) L contains no semi-definite element except 0.
- (ii) $L^\perp$ contains a (positive) definite element.

Here we have put $L^\perp = \{F_e \ni A ; (A, B) = 0 \quad \forall B \in L\}$.

Proof of (i) $\Rightarrow$ (ii). Assume that $L^\perp$ contains no definite element. Note that the set of positive definite Hermitian endomorphism P is an open convex cone. Since

the linear space $L^\perp$ does not intersect with P , there exists a hyperplane H of F_e containing $L^\perp$ such that H does not meet P , in view of a Theorem of Minkowsky [5]. Since $H^\perp \subseteq L$, a generator A of $H^\perp$ is not semi-definite. Let $e_1, \dots, e_n$ be the unit eigen vectors of A and let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues. We may assume $\lambda_1 \geq \dots \geq \lambda_n$. Then $\lambda_1 > 0$, $\lambda_n < 0$. Therefore there exist positive numbers $\mu_1, \dots, \mu_n$ such that $\lambda_1 \mu_1 + \dots + \lambda_n \mu_n = 0$. If we define a Hermitian endomorphism B by setting

$$Be_j = \mu_j e_j \quad (j=1, 2, \dots, n),$$

then $(B, A) = 0$ and hence $B \in (H^\perp)^\perp = H$. But the positive definiteness of B contradicts to $H \cap P = \emptyset$, thus (i) $\Rightarrow$ (ii) is proved.

Proof of (ii) $\Rightarrow$ (i). Let B be a positive definite element of $L^\perp$ and let A be a semi-definite element of L . Let $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ be the eigenvalues of A and let $e_1, \dots, e_n$ be the corresponding eigen vectors. If we set $(Be_j, e_j) = \mu_j$ ($j=1, 2, \dots, n$), we obtain

$$\lambda_1 \mu_1 + \dots + \lambda_n \mu_n = 0.$$

But this is impossible unless $\lambda_1 = \dots = \lambda_n = 0$ since $\mu_j > 0$ by the positive definiteness of B . Thus $A = 0$ and (ii) $\Rightarrow$ (i) is proved.

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