

Dynamics of skew tent maps

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1 Introduction

We consider skew tent maps defined by

$$f_{a,b}(x) = \begin{cases} ax + 1, & (x \leq 0) \\ -bx + 1, & (x \geq 0) \end{cases}$$

$$(a, b) \in D := \{(a, b) : a > 0, b > 1, a + b \geq ab\}.$$

By the dynamical behavior of $f_{a,b}$, the parameter domain D is divided into subdomains D_k defined by some algebraic curves : $D = \sum_{k=2}^{\infty} D_k$. In each D_k there are subdomains D_k^A where $f_{a,b}$ has unique attracting periodic orbit of period k and D_k^B where $f_{a,b}$ has $2k$ or k chaotic intervals, which is analyzed by the method of renormalization ([Ich][Ito]). In this paper we give the relation between $\star$ -product for kneading sequence and renormalization as Theorem 1. We also give an explicit proof of Theorem A in [MV91], which states monotonicity of kneading sequence of this family. Moreover we correct Corollary of Theorem 2 in [MV92] as Proposition 4.

We denote kneading sequence of $f_{a,b}$ by $K(a, b)$ and topological entropy of $f_{a,b}$ by $h(a, b)$. We refer basic definitions and notations from [Ich][CE80].

2 Renormalization and $\star$ -product

We denote $f_{a,b}$ by f in this section. For getting maximal level of renormalization, we assume sequence $\underline{B}$ is prime. Let $|\underline{A}|$ be the length of sequence $\underline{A}$ and $\text{int}(J)$ interior of an interval J . For the definitions of renormalization and $\star$ -product, see [Ito] and [CE80] respectively.

Definition $\underline{S}$ is called *prime* if $\underline{S}$ does not have any finite sequence $\underline{A} (\neq 0)$ of L 's and R 's and any finite or infinite sequence $\underline{B} (\neq C)$ such that $\underline{S} = \underline{A} \star \underline{B}$.

Theorem 1 $K(a, b) = \underline{A} \star \underline{B}$ where $\underline{A} (\neq \emptyset)$ is finite sequence of L 's and R 's and $\underline{B} (\neq C)$ is prime if and only if there exist invariant closed intervals $\{J_i\}_{i=0, \dots, |\underline{A}|}$ such that $J_{|\underline{A}|} \ni 0$, $fJ_i = J_{i+1}$ ($i = 0, \dots, |\underline{A}| - 1$), $fJ_{|\underline{A}|} = J_0$ and $\text{int}(J_i) \cap \text{int}(J_{i'}) = \emptyset$ ($i \neq i'$). f can not have any refinement of $\{J_i\}$.

Proof Assume $K(a, b) = \underline{A} \star \underline{B}$ where $\underline{A} (\neq \emptyset)$ is finite sequence of L 's or R 's and $\underline{B} (\neq C)$ is prime. Set $x_n = f^n(1)$ ($n \geq 0$), $p = |\underline{A}|$ and $\underline{A} = A_0 A_1 \dots A_{p-1}$. Let J_i be convex hull of $\{x_{i+k(p+1)} : k = 0, 1, \dots\}$ for $i = 0, \dots, p$. Then, we have $fJ_i = J_{i+1}$ ($i = 0, \dots, p-1$) and f is monotone on each J_i except of $i = p$. Remark that f^{p+1} on each J_i has same slopes. It follows that $f^{p+1}|_{J_i} \sim f^{p+1}|_{J_{i'}}$ ($i \neq i'$). We consider the following two cases.

The first case : $\underline{B}$ does not contain both L and R .

$\underline{B}$ is finite in this case. It follows that J_p contains a turning point 0 as an end point of it. Hence, $fJ_p = J_0$. As f is monotone on J_i for all i ($0 \leq i \leq p$), f^{p+1} restricted on J_i is monotone and surjective on J_i . Hence, its slope is -1 . Then $\{J_i\}$'s are disjoint or there would exist some i, i' such that $J_i = J_{i'}$ from continuity of f . The latter can not occur

because of the assumption of $\underline{A}$. Therefore $\{J_i\}$ are disjoint. Notice that the first case corresponds to boundary curve of D_k^A and D_k^B .

The second case : $\underline{B}$ contains both L and R .

In this case $fJ_p = J_0$ and $f^{p+1}|_{J_i}$ has unique turning point c_i inside J_i . We set two slopes of $f^{p+1}|_{J_i}$ $\alpha (> 0)$, $\beta (< 0)$. We divide J_i into two subinterval I_{α_i} and I_{β_i} corresponding to slope α and β . As $f^{p+1}|_{J_i}$ is surjective on J_i , we have that $\sup\{|\alpha|, |\beta|\} > 1$.

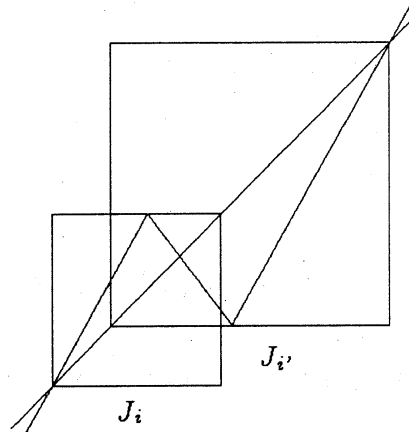


Figure 1: The graph of f^{p+1} on $J_{i'}$ having two turning points.

If $|\beta| < 1$, then turning point is attracted to a fixed point on I_β . It follows that $\underline{B} = L^\infty$ or R^∞ . This contradicts assumption of this case.

If $|\beta| = 1$, we reduce to the first case.

If $|\beta| > 1$ and $\text{int}(J_i) \cap \text{int}(J_{i'}) \neq \emptyset$, there exists $J_{i'}$ such that $f^{p+1}|_{J_{i'}}$ has two turning points (see Figure 1) or there exist i, i' such that $J_i = J_{i'}$. In the latter case we have J_m equals J_p for some $m (m \neq p)$. This contradicts that f is monotone on J_m because J_p includes turning point in it. Hence we obtain $\text{int}(J_i) \cap \text{int}(J_{i'}) = \emptyset$. Notice that the second case corresponds to D_k^B .

Conversely, if there exist disjoint invariant closed intervals $\{J_i\}_{i=0, \dots, |\underline{A}|}$ in theorem, we have $K(a, b) = \underline{A} \star \underline{B}$ with $\underline{A} = A_{J_0} A_{J_1} \cdots A_{J_{p-1}}$. If $\underline{B}$ is not prime, f has refinement of $\{J_i\}$. Hence, $\underline{B}$ is prime. $\square$

Now we have the relation of our renormalization (i.e., $(|\underline{A}|+1)$ - renormalization is a skew tent map of D) and $\star$ -product.

Corollary 1 If $|\underline{B}| \neq 2$ in above theorem, then f is renormalizable of level $|\underline{A}| + 1$.

Proof Let p be $|\underline{A}|$. In the first case, we have $|\underline{B}| = 2$ because a turning point of f on J_p is 2-periodic point of f^{p+1} . In the second case, we have $|\beta| > 1$ and $f^{p+1}J_i = J_i$. It follows $(\alpha, \beta) \in D$. Therefore f is $(p+1)$ -renormalizable on $[c_i, f^{p+1}(c_i)]$ (resp. $[f^{p+1}(c_i), c_i]$) if $c_i < f^{p+1}(c_i)$ (resp. $f^{p+1}(c_i) < c_i$). $\square$

It is well known that for a smooth unimodal map g , n -periodic g -admissible sequence implies the existence of n or $2n$ -periodic point ([Dev89]). This fact is proved by Schwarzian derivative. But we have the following analogous fact for skew tent maps.

Corollary 2 If $K(a, b) = \underline{A} \star \underline{B}$ where $\underline{A} (\neq \emptyset)$ is finite sequence of L 's and R 's and $\underline{B} (\neq C)$ is prime, then f has periodic points of period $|\underline{A}| + 1$. Moreover if $|\underline{B}| = 2$, then f also has periodic points of period $2(|\underline{A}| + 1)$.

Remark For showing Corollary 1 and 2, we need only the assumption $\underline{B} \neq C, L^\infty, R^\infty$ instead of primarity of $\underline{B}$.

3 Monotonicity of kneading sequences

In this section we will mention the monotonicity property of kneading sequence in the domain

$$\tilde{D} = \{(a, b) \in D; a \geq 1\}.$$

Let us define the order for parameter pairs as follows, according to M. Misiurewicz and E. Visinescu [MV91] :

$(a, b) \succ (a', b') \Leftrightarrow a' \geq a, b' \geq b$, and at least one of these inequalities is strict.

Kneading sequences are monotone increasing with respect to this order.

Monotonicity Theorem (Theorem A in [MV91]) For $(a', b'), (a, b)$ in $\tilde{D}$ with $(a', b') \succ (a, b)$, it holds that $K(a', b') > K(a, b)$.

This theorem is already proved in [MV91]. M. Misiurewicz and E. Visinescu showed the claim by using the estimation of topological entropy. But we shall reprove it by using only thier results for D^* in [MV91], and renormalization method, not via the topological entropy. For that purpose, we prepare Proposition 1, Proposition 2 and Proposition 3 (for the detailed proofs, see [Ich]).

As to $\star$ -product, we have the following.

Proposition 1 Let $\underline{A}$ and $\underline{B}$ be symbolic sequences of L, R , and C with $\underline{A} \succ \underline{B}$. Then for all $n \geq 1$, $R^{*n} \star \underline{A} \succ R^{*n} \star \underline{B}$.

3.1 Monotonicity in D^*

Let D^* be the domain $\{(a, b) \in D; a + b < ab^2, a > 1\}$.

M. Misiurewicz and E. Visinescu proved in [MV91] that $K(a', b') > K(a, b)$ for $(a', b'), (a, b) \in D^*$ such that $(a', b') \succ (a, b)$. This domain D^* is characterlized by the following.

Fact 1 (Lemma 2.1 in [MV91]) $(a, b) \in D^* \Leftrightarrow K(a, b) \succ RLR^\infty$.

First, monotone increasing property of kneading sequence is proved in D^* .

Fact 2 (Proposition 4.3 in [MV91]) If (a, b) and (a', b') are in D^* with $(a, b) \prec (a', b')$, then $K(a, b) \prec K(a', b')$.

3.2 Renormalization and $\star$ -product (for $\tilde{D}$)

Proposition 2 Let (a, b) be in D . The following three conditions are equivalent mutually.

(i) $(a, b) \in D_0$.

- (ii) There exists a unique number $m \geq 1$ and a prime sequence $\underline{B}$ whose length is longer than 2 such that $K(a, b) = R^{\star m} \star \underline{B}$.
- (iii) There exists some number $m \geq 1$ such that $\varphi^m(a, b) \in D^*$, where $\varphi(a, b) = (b^2, ab)$.

Furthermore, there exist closed subintervals of $I_{a,b}$, $\{I_i\}_{i=0, \dots, 2^m-1}$ such that their interiors are disjoint mutually, $f_{a,b}I_i = I_{i+1}$ for $0 \leq i \leq 2^m - 2$ and $f_{a,b}I_{2^m-1} = I_0$, $I_{2^m-1} \ni 0$, and $f_{a,b}^{2^m}|_{I_i} \sim f_{\varphi^m(a,b)}$.

Proposition 3 Let $(a, b), (a', b') \in \tilde{D} \setminus D^*$ such that $(a, b) \prec (a', b')$. If $\varphi^m(a, b) \in D^*$ and $\varphi^n(a', b') \in D^*$, then $m \geq n$.

3.3 Proof of Monotonicity Theorem

Assume that $(a, b) \prec (a', b')$.

- (i) If both (a, b) and (a', b') belong to D^* , then the proof is already given by Fact 2.
- (ii) Assume that either (a, b) or (a', b') belongs to D^* . Then (a', b') is in D^* because $(a, b) \prec (a', b')$. By virtue of Fact 1, $K(\varphi^n(a, b)) \preceq RLR^\infty \prec K(\varphi^n(a', b'))$. We have that $K(a, b) \prec K(a', b')$ as an order relation " $\prec$ " is total.
- (iii) Assume that (a, b) and (a', b') both belong to $\tilde{D} \setminus D^*$. Then, by Proposition 3, their kneading sequences are written as, for some $n \leq m$,

$$K(a, b) = R^{\star m} \star K(\varphi^m(a, b)) \quad \text{and} \quad K(a', b') = R^{\star n} \star K(\varphi^n(a, b)).$$

If $m = n$, then we have that $\varphi^n(a, b) \prec \varphi^n(a', b')$ since φ is an increasing function. Because $K(\varphi^n(a, b)) \prec K(\varphi^n(a', b'))$ and from Proposition 1, we have that $K(a, b) \prec K(a', b')$.

If $n < m$, then we have that $\varphi^n(a, b) \notin D^*$ and $\varphi^n(a', b') \in D^*$. By virtue of Fact 1, it follows that

$$K(\varphi^n(a, b)) \preceq RLR^\infty \prec K(\varphi^n(a', b')).$$

By Proposition 1, we have that $K(a, b) \prec K(a', b')$. □

4 Renormalization and topological entropy

Now we correct two statements of [MV92].

First : kneading sequence for boundary curve of $A_m (= D_{m+1}^A)$ and $B_m (= D_{m+1}^B)$.

In Theorem 1 of the paper [MV92], they say ;

$$(\lambda, \beta) (= (a, b)) \in A_m \Leftrightarrow K(\lambda, \beta) = (RL^m)^\infty,$$

$$(\lambda, \beta) \in B_m \Leftrightarrow K(\lambda, \beta) = RL^{m-1} \star \underline{B} \text{ with } \underline{B} \in M$$

where M is set of kneading sequence for tent map $f_{\lambda, \lambda}$ ($1 < \lambda \leq 2$).

A_m and B_m have common boundary curve : $\lambda^m \mu = 1$. In our opinion this curve should be discussed separately from A_m and from B_m . We find our reason in the fact that the kneading sequence on this curve is $RL^m RL^{m-1} C$, not admitted by one on A_m and on B_m .

Second : topological entropy of $B_1 (= D_0)$ is not constant.

In Corollary in [MV92], they say ;

let $(\lambda, \beta), (\lambda', \beta') \in \{(\lambda, \beta) \in D; \lambda \leq 1\}$ such that $(\lambda, \beta) < (\lambda', \beta')$,

$$(\lambda, \beta), (\lambda', \beta') \in A_m \cup B_m \Rightarrow h(\lambda, \beta) = h(\lambda', \beta').$$

Namely, topological entropy on B_m is constant for all $m (\geq 1)$. But we can show the followings :

Proposition 4 Let $(\lambda, \beta), (\lambda', \beta') \in \{(\lambda, \beta) \in D; \lambda \leq 1\}$. If $(\lambda, \beta) < (\lambda', \beta')$,

$$h(\lambda, \beta) < h(\lambda', \beta').$$

Proof From [MT88] we obtain that topological entropy of $f_{a,b}$ for B_1 naturally follows from one of its renormalized map of subdomain $a \geq 1$ where the strictly monotonicity holds. $\square$

A counter example to this statement is given in [Ich].

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