

the generalized Numerical Range and its Boundary

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Abstract. We consider the boundary of C -numerical range of a matrix and its related topics in the theory of invariant polynomials

1. the generalized numerical range of a matrix as a semi-algebraic set.

We recall the definition of the C -numerical range. Denote by $M_n(\mathbb{C})$ the set of all $n \times n$ complex matrices.

Def. Let A, C be $n \times n$ complex matrices. Set

$$W_C(A) = \{tr(CUAU^*) : U \in U(n)\} \quad (1.1)$$

where $U(n)$ is the compact group of all unitary matrices of order n . $W_C(A)$ is called the C -numerical range of A .

It is clear that C -numerical range is a unitary invariant of the square matrix A : If B is unitarily similar to A , i.e., $B = UAU^*$, then $W_C(B) = W_C(A)$. If C is a rank one orthogonal projection, then the C -numerical range $W_C(A)$ coincides with the classical numerical range $W(A)$,

$$W(A) = \{(A\xi, \xi) : \xi \in \mathbb{C}^n, \|\xi\| = 1\}.$$

The generalized numerical range $W_C(A)$ is the range of the real algebraic variety

$$U(n) = \{U = \{u_{i,j}\}_{1 \leq i,j \leq n} \in \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2} : \sum_{k=1}^n u_{i,k} \overline{u_{j,k}} = \delta_{i,j} \ (1 \leq i,j \leq n)\} \quad (1.2)$$

under the polynomial mapping

$$(1.3). \quad X \in M_n(\mathbb{C}) = \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2} \mapsto tr(CXAX^*) \in \mathbb{C} \cong \mathbb{R}^2$$

We recall Tarski-Seidenberg's theorem.

Def. A subset Λ of the n -dimensional affine space $\mathbb{R}^n$ is said to be semi-algebraic if Λ is an element of the Boolean algebra generated by the family of sets

$$Z_0(f) = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : f(x_1, x_2, \dots, x_n) = 0\},$$

and

$$Z_1(f) = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : f(x_1, x_2, \dots, x_n) > 0\}$$

where f varies over the real polynomial ring $\mathbb{R}[X_1, X_2, \dots, X_n]$.

In other words, Λ is a semi-algebraic set $(\subset \mathbb{R}^n) \Leftrightarrow_{\text{definition}}$

$$\Lambda = \bigcup_{j=1}^m \bigcap_{k=1}^{l_j} Z_{\epsilon_{j,k}}(f_{j,k})$$

where $\epsilon_{j,k} \in \{0, 1\}$, $f_{j,k} \in \mathbb{R}[X_1, X_2, \dots, X_n]$.

Of course, an algebraic set $(\subset \mathbb{R}^n)$ is semi-algebraic.

a Corollary of Tarski-Seidenberg's theorem.

The range $\phi(\Lambda)$ of a semi-algebraic set $\Lambda \subset \mathbb{R}^m$ under a polynomial mapping $\phi = (\phi_1, \phi_2, \dots, \phi_n) : \mathbb{R}^m \rightarrow \mathbb{R}^n$, is semi-algebraic, where $\phi_j \in \mathbb{R}[X_1, X_2, \dots, X_m]$ ($1 \leq j \leq n$).

A Simple Example

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$$

↓ projection

A closed disc

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}.$$

We remark that the closed disc is semi-algebraic and not algebraic.

Another Result about Semi-Algebraic Sets. The boundary $\partial\Lambda$ of a semi-algebraic set $\Lambda(\subset \mathbb{R}^n)$ is also semi-algebraic.

Using the result above, Takaguchi, Nishikawa and I proved the following.

Theorem. [Takaguchi=Nishikawa=Nakazato[1]] The boundary $\partial W_C(A)$ of C -numerical range $W_C(A) \subset \mathbb{C} \cong \mathbb{R}^2$ is the union of a finite number of real algebraic arcs.

Furthermore, using the theorem above and Bezout's theorem, we proved the following.

Theorem. The number of non ω -regular points of $\partial W_C(A)$ is finite.

2. Concrete C -numerical ranges

Second I consider the q -numerical ranges of matrices.

We set

$$C = C_q = \begin{pmatrix} q & \sqrt{1-|q|^2} & 0 & 0 \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We denote $W_{C_q}(A)$ by $W(A : q)$. We recall N.K.Tsing's formula.

A formula due to Tsing.

$$W(A : q) = \cup \{ q(A\xi, \xi) + \sqrt{1-|q|^2} r \exp(i\theta) \sqrt{||A\xi||^2 - |(A\xi, \xi)|^2} : \xi \in \mathbb{C}^n, ||\xi|| = 1, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi \} \quad (2.2).$$

This formula is rewritten as

$$W(A : q) = \cup \{ B(qz : \sqrt{\phi_A(z)}) : z \in W(A : 1) \} \quad (2.3),$$

where $\phi_A(z) = \max\{ ||A\xi||^2 - |z|^2 : \xi \in \mathbb{C}^n, ||\xi|| = 1, (A\xi, \xi) = z \}$ (2.5),
for $z \in W(A) = W(A : 1)$.

We set

$$\psi_A(z) = \max\{ ||A\xi||^2 : \xi \in \mathbb{C}^n, ||\xi|| = 1, (A\xi, \xi) = z \}.$$

We review N.K.Tsin's results.

Theorem. [N.K.Tsing [6]] ψ_A is a concave function on $W(A)$ and hence ϕ_A and $\sqrt{\phi_A}$ are also concave on $W(A)$.

Using the theorem above we can prove the following two claims.

Proposition. $A, B_1, \dots, B_m$: matrices. $W(A) = \cup_{j=1}^m W(B_j)$, $\phi_A(z) = \max\{ \phi_{B_j}(z) : 1 \leq j \leq m \}$,
 $\Rightarrow W(A : q) = \cup_{j=1}^m W(B_j : q)$

for every q with $|q| \leq 1$.

Theorem. [2] (1) Suppose that $\lambda_1, \lambda_2, \dots, \lambda_n$ are arbitrary complex numbers ($n \geq 3$). Then the following equation holds for every $z \in \text{Conv}(\{\lambda_1, \lambda_2, \dots, \lambda_n\})$:

$$\phi_{\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)}(z) = \max\{ \phi_{\text{diag}(\lambda_{j_1}, \lambda_{j_2}, \lambda_{j_3})}(z) : 1 \leq j_1 < j_2 < j_3 \leq n \}.$$

Hence

$$W(\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) : q) = \cup \{ W(\text{diag}(\lambda_{j_1}, \lambda_{j_2}, \lambda_{j_3}) : q) : 1 \leq j_1 < j_2 < j_3 \leq n \}.$$

(2) If $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ and $|\lambda_1| = |\lambda_2| = |\lambda_3| = 1$, then

$$\partial W(\text{diag}(\lambda_1, \lambda_2, \lambda_3) : q) \subseteq \{ z \in \mathbb{C} : |z| = 1 \} \cup \{ \partial W(\text{diag}(\lambda_j, \lambda_k) : q) : 1 \leq j < k \leq 3 \}.$$

3. C-numerical ranges of 2 by 2 matrices

The study of C -numerical ranges of 2×2 matrices is reduces to that of

$$W\left(\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} : q\right)$$

for $0 \leq \beta \leq \alpha, 0 \leq q \leq 1$.

We have the following formula(cf.[3]):

$$W\left(\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} : q\right)$$

$$= \{r[(\alpha + \beta)/2 + \sqrt{1 - q^2}(\alpha - \beta)/2]\cos(\theta) + \sqrt{-1}r[(\alpha - \beta)/2 + \sqrt{1 - q^2}(\alpha + \beta)/2]\sin(\theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}, \text{ where } 0 \leq \beta \leq \alpha, 0 \leq q \leq 1.$$

4. Relation between generalized numerical ranges and invariant polynomials

It is well known that the following three conditions for 2×2 matrices x, y are mutually equivalent: (i) $W(x) = W(y)$; (ii) $\text{tr}(x) = \text{tr}(y), \text{tr}(x^2) = \text{tr}(y^2), \text{tr}(x^*x) = \text{tr}(y^*y)$; (iii) $y = UxU^*$ for some unitary matrix.

For $n \times n$ matrices x, y the following three conditions are mutually equivalent: (iv) $W_P(x) = W_P(y)$ for every orthogonal projection P ; (v) $W_H(x) = W_H(y)$ for every Hermitian matrix H ; (vi) $\det(\lambda I_n - (\alpha x + \beta x^*)) = \det(\lambda I_n - (\alpha y + \beta y^*))$ for every $\lambda, \alpha, \beta \in \mathbb{C}$.

In the case $n = 3$, the condition (iv) is equivalent to the condition: $W(x) = W(y), \text{tr}(x) = \text{tr}(y)$.

Question. x, y : 3×3 matrices such that

$$\text{tr}(x) = \text{tr}(y), W(x : q) = W(y : q)$$

for every $0 \leq q \leq 1$.

$$\Rightarrow ? \quad \exists U : \text{a unitary matrix such that } y = UxU^* \text{ or } y = U({}^t x)U^*.$$

I had the question above under the influence of the following theorem.

Theorem. (S.Teranishi, Nagoya Math.J.1986[8]) . Let x, y be arbitrary 3×3 complex matrices. (1) If

$$\text{tr}(x^k) = \text{tr}(y^k) \quad (k = 1, 2, 3) \tag{4.1},$$

$$\text{tr}(x^*x) = \text{tr}(y^*y) \tag{4.2},$$

$$\text{tr}(x^{*2}x) = \text{tr}(y^{*2}y) \tag{4.3},$$

$$\text{tr}(x^{*2}x^2) = \text{tr}(y^{*2}y^2) \tag{4.4},$$

then there exists a unitary matrix U such that

$$y = UxU^* \text{ or } y = U({}^t x)U^*$$

(2) If the conditions (4.1),(4.2),(4.3),(4.4) holds and if

$$\text{tr}(x^{*2}x^2x^*x) = \text{tr}(y^{*2}y^2y^*y) \tag{4.5},$$

then there exists a unitary matrix U such that $y = UxU^*$.

We denote by $M_n(\mathbb{C})$ the C^* -algebra of all $n \times n$ complex matrices. I want to find a finite subset F of $M_n(\mathbb{C})$ such that $\{W_C(\cdot) : C \in F\}$ is a complete unitary invariant system(cf.[7],Theorem 10). This desire is influenced by the following theorem.

Theorem (cf.[9]) Let x, y be $n \times n$ complex matrices. Set $x_1 = x, x_2 = x^*, y_1 = y, y_2 = y^*$. If

$$\text{tr}(x_{i_1} x_{i_2} \cdots x_{i_p}) = \text{tr}(y_{i_1} y_{i_2} \cdots y_{i_p})$$

for every $1 \leq p \leq 2^n - 1, 1 \leq i_1, i_2, \dots, i_p \leq 2$, then there exists a unitary matrix U such that $y = UxU^*$.

We can obtain another criterion by using the following lemma.

Lemma. Let A, B be C^* -subalgebras of $M_n(\mathbb{C})$ such that $I_n \in A, I_n \in B$. If ϕ is a $*$ -isomorphism of A onto B satisfying $\text{tr}(\phi(x)) = \text{tr}(x)$ for every $x \in A$, then there exists a unitary matrix U such that $\phi(x) = UxU^*$ for every $x \in A$.

Proposition. If

$$\text{tr}(x_{i_1} x_{i_2} \cdots x_{i_p}) = \text{tr}(y_{i_1} y_{i_2} \cdots y_{i_p})$$

for every $1 \leq p \leq 4(n^2 - 3), 1 \leq i_1, i_2, \dots, i_p \leq 2$, then there exists a unitary matrix U , such that $y = UxU^*$.

This proposition is stronger than the theorem above under the condition

$$n \geq 8.$$

We have an analogous result.

Proposition. If

$$\text{tr}(x_{i_1} x_{i_2} \cdots x_{i_q}) = \text{tr}(y_{i_1} y_{i_2} \cdots y_{i_q})$$

for every $1 \leq q \leq 2(n^2 - 3), 1 \leq i_1, i_2, \dots, i_q \leq 2$, then there exists a $*$ -preserving, Hilbert-Schmidt norm preserving linear bijection of the C^* -algebra $C^*(x)$ onto the C^* -algebra $C^*(y)$.

5. 3 by 3 orthostochastic matrices

We consider a 3×3 real matrix

$$A(x, y, z : u, v, w) = \begin{pmatrix} x+w & y+u & z+v \\ z+u & x+v & y+w \\ y+v & z+w & x+u \end{pmatrix} = xe_1 + ye_2 + ze_3 + ue_4 + ve_5 + we_6,$$

for $x \geq 0, y \geq 0, z \geq 0, u \geq 0, v \geq 0, w \geq 0, x+y+z+u+v+w = 1$. Here

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Such a matrix $A(x, y, z : u, v, w)$ is called a 3×3 doubly stochastic matrix. A (3×3) doubly stochastic matrix D is said to be orthostochastic if it is the Hadamard product (entrywise product) of some (3×3) unitary matrix U and its conjugate matrix $\bar{U}$. The necessary and sufficient condition of Au-Yeng and Poon for a doubly stochastic matrix

$$A(x, y, z : u, v, w)$$

to be orthostochastic (cf.[5]) is stated in another fashion (cf.[4]):

$$x^2y^2 + x^2z^2 + y^2z^2 - 2xyz(x+y+z) + u^2v^2 + u^2w^2 + v^2w^2 - 2uvw(u+v+w) - 4uvw(x+y+z) - 4xyz(u+v+w) - 2(xy+xz+yz)(uv+uw+vw) \leq 0. \quad (5.1)$$

Using this characterization we can obtain the following proposition.

Proposition ([4]) Suppose that A_3 is the affine hull of the compact convex set of all 3×3 doubly stochastic matrices and O_3 is the compact set of all 3×3 orthostochastic matrices. Denote by ∂O_3 the boundary of O_3 in the affine space A_3 . Then the following equation holds:

$$\partial O_3 = \{g \circ g : g \in SO(3)\},$$

where $SO(3)$ is the group of rotations in the 3-dimensional Euclidean space.

Consider a linear functional Ψ of the complex vector space $M_3(\mathbb{C})$,

$$\Psi : X = \{x_{ij} : 1 \leq i, j \leq 3\} \in M_3(\mathbb{C}) \mapsto (x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33}).$$

Suppose that A and C are complex 3×3 diagonal matrices with diagonal entries

$$a = \{a_1, a_2, a_3\}, c = \{c_1, c_2, c_3\}$$

respectively. Consider the 3×3 matrix $[A : C]$ given as the product $({}^t a) c$, that is,

$$[A : C] = \begin{pmatrix} a_1 c_1 & a_1 c_2 & a_1 c_3 \\ a_2 c_1 & a_2 c_2 & a_2 c_3 \\ a_3 c_1 & a_3 c_2 & a_3 c_3 \end{pmatrix}.$$

Then the C -numerical range of A coincides with the set

$$\{\Psi([A : C] \circ G) : G \in O_3\}.$$

The following proposition is effective to determine the C -numerical range of a matrix A under the condition $A = C$ and A is a 3×3 complex diagonal matrix.

Proposition ([4]) Suppose that A is a complex 3×3 complex diagonal matrix. Then the A -numerical range $W_A(A)$ coincides with the set

$$\{\Psi([A : A] \circ G) : G \in O_3 \text{ and } ({}^t G) = G\}.$$

Under the condition

$$A = \text{diag}\{1, 0, \alpha\},$$

the description of the boundary of the range $W_A(A)$ concerns deeply the curve " (Steiner's) Deltoid". We define the Deltoid $D[z_0 : K : \exp[i\eta]]$ (z_0 is a complex number, K is a positive number, η is a real number) in the complex plane $\mathbb{C}$ by the following: Choose $\exp[i\theta]$ (θ is a real number) is an arbitrary cubic root of $\exp[i\eta]$.

$$D[z_0 : K : \exp[i\eta]] = \{z_0 + (K/3) \exp[i\theta](2 \exp[2i\phi] + \exp[-i\phi]) : 0 \leq \phi \leq 2\pi\}.$$

We obtain the following theorem.

Theorem ([4]) Suppose that $A = \text{diag}\{1, 0, \alpha\}$ and $\alpha = a + ib$ where a, b are real numbers with $b \neq 0$. Then the boundary of the range $W_A(A)$ is a subset of the following set

$$\{(1-t)(1+\alpha^2) + t : 0 \leq t \leq 1\} \cup \{(1-t)(1+\alpha^2) + 2t\alpha : 0 \leq t \leq 1\} \cup \{(1-t)(1+\alpha^2) + t(\alpha^2) : 0 \leq t \leq 1\}$$

$$\cup D[z_0 : K : \exp[i\eta]]$$

where

$$z_0 = (1/(4b^2))\alpha(\bar{\alpha} - 1)(3a^2 - 3a - b^2 + i[b + 4ab]),$$

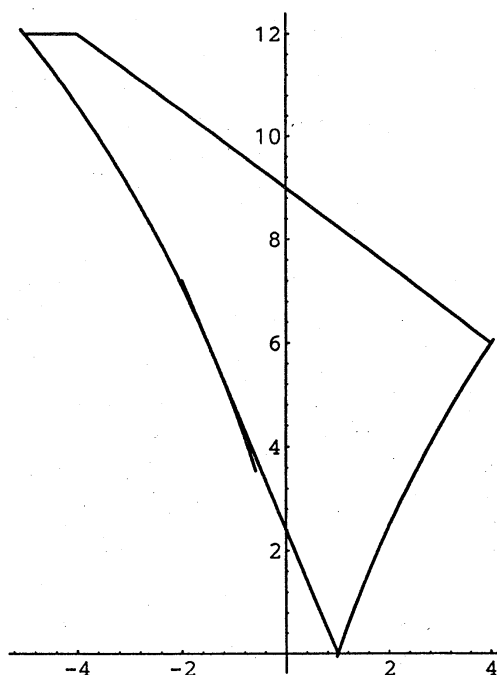
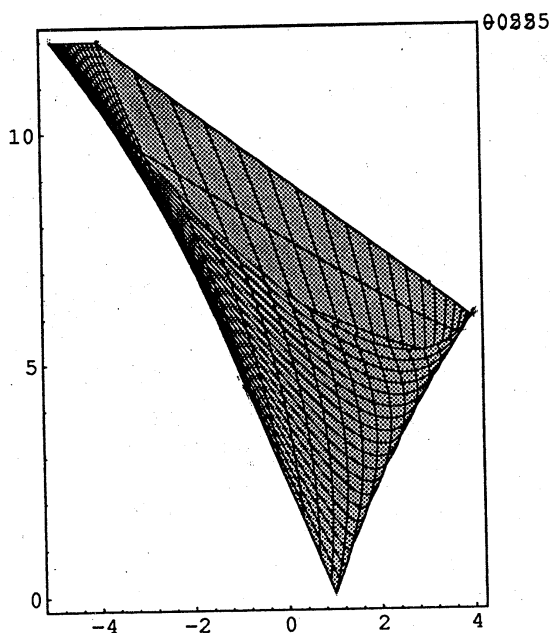
$$K = (3/(4b^2))|\alpha - 1|^2 |\alpha|^2,$$

$$|\alpha|^2 |\alpha - 1|^2 \exp[i \eta] = -(a - a^2 + b - 2a b + b^2)(a - a^2 - b + 2a b + b^2) + i(-2 b)(2a - 1)(a^2 - a - b^2).$$

We give you some graphics about C -numerical ranges after this document.

References

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$W_c(C)$

(II-3) $C = \text{diag}\{1, 0, 2 + 3i\}$.

Graphics of C-numerical Ranges of Matrices and the set of
3 x 3 symmetric orthostochastic matrices

(I) The q -numerical range of a 3 x 3 complex diagonal matrix A .

(i) The diagonal entries of A are mutually distinct cubic roots of 1.

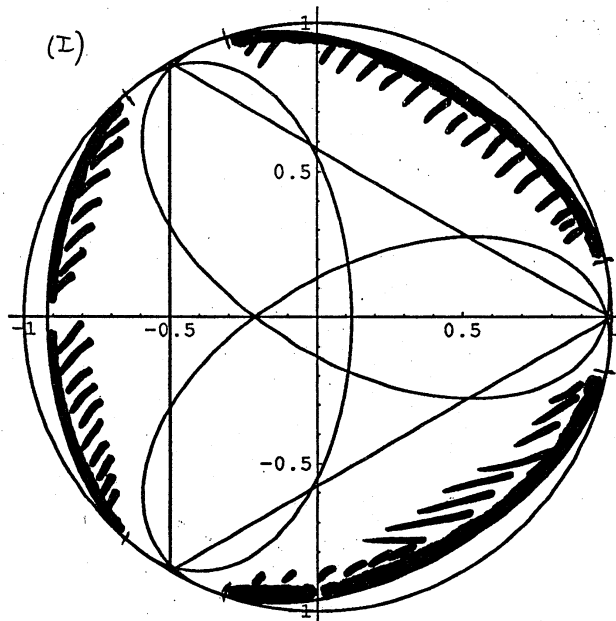
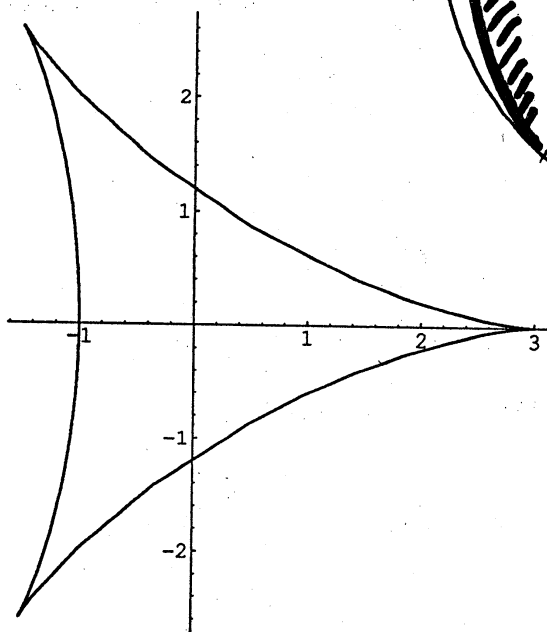
(ii) $q=4/5$.

(II) The C -numerical range $W_c(C)$ of the 3 x 3 complex diagonal matrix.

(II-1) The diagonal entries of C are mutually distinct cubic roots of 1. This generalized numerical range coincides with the closed domain surrounded by the Deltoid $D(0,3,1)$:

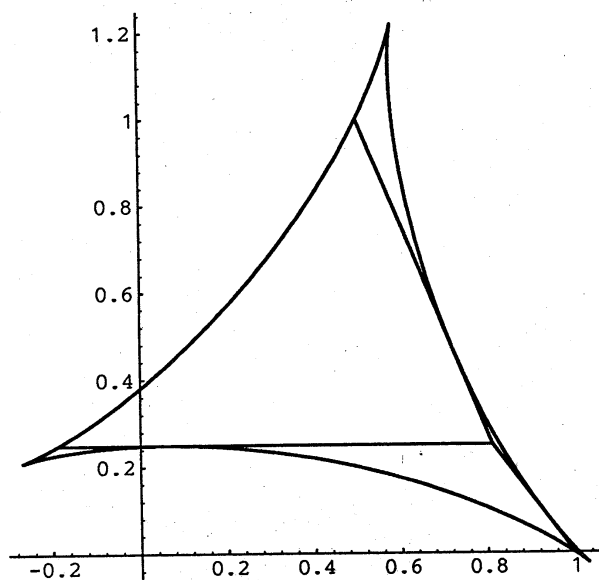
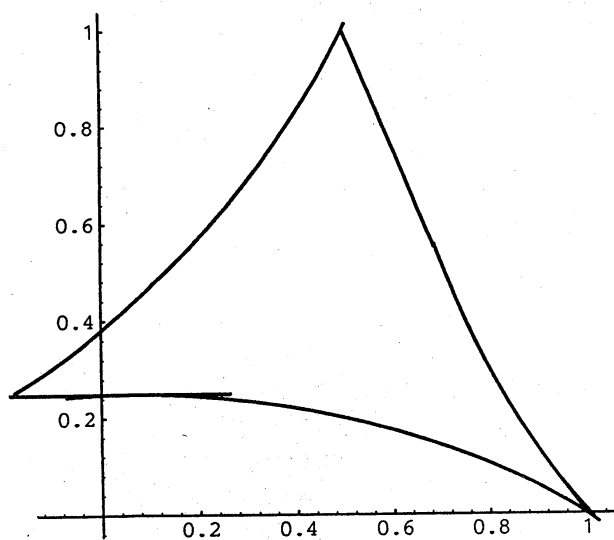
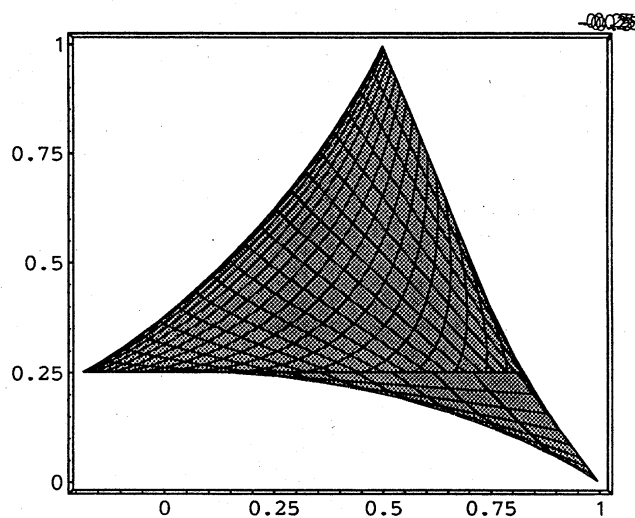
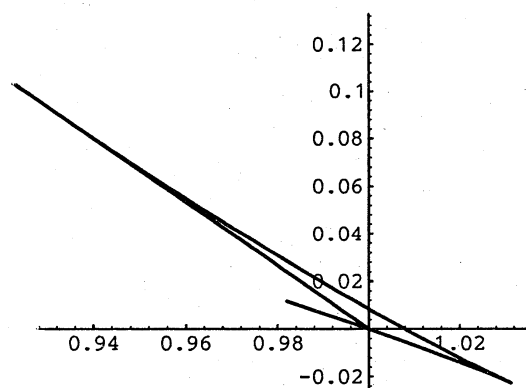
$$x=2 \cos(2t) + \cos(2t), y=2\sin(2t) - \sin(2t)$$

(II-1)



(II-2) $C = \text{diag}\{1, 0, (1/4) + i(1/2)\}$.

One graphic describes the range $W_c(C)$ based on the command
`ParametricPlot3D` (2-parameter plot) in `Mathematica` (ver2.0,
 hardware : Apple,Quadra 900) . Other three graphics describe
 its boundary based on the command `ParametricPlot` (1-parameter
 plot on the plane).



(III) The set of 3×3 symmetric orthostochastic matrices in the 3-dimensional affine space H of all 3×3 symmetric real matrices $\{(a_{11}, a_{12}, a_{13}), (a_{12}, a_{22}, a_{23}), (a_{13}, a_{23}, a_{33})\}$ such that

$$a_{11} + a_{12} + a_{13} = a_{12} + a_{22} + a_{23} = a_{13} + a_{23} + a_{33} = 1.$$

We use a non-orthogonal coordinates system $\langle \cdot, \cdot \rangle$ in H for which:

$$\{(1/3, 1/3, 1/3), (1/3, 1/3, 1/3), (1/3, 1/3, 1/3)\} = \langle 0, 0, 0 \rangle,$$

$$\{(0, 1, 0), (1, 0, 0), (0, 0, 1)\} = \langle 1, 0, 0 \rangle,$$

$$\{(0, 0, 1), (0, 1, 0), (1, 0, 0)\} = \langle -1/2, \sqrt{3}/2, 0 \rangle,$$

$$\{(1, 0, 0), (0, 0, 1), (0, 1, 0)\} = \langle -1/2, -\sqrt{3}/2, 0 \rangle,$$

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} = \langle 0, 0, b \rangle$$

for some $b > 0$.

