

Spectral theory and eigenfunction expansions for uniformly propagative systems

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1. Introduction.

In a recent year C.H.Wilcox showed that linear partial differential systems describing the wave propagation phenomena of classical physics in inhomogeneous anisotropic media filling the whole space can be written in the form

$$(1.1) \quad \frac{\partial u}{\partial t} = M(x)^{-1} \sum_{j=1}^n A_j \frac{\partial u}{\partial x_j} + f(x, t),$$

where $t \in \mathbb{R}^1$, $x \in \mathbb{R}^n$, $u(x, t)$ is $\mathbb{C}^m$ -valued function, $M(x)$ is an $m \times m$ positive definite hermitian matrix depending on $x \in \mathbb{R}^n$ and A_j 's are $m \times m$ constant hermitian matrices.

In this lecture we shall study some spectral properties, especially eigenfunction expansions, for the operator

$$(1.2) \quad L = M(x)^{-1} \sum_{j=1}^n A_j D_j \quad (D_j = \frac{1}{i} \frac{\partial}{\partial x_j})$$

in a suitable Hilbert space and scattering theory between system (1.2) and system

$$(1.3) \quad L_0 = \sum_{j=1}^n A_j D_j.$$

2. Preliminaries.

Our assumptions imposed on the operators are as follows:

(A.1) L_0 is uniformly propagative system in the sense of Wilcox,

i.e., the roots of the characteristic equation $p(\lambda, \xi) =$

$$\det \left(\lambda I - \sum_{j=1}^n A_j \xi_j \right) = 0 \text{ have constant multiplicities}$$

and never vanish unless vanish identically;

(A.2) There exist constants $C_1, C_2 > 0$ and $\delta > 1$ such that

$$(2.1) \quad C_1 |\xi|^2 \leq (\xi, M(x)\xi) \leq C_1^{-1} |\xi|^2 \quad \text{for all } x \text{ and } \xi \in \mathbb{R}^n,$$

$$(2.2) \quad \sup_{1 \leq i, j \leq n} |m_{ij}(x) - \delta_{ij}| \leq C_2 (1 + |x|^2)^{-\frac{\delta}{2}} \text{ for all } x \in \mathbb{R}^n,$$

where $m_{ij}(x)$ is the (i, j) -component of $M(x)$.

By assumption (A.1) the roots $\lambda_l(\xi)$ of $p(\lambda, \xi) = 0$ can be

enumerated as $\lambda_{\mu}(\xi) < \dots < \lambda_{-1}(\xi) < \lambda_0(\xi) < \lambda_1(\xi) < \dots < \lambda_{\mu}(\xi)$, where $\lambda_0(\xi) \equiv 0$

if it exists and will be omitted otherwise. Let us take $\delta_l(\xi) > 0$ so

small that $\Gamma_l(\xi) = \{ \zeta \in \mathbb{C}^1 : |\zeta - \lambda_l(\xi)| = \delta_l(\xi) \}$ does not enclose any

root of $p(\lambda, \xi) = 0$ except $\lambda_l(\xi)$ and put

$$(2.3) \quad P_l(\xi) = -\frac{1}{2\pi i} \oint_{\Gamma_l(\xi)} \left(\sum A_j \xi_j - \zeta I \right)^{-1} d\zeta.$$

Let $\hat{P}_l$ be the operator determined by the multiplication by $P_l(\xi)$

and put $P_l = \mathcal{F}^{-1} \hat{P}_l \mathcal{F}$. We put $S_l = \{ \xi \in \mathbb{R}^n; \lambda_l(\xi) = \text{sign } l \}$.

Let ds_l be the surface element of S_l and $d\sigma_l = ds_l / |\nabla \lambda_l|$.

We need some auxiliary spaces. For $\sigma \in \mathbb{R}^1$ and $s \in \mathbb{R}^1$ we put

$$H_{0,\sigma}^s = \{f \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^m) : \|f\|_{H_{0,\sigma}^s}^2 = \int_{\mathbb{R}^n} |\mathcal{F}^{-1}((1+|\xi|^2)^{\frac{\sigma}{2}} \mathcal{F}u)(\xi))|^2 (1+|x|^2)^s dx < \infty\},$$

$$H_{1,\sigma}^s = \{f \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^m) : \|f\|_{H_{1,\sigma}^s}^2 = \int_{\mathbb{R}^n} |\mathcal{F}^{-1}((1+|\xi|^2)^{\frac{\sigma}{2}} \mathcal{F}u)(\xi))|_M^2 (1+|x|^2)^s dx < \infty\},$$

where we put $(u(x), v(x))_M = (u(x), M(x)v(x))$;

$$H_0 = H_{0,0}^0, \quad H_1 = H_{1,0}^0.$$

By assumption (A.2) $H_{0,\sigma}^s = H_{1,\sigma}^s$ as a set. We define the identification operator $J: H_{1,\sigma}^s \rightarrow H_{0,\sigma}^s$ by the equation $Ju(x) = u(x)$.

We can define the selfadjoint realization L_0 naturally.

since it has constant coefficients. We put $L = J^* L_0 J$. L is obviously a selfadjoint operator in H_1 .

3. Theorems.

Our results are summarized in the following theorems.

Theorem 1. (Limiting absorption principle for L_0)

Let assumption (A.1) be satisfied. Let $I_0 = \mathbb{R}^1 \setminus \{0\}$ and let

$$\Pi^\pm = \{\xi \in \mathbb{C}^1 : \operatorname{Im} \xi \gtrless 0\}.$$

Let ε be any positive constant. Then the following statements hold:

(1) The resolvent $R_0(\xi)$ ($\operatorname{Im} \xi \neq 0$) of L_0 can be extended to

$$\Pi^\pm \cup I_0 \quad \text{as a } B(H_{0, \frac{1+\varepsilon}{2}}, H_{0, -\frac{1+\varepsilon}{2}})\text{-valued locally Hölder}$$

continuous function. Moreover $R_0(\xi)(I - P_0)$ can be extended to the

same regions as a $B(H_{0, \frac{1+\varepsilon}{2}}, H_{0, -\frac{1+\varepsilon}{2}}^1)$ -valued locally Hölder

continuous function. We denote their boundary values on I_0

as $R_0(\lambda \pm i0)$ and $R'_0(\lambda \pm i0)$, respectively.

(2) For any $u \in H_{0, (1+\varepsilon)/2}$ and $\lambda \in I_0$, $(L_0 - \lambda)R_0(\lambda \pm i0)u = u$.

Theorem 2. (limiting absorption principle for L) Let

assumptions (A.1) and (A.2) be satisfied. Let $I_1 = \mathbb{R}^1 \setminus (\sigma_p(L) \cup \{0\})$,

where $\sigma_p(L)$ is the point spectrum of L . Then the following

statements hold:

(1) The resolvent $R(\zeta)$ of L ($\text{Im} \zeta \neq 0$) can be extended to

$\mathbb{R}^1 \cup I_1$ as a $B(H_{1, \delta/2}, H_{1, -\delta/2})$ -valued locally Hölder continuous function.

(2) For any $u \in H_{1, \delta/2}$ and $\lambda \in I_1$, $(L - \lambda)R(\lambda \pm i0)u = u$.

Theorem 3. (Discreteness of the point spectrum) Let assumptions

(A.1) and (A.2) be satisfied. Then $\sigma_p(L) \setminus \{0\}$ is discrete

and only possible accumulation point is the origin.

Lemma 4. For each $l = 0$, let $\varphi_l^{(k)}(\omega)$, $k = 1, 2, 3, \dots$

be a C^∞ -class complete orthonormal system of $L^2(S_l, \mathbb{C}^1, d\sigma_l)$

and let $h_l(x, \lambda, k)$ be the matrix depending on $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}_{\text{sign } l}$

and $k \in \mathbb{N}$ (= the set of all integers) defined by

$$(3.1) \quad h_l(x, \lambda, k) = (2\pi)^{-\frac{n}{2}} \int_{S_l} e^{i|\lambda|\omega_l \cdot x} \varphi_l^{(k)}(\omega_l) \hat{P}_l(\omega_l) d\sigma_l(\omega_l).$$

Then $h_l(x, \lambda, k)$ is a bounded function of all variables. Further-

more for each fixed $\lambda \in \mathbb{R}_{\text{sign } j}$, $k \in \mathbb{N}$ and for any $\varepsilon > 0$ each column $h_j^{(i)}(x, \lambda, k)$ of $h_j(x, \lambda, k)$ satisfies the following relations:

(3.2) For any n -tuple of nonnegative integers $(\alpha_1, \alpha_2, \dots, \alpha_n)$,

$$D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n} h_j^{(i)}(x, \lambda, k) \in H_{0, -(1+\varepsilon)/2};$$

$$(3.3) \quad (L_0 - \lambda I) h_j^{(i)}(x, \lambda, k) = 0.$$

Theorem 5. (Eigenfunction expansion for the operator L_0)

Let $h_j(x, \lambda, k)$ ($j \neq 0$) be the function defined in Lemma 4.

Then for any $f \in H_0$ and $K > 0$, $\int_{|\lambda| < K} h_j(x, \lambda, k)^* f(x) dx$ belongs to $L^2(\mathbb{R}_{\text{sign } j}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$ and

$$(T_j f)(\lambda, k) = \lim_{K \rightarrow \infty} \int_{|\lambda| < K} h_j(x, \lambda, k)^* f(x) dx$$

exists. Let

$$T : H_0 \longrightarrow \sum_{j \neq 0} \oplus L^2(\mathbb{R}_{\text{sign } j}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$$

be the operator defined by

$$Tf = \sum_{j \neq 0} \oplus T_j f, \quad f \in H_0.$$

The operator $T' : \sum_{j \neq 0} \oplus L^2(\mathbb{R}_{\text{sign } j}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda) \rightarrow H_0$ can be defined by

$$T'(\sum_{j \neq 0} \oplus \hat{f}_j)(x) = \sum_{j \neq 0} \lim_{K \rightarrow \infty} \int_{\mathbb{R}_{\text{sign } j} \cap \{|\lambda| < K\}} \sum_{k=1}^K h_j(x, \lambda, k) \hat{f}_j(\lambda, k) |\lambda|^{\frac{n-1}{2}} d\lambda.$$

Furthermore the following statements hold:

(1) T is a partially isometric operator from H_0 into

$\sum_{j \neq 0} \oplus L^2(\mathbb{R}_{\text{sign } j}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$ and T' is its adjoint operator;

(2) (Expansion formula) $(I - P_0)f = T'T f$ for all $f \in H_0$;

(3) (Diagonal representation formula) $f \in D(L_0)$ if and only if

$$\lambda(T_0 f)(\lambda, k) \in L^2(\mathbb{R}_{\text{signe}}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda) \quad \text{for all } k \neq 0. \quad \text{If}$$

$$f \in D(L_0), \quad (T_0 L_0 f)(\lambda, k) = \lambda (T_0 f)(\lambda, k).$$

Lemma 6. Put $G(\lambda \pm i0) = J^{*-1} + \lambda(J^{*-1} - J)R(\lambda \pm i0) \in$

$B(H_1, \delta/2, H_0, +\delta/2)$. Let $\alpha_{\mathbf{q}}^{\pm}(x, \lambda, k)$ ($\mathbf{q} \neq 0$) be an $m \times m$ -matrix

depending on $x \in \mathbb{R}^n$, $\lambda \in I_1$ and $k \in \mathbb{N}$ defined by

$$(3.4) \quad \alpha_{\mathbf{q}}^{\pm}(x, \lambda, k) = G(\lambda \pm i0)^* h_{\mathbf{q}}(x, \lambda, k),$$

where $G(\lambda \pm i0)^*$ is applied to $h_{\mathbf{q}}(x, \lambda, k)$ by matrix multiplication

rule. Then the following statements hold:

(1) Each column $\alpha_{\mathbf{q}}^{\pm(i)}(x, \lambda, k)$ of $\alpha_{\mathbf{q}}^{\pm}(x, \lambda, k)$ is an $H_{1, -\delta/2^-}$

valued locally Hölder continuous function of $\lambda \in I_1$ for each fixed

$k \in \mathbb{N}$;

(2) $\alpha_{\mathbf{q}}^{\pm}(x, \lambda, k)$ can be decomposed into three parts as

$$(3.5) \quad \alpha_{\mathbf{q}}^{\pm}(x, \lambda, k) = J^{-1} h_{\mathbf{q}}(x, \lambda, k) + t_{\mathbf{q}}^{\pm}(x, \lambda, k) + w_{\mathbf{q}}^{\pm}(x, \lambda, k)$$

where $h_{\mathbf{q}}(x, \lambda, k)$ is the matrix defined by (3.1), $t_{\mathbf{q}}^{\pm}(x, \lambda, k)$

is an $H_{1, -(1+\varepsilon)/2^-}$ -valued continuous function of $\lambda \in I_1$ for

each fixed $k \in \mathbb{N}$ and any $\varepsilon > 0$, and is given by

$$(3.6) \quad t_{\mathbf{q}}^{\pm}(x, \lambda, k) = -\lambda J^{-1} \sum_{\text{sign } k = \text{sign } \eta} \mathfrak{F}^{-1} \left\{ \lim_{\eta \downarrow 0} \frac{[\hat{P}_{\mathbf{q}}(\xi) \mathfrak{F}((1 - M(\eta)) \alpha_{\mathbf{q}}^{\pm}(y, \lambda, k))(\xi)]|_{\lambda_k(\xi) = \lambda}}{\lambda_k(\xi) - (\lambda \pm i\eta)} \right\}$$

using $\alpha_{\mathbf{q}}^{\pm}(x, \lambda, k)$, and $w_{\mathbf{q}}^{\pm}(x, \lambda, k)$ is an $H_{1, (\delta-2)/2^-}$ -valued continuous

function of $\lambda \in L_1$ for each fixed $k \in N$;

$$(3) \quad L(D)\alpha_{\mathbf{j}}^{\pm}(x, \lambda, k) = \lambda \alpha_{\mathbf{j}}^{\pm}(x, \lambda, k), \text{ where } L(D) \text{ is applied}$$

by matrix multiplication rule and the differentiation is in the sense of distributions.

Remark. In the decomposition (3.5) of the function $\alpha_{\mathbf{j}}^{\pm}(x, \lambda, k)$ $J^{-1}h_{\mathbf{j}}(x, \lambda, k)$, $t_{\mathbf{j}}^{\pm}(x, \lambda, k)$ and $w_{\mathbf{j}}^{\pm}(x, \lambda, k)$ are considered to describe the state of the incident wave, outgoing (incoming) spherical wave, and the wave damping rapidly at infinity, respectively, in the description of the stationary scattering process. Furthermore $t_{\mathbf{j}}^{\pm}(x, \lambda, k)$'s are the quantities which are connected with the scattering amplitudes. (See formulas (3.6) and (3.8).)

Theorem 7. Let $\sigma_p^K(L) = \{\lambda \in \mathbb{R}^1: |\lambda - \mu| < K^{-1} \text{ for some } \mu \in \sigma_p(L)\}$ and let $I_{\mathbf{j}, K} = \mathbb{R}_{\text{sign } \mathbf{j}} \setminus [\{|\lambda| > K\} \cup \{|\lambda| < K^{-1}\} \cup \sigma_p^K(L)]$. Let $\alpha_{\mathbf{j}}^{\pm}(x, \lambda, k)$ be the matrix-valued function defined in Lemma 6.

Then for any $f \in H_1$ and $K > 0$, $\int_{|\lambda| < K} \alpha_{\mathbf{j}}^{\pm}(x, \lambda, k)^* M(x) f(x) dx$ belongs to $L^2(\mathbb{R}_{\text{sign } \mathbf{j}}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$ and

$$(Z_{\mathbf{j}}^{\pm} f)(\lambda, k) = \text{l.i.m.}_{K \rightarrow \infty} \int_{|\lambda| < K} \alpha_{\mathbf{j}}^{\pm}(x, \lambda, k)^* M(x) f(x) dx$$

exists. For any $\hat{f}_{\mathbf{j}}(\lambda, k) \in L^2(\mathbb{R}_{\text{sign } \mathbf{j}}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$ and $K > 0$,

$$\int_{I_{\mathbf{j}, K}} \sum_{k=1}^K \alpha_{\mathbf{j}}^{\pm}(x, \lambda, k) \hat{f}_{\mathbf{j}}(\lambda, k) |\lambda|^{\frac{n-1}{2}} d\lambda \text{ belongs to } H_1 \text{ and}$$

$$(Z_j^\pm \hat{f}_j)(x) = \lim_{K \rightarrow \infty} \int_{I_{j,K}} \sum_{k=1}^K \alpha_j^\pm(x, \lambda, k) \hat{f}_j(\lambda, k) |\lambda|^{\frac{n-1}{2}} d\lambda$$

exists, where l.i.m. stands for the convergence in H_1 and the

integration is Bochner integral. Let

$$Z^\pm : H_1 \longrightarrow \sum_{j \neq 0} \oplus L^2(\mathbb{R}_{\text{sign } j}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$$

and

$$Z'^\pm : \sum_{j \neq 0} \oplus L^2(\mathbb{R}_{\text{sign } j}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda) \longrightarrow H_1$$

be the operators defined by

$$Z^\pm f = \sum_{j \neq 0} \oplus Z_j^\pm f, \quad f \in H_1$$

and

$$Z'(\hat{f}_{-\mu}, \dots, \hat{f}_{-1}, \hat{f}_1, \dots, \hat{f}_\mu) = \sum_{j \neq 0} \oplus Z_j^\pm \hat{f}_j,$$

$$(\hat{f}_{-\mu}, \dots, \hat{f}_{-1}, \hat{f}_1, \dots, \hat{f}_\mu) \in \sum_{j \neq 0} \oplus L^2(\mathbb{R}_{\text{sign } j}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda),$$

respectively. Then the following statements hold:

(1) Let P_{ac} be the projection operator in H_1 onto the absolutely

continuous subspace with respect to L . Then $Z^\pm$ is a partially

isometric operator from H_1 into $\sum_{j \neq 0} \oplus L^2(\mathbb{R}_{\text{sign } j}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$

with initial set $P_{ac} H_1$ and $Z'^\pm$ is its adjoint operator;

(2) (Expansion formulas for L) For any $f \in H_1$

$$P_{ac} f = Z'^\pm Z^\pm f$$

$$= \sum_{j \neq 0} \lim_{K \rightarrow \infty} \left[\int_{I_{j,K}} \sum_{k=1}^K \alpha_j^\pm(x, \lambda, k) \times \right. \\ \left. \left\{ \lim_{K' \rightarrow \infty} \int_{|y| < K'} \alpha_j(y, \lambda, k)^* M(y) f(y) dy \right\} |\lambda|^{\frac{n-1}{2}} d\lambda \right];$$

(8)

(3) (Diagonal representations) For $f \in H_1$, $P_{ac} f \in D(L)$ if and only if $\lambda Z_{\mathfrak{f}}^{\pm} f(\lambda, \cdot) \in L^2(\mathbb{R}_{\text{sign } \mathfrak{f}}, \ell^2(\mathbb{C}^m), |\lambda|^{\frac{n-1}{2}} d\lambda)$ for each $\mathfrak{f} \neq 0$. Furthermore for $f \in H_1$ with $P_{ac} f \in D(L)$

$$(Z_{\mathfrak{f}}^{\pm} Lf)(\lambda, k) = \lambda (Z_{\mathfrak{f}} f)(\lambda, k), \quad \lambda \in I_{\mathfrak{f}}.$$

Theorem 8. (Orthogonality of eigenfunctions) The range $R(Z^{\pm})$ of $Z^{\pm}$ is equal to the range $R(T)$ of T , where T is the operator defined in Theorem 5.

Next we shall give the theorems concerning the applications of the previous theorems.

Theorem 9. (Existence of the wave operator and its completeness)

Let assumptions (A.1) and (A.2) be satisfied. Then the wave operators

$$W_{\pm}(L, L_0, J^*) = s\text{-}\lim_{t \rightarrow \pm\infty} e^{itL} J^* e^{-itL_0} (I - P_0),$$

$$W_{\pm}(L_0, L, J) = s\text{-}\lim_{t \rightarrow \pm\infty} e^{itL_0} J e^{-itL} P_{ac}$$

exist and therefore they are complete.

Theorem 10. Let assumptions (A.1) and (A.2) be satisfied.

Let $\Delta \subset I_0$ (or $\Delta \subset I_1$), then

$$W_{\pm}(L, L_0, J^*) E_0(\Delta) = Z^{**} T E_0(\Delta), \quad \Delta \subset I_0 \quad \text{and}$$

$$W_{\pm}(L_0, L, J) E(\Delta) = T^* Z^{\pm} E(\Delta), \quad \Delta \subset I_1.$$

Scattering operator S is customarily defined by the formula

$$(3.7) \quad S = W_+(L, L_0, J^*)^* W_-(L, L_0, J^*) .$$

Put $\hat{S} = TST^*$. Then S is a unitary operator $(I-P_0)H_0 \rightarrow P_{ac}H_1$

and $\hat{S}$ is a unitary operator $M \rightarrow M$, where $M = R(T) = R(Z^\pm)$.

Theorem 11. (Representation formula for the scattering

operator) For $l \neq 0$, let $F_{l,m}(\lambda, k, k')$ be the matrix depending

on $\lambda \in R_{\text{sign } l \cap I_1}$, $k \in N$, $k' \in N$, and $m(\text{sign } m = \text{sign } l$,

$m = -\mu, -\mu+1, \dots, -1, 1, \dots, \mu$) defined by

$$(3.8) \quad F_{l,m}(\lambda, k, k') = \int_{S_m} \varphi_m^{(k')}(\omega_m) P_l(\omega_m) \left[\mathcal{F}(\lambda)^{\frac{n-1}{2}} \lambda (M(x)-1) \alpha_l^+(x, \lambda, k) \right] \Big|_{\lambda_m(\xi)=\lambda} d\sigma_m(\omega_m),$$

Then for any fixed λ and k , each column of $F_{j,m}(\lambda, k, k')$

belongs to $\ell^2(\mathbb{C}^m)$ with respect to $k' \in N$ and for any $\hat{f}(k') \in$

$\ell^2(\mathbb{C}^m)$ and any fixed $\lambda \in R_{\text{sign } l \cap I_1}$,

$$\sum_{k'=1}^{\infty} F_{l,m}(\lambda, k, k') \hat{f}(k') \in \ell^2(\mathbb{C}^m)$$

with respect to k . Let

$$\hat{t}(\lambda) : \sum_{\text{sign } l = \text{sign } \lambda} \oplus \ell^2(\mathbb{C}^m) \rightarrow \sum_{\text{sign } l = \text{sign } \lambda} \oplus \ell^2(\mathbb{C}^m)$$

be the operator defined by

$$\begin{aligned} & \hat{t}(\lambda) \left(\sum_{\text{sign } l = \text{sign } \lambda} \oplus \hat{f}_j \right) (k) \\ &= \sum_{\text{sign } l = \text{sign } \lambda} \oplus \left(\sum_{\text{sign } m = \text{sign } \lambda} \sum_{k=1}^{\infty} F_{l,m}(\lambda, k, k') \hat{f}_m(k') \right). \end{aligned}$$

Then $\hat{t}(\lambda)$ is a compact operator on $\sum_{\text{sign } \mathfrak{l} = \text{sign } \lambda} \oplus \ell^2(\mathfrak{e}^{\mathfrak{m}})$ and the scattering operator $\hat{S}$ can be written in terms of the operator $t(\lambda)$ as follows: For any $f \in M = R(T) = R(Z^{\pm})$

$$(\hat{S}f)(\lambda, k) = f(\lambda, k) - 2\pi i [\hat{t}(\lambda)f(\lambda)](k) \quad \text{a.e. } \lambda \in \mathbb{R}^1.$$

Remark 1. $F_{\mathfrak{l}, \mathfrak{m}}(\lambda, k, k')$ is the k' -th Fourier coefficient of the numerator in (3.6) with respect to the complete orthonormal basis $\varphi_{\mathfrak{m}}^{(k')}(\omega_{\mathfrak{m}})$ as a element of $L^2(S_{\mathfrak{m}})$.

Remark 2. The operator

$$I - 2\pi i \hat{t}(\lambda) : \sum_{\text{sign } \mathfrak{l} = \text{sign } \lambda} \oplus \ell^2(\mathfrak{e}^{\mathfrak{m}}) \rightarrow \sum_{\text{sign } \mathfrak{l} = \text{sign } \lambda} \oplus \ell^2(\mathfrak{e}^{\mathfrak{m}})$$

is a strongly continuous unitary operator valued function of

$$\lambda \in \mathbb{R}^+ \cap I_1 \quad (\text{or } \mathbb{R}^- \cap I_1).$$

Remark 3. If $M(x) - 1$ decreases sufficiently rapidly at infinity we can prove that the operator $\hat{t}(\lambda)$ is Hilbert-Schmit type.

4. Concluding remark.

Our method for proving limiting absorption principles is the method of Agmon and the method for proving the expansion formulas are the method of Kato-Kuroda. In using these methods the following lemma proves to ^{play a} essential role:

Lemma 12. Let K be an arbitrary Hilbert space and

let $H^s(\mathbb{R}^1, K)$ be K -valued Sobolev space of order s in the usual

sense. Let $u \in H^s(\mathbb{R}^1, K)$ with $s > 1/2$. For $\mu \in \mathbb{R}^1$ put

$$u_1(\lambda, \mu) = \frac{u(\lambda) - u(\mu)}{\lambda - \mu} \quad \text{and} \quad u_2^\pm(\lambda, \mu) = \lim_{\eta \downarrow 0} \frac{u(\mu)}{\lambda - (\mu \pm i\eta)}.$$

We fix μ and regard u_1 and u_2 as K -valued distributions of

the variable λ . Then $u_1 \in H^{s-1}(\mathbb{R}^1, K)$ and $u_2^\pm \in H^{-(1+\varepsilon)/2}(\mathbb{R}^1, K)$

for any $\varepsilon > 0$, and there exist constants C_1 and C_2 such that

$$\|u_1(\cdot, \mu)\|_{H^{s-1}} \leq C_1 \|u\|_{H^s},$$

$$\|u_2^\pm(\cdot, \mu)\|_{H^{-(1+\varepsilon)/2}} \leq C_2 \|u\|_{H^s}.$$

$u_1(\cdot, \mu)$ (or $u_2^\pm(\cdot, \mu)$) are $H^{s-1}(\mathbb{R}^1, K)$ -valued (or $H^{-(1+\varepsilon)/2}$ -valued) continuous function of μ .

Furthermore the following equation holds in $H^{-(1+\varepsilon)/2}$:

$$\lim_{\eta \downarrow 0} \frac{u(\lambda)}{\lambda - (\mu \pm i\eta)} = \frac{u(\lambda) - u(\mu)}{\lambda - \mu} + \lim_{\eta \downarrow 0} \frac{u(\mu)}{\lambda - (\mu \pm i\eta)}.$$

For the proof of the theorems and lemmas we refer the readers to the papers [1] and [2].

5. References.

- [1] Yajima, K., The limiting absorption principle for uniformly propagative systems., J. Fac. Sci. Univ. Tokyo, Sec. 1A, 21, No. 1, (1974), 119-131.

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