

MEASURE-PRESERVING HOMEOMORPHISMS OF NONCOMPACT MANIFOLDS AND MASS FLOW TOWARD ENDS

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1. INTRODUCTION

This article is concerned with groups of measure-preserving homeomorphisms of non-compact topological manifolds. Suppose M is a connected n -manifold and ω is a good Radon measure of M with $\omega(\partial M) = 0$. Let $\mathcal{H}(M)$ denote the group of homeomorphisms of M equipped with the compact-open topology, and by $\mathcal{H}(M; \omega) \subset \mathcal{H}(M; \omega\text{-end-reg})$ we denote the subgroups consisting of ω -preserving homeomorphisms and ' ω -end-biregular' homeomorphisms of M . (When M is compact, the conditions related to ends are redundant and are suppressed from the notations.) For any subgroup $\mathcal{G}$ of $\mathcal{H}(M)$, the symbol $\mathcal{G}_0$ denotes the connected component of id_M in $\mathcal{G}$.

Relations of these groups are studied in [6, 2, 3, 4, 8]. When M is compact, A. Fathi [6] showed that $\mathcal{H}(M; \omega)$ is a SDR (strong deformation retract) of $\mathcal{H}(M; \omega\text{-reg})$ and that $\mathcal{H}(M; \omega\text{-reg})$ is HD (homotopy dense) in finite dimension in $\mathcal{H}(M)$. In case $n = 2$, since $\mathcal{H}(M)$ is an ANR, this implies that $\mathcal{H}(M; \omega\text{-reg})$ is HD in $\mathcal{H}(M)$ and $\mathcal{H}(M; \omega)$ is a SDR of $\mathcal{H}(M)$. When M is non-compact, R. Berlanga [2, 3, 4] extended Fathi's arguments and showed that $\mathcal{H}(M; \omega)$ is a SDR of $\mathcal{H}(M; \omega\text{-end-reg})$. In case $n = 2$, we have shown that $\mathcal{H}(M; \omega\text{-end-reg})_0$ is HD in $\mathcal{H}(M)_0$ and thus $\mathcal{H}(M; \omega)_0$ is a SDR of $\mathcal{H}(M)_0$ [8]. However, we have no general results on relations between $\mathcal{H}(M; \omega\text{-end-reg})$ and $\mathcal{H}(M)$ in dimension $n \geq 3$.

A. Fathi [6] also studied the internal structure of $\mathcal{H}(M; \omega)$. When M is compact, he defined a mass flow homomorphism $\tilde{\theta} : \tilde{\mathcal{H}}_0(M, \omega) \rightarrow H_1(M, \mathbb{R})$ or $\theta : \mathcal{H}_0(M, \omega) \rightarrow H_1(M, \mathbb{R})/\Gamma$ and studied the existence of a section of $\tilde{\theta}$ and the perfectness of $\text{Ker } \theta$. In this article we consider the non-compact case and study a mass flow homomorphism toward ends [9]. Let $\mathcal{H}_E(M; \omega)$ denote the subgroup consisting of all $h \in \mathcal{H}(M; \omega)$ which fix the ends of M . There is a natural continuous homomorphism $J : \mathcal{H}_E(M; \omega) \rightarrow V_\omega$ which measures mass flow toward ends. This quantity has been introduced in [1] as the end charge c_h ($h \in \mathcal{H}_E(M; \omega)$), which are finitely additive signed measure on the ends of M . We use the following presentation of this notion: If $h \in \mathcal{H}_E(M; \omega)$ and C is a Borel subset of M such that $\text{Fr } C$ is compact, then the mass transferred into C by h is counted

by $J_h(C) = \omega(C - h(C)) - \omega(h(C) - C)$. The range V_ω is the topological vector space of functions $J_h : C \mapsto J_h(C)$, which parametrize mass flow toward ends.

We use deformation of measures by engulfing isotopy in M and show that the mass flow homomorphism J has a continuous (non-homomorphic) section.

Theorem 1.1. *There exists a continuous map $s : V_\omega \rightarrow \mathcal{H}_\partial(M, \omega)_1$ such that $Js = id_{V_\omega}$ and $s(0) = id_M$.*

The topological group $\mathcal{H}_E(M, \omega)$ acts continuously on V_ω by $h \cdot a = J_h + a$ ($h \in \mathcal{H}_E(M, \omega)$, $a \in V_\omega$). The mass flow homomorphism $J : \mathcal{H}_E(M, \omega) \rightarrow V_\omega$ coincides with the orbit map at $0 \in V_\omega$. The existence of section for this orbit map and the contractibility of the base space V_ω implies the following consequences.

Corollary 1.1. (1) $\mathcal{H}_E(M; \omega) \cong \text{Ker } J \times V_\omega$.

(2) $\text{Ker } J$ is a strong deformation retract of $\mathcal{H}_E(M; \omega)$.

In [10] we have obtained a version of Theorem 1.1 for smooth manifolds and volume-preserving diffeomorphisms. In the succeeding sections we explain definition of the mass flow homomorphism J toward ends (§§ 2-4) and give some details of arguments to deduce Theorem 1.1 (§§ 5-6).

2. END COMPACTIFICATIONS

2.1. Conventions. Throughout the paper, X denotes a connected, locally connected, locally compact, separable metrizable space, and the symbols $\mathcal{O}(X)$, $\mathcal{F}(X)$, $\mathcal{K}(X)$, and $\mathcal{C}(X)$ denote the sets of open subsets, closed subsets, compact subsets, and connected components of X respectively. When A is a subset of X , the symbols $\text{Fr}_X A$, $\text{cl}_X A$ and $\text{Int}_X A$ denote the frontier, closure and interior of A relative to X .

The symbol $\mathcal{H}_A(X)$ denotes the group of homeomorphisms h of X onto itself with $h|_A = id_A$, equipped with the compact-open topology. This group includes various subgroups. $\mathcal{H}_A^c(X)$ denotes the subgroup consisting of homeomorphisms with compact support. When X is a polyhedron, $\mathcal{H}^{\text{PL}}(X)$ denotes the subgroup of PL-homeomorphisms of X . For any subgroup G of $\mathcal{H}(X)$, the symbols G_0 and G_1 denote the connected component and the path-component of id_M in G respectively. When $G \subset \mathcal{H}^c(X)$, by G_1^* we denote the subgroup of G_1 consisting of $h \in G$ which admits an isotopy $h_t \in G$ ($t \in [0, 1]$) such that $h_0 = id_X$, $h_1 = h$ and there exists $K \in \mathcal{K}(X)$ with $\text{Supp } h_t \subset K$ ($t \in [0, 1]$).

2.2. End compactifications. (cf. [4])

Suppose X is a noncompact, connected, locally connected, locally compact, separable metrizable space. An end of X is a function e which assigns an $e(K) \in \mathcal{C}(X - K)$ to each $K \in \mathcal{K}(X)$ such that $e(K_1) \supset e(K_2)$ if $K_1 \subset K_2$. The set of ends of X is denoted

by $E = E_X$. The end compactification of X is the space $\overline{X} = X \cup E$ equipped with the topology defined by the following conditions:

- (i) X is an open subspace of $\overline{X}$,
- (ii) the fundamental open neighborhoods of $e \in E$ are given by

$$N(e, K) = e(K) \cup \{e' \in E \mid e'(K) = e(K)\} \quad (K \in \mathcal{K}(X)).$$

Then, $\overline{X}$ is a connected, locally connected, compact, metrizable space, X is a dense open subset of $\overline{X}$ and E is a compact 0-dimensional subset of $\overline{X}$. We fix a metric d on $\overline{X}$. For any $\varepsilon > 0$ there exists a neighborhood U of E in $\overline{X}$ such that $\text{diam}_d C < \varepsilon$ ($C \in \mathcal{C}(U)$).

Consider the family $\mathcal{S} = \mathcal{S}(X) = \{C \subset X \mid \text{Fr}_X C : \text{compact}\}$. For each $C \in \mathcal{S}$ we set

$$\overline{C} = C \cup E_C, \quad E_C = \{e \in E_X \mid e(K) \subset C \text{ for some } K \in \mathcal{K}(X)\}.$$

Then, E_C is open and closed in E_X and $\overline{C}$ is a neighborhood of E_C in $\overline{X}$.

Lemma 2.1. *Let $C, D \in \mathcal{S}(X)$.*

- (1) (i) $C \cup D \in \mathcal{S}(X)$ and $E_{C \cup D} = E_C \cup E_D$.
- (ii) $C \cap D \in \mathcal{S}(X)$ and $E_{C \cap D} = E_C \cap E_D$.
- (iii) $X - C \in \mathcal{S}(X)$ and $E_{X-C} = E_X - E_C$.
- (2) (i) $E_C \subset E_D$ iff $C - D$ is relatively compact in X (i.e., has the compact closure in X).
- (ii) $E_C = E_D$ iff the symmetric difference $C \Delta D = (C - D) \cup (D - C)$ is relatively compact in X .

Each $h \in \mathcal{H}(X)$ has a unique extension $\overline{h} \in \mathcal{H}(\overline{X})$. The map $\mathcal{H}(X) \rightarrow \mathcal{H}(\overline{X}) : h \mapsto \overline{h}$ is a continuous group homomorphism. We set $\mathcal{H}_{A \cup E}(X) = \{h \in \mathcal{H}_A(X) \mid \overline{h}|_E = \text{id}_E\}$. Then $\mathcal{H}_{A \cup E}(X)_0 = \mathcal{H}_A(X)_0$, and if $C \in \mathcal{S}(X)$ and $h \in \mathcal{H}_E(X)$, then $h(C) \in \mathcal{S}(X)$ and $E_{h(C)} = E_C$.

3. FUNDAMENTAL FACTS ON RADON MEASURES

Next we recall general facts on spaces of Radon measures cf. [4, 6]. Suppose X is a connected, locally connected, locally compact, separable metrizable space.

3.1. Spaces of Radon measures.

Let $\mathcal{B}(X)$ denote the σ -algebra of Borel subsets of X . A Radon measure on X is a measure μ on the measurable space $(X, \mathcal{B}(X))$ such that $\mu(K) < \infty$ for any compact subset K of X . Let $\mathcal{M}(X)$ denote the set of Radon measures on X . We say that $\mu \in \mathcal{M}(X)$ is good if $\mu(p) = 0$ for any point $p \in X$ and $\mu(U) > 0$ for any nonempty open subset U of X . For $A \in \mathcal{B}(X)$ let $\mathcal{M}_g^A(X)$ denote the set of good Radon measures μ on X with $\mu(A) = 0$.

The weak topology w on $\mathcal{M}(X)$ is the weakest topology such that the function

$$\Phi_f : \mathcal{M}(X) \rightarrow \mathbb{R} : \Phi_f(\mu) = \int_X f d\mu$$

is continuous for any continuous function $f : X \rightarrow \mathbb{R}$ with compact support. The notation $\mathcal{M}(X)_w$ denotes the space $\mathcal{M}(X)$ equipped with the weak topology w .

For $\mu \in \mathcal{M}(X)$ and $A \in \mathcal{B}(X)$ the restriction $\mu|_A \in \mathcal{M}(A)$ is defined by $(\mu|_A)(B) = \mu(B)$ ($B \in \mathcal{B}(A)$). For any $A \in \mathcal{F}(X)$ the restriction map $\mathcal{M}^{\text{Fr}A}(X)_w \rightarrow \mathcal{M}(A)_w : \mu \mapsto \mu|_A$ is continuous, and for any $K \in \mathcal{K}(X)$ the map $\mathcal{M}^{\text{Fr}K}(X)_w \rightarrow \mathbb{R} : \mu \mapsto \mu(K)$ is continuous ([4, Lemma 2.2]).

3.2. Action of homeomorphism groups.

Suppose $A \in \mathcal{B}(X)$ and $\omega \in \mathcal{M}(X)$.

Definition 3.1. $\mu \in \mathcal{M}(X)$ is said to be

- (i) ω -biregular if μ and ω have same null sets (i.e., $\mu(B) = 0$ iff $\omega(B) = 0$ for any $B \in \mathcal{B}(X)$),
- (ii) ω -mass-biregular if μ is ω -biregular and $\mu(X) = \omega(X)$,
- (iii) ω -cpt-biregular if μ is ω -biregular and $\mu|_{X-K} = \omega|_{X-K}$ for some $K \in \mathcal{K}(X)$.

The corresponding subspaces are denoted by the following symbols respectively:

$$\mathcal{M}(X, \omega\text{-reg}), \quad \mathcal{M}(X, \omega\text{-mass-reg}), \quad \mathcal{M}(X, \omega\text{-cpt-reg}).$$

Definition 3.2. $h \in \mathcal{H}(X)$ is said to be

- (i) ω -preserving if $h_*\omega = \omega$ (i.e., $\omega(h(B)) = \omega(B)$ for any $B \in \mathcal{B}(X)$),
- (ii) ω -biregular if $h_*\omega$ and ω have the same null sets
(i.e., $\omega(h(B)) = 0$ iff $\omega(B) = 0$ for any $B \in \mathcal{B}(X)$) ([6]).

The corresponding subgroups are denoted by the following symbols:

$$\mathcal{H}(X; \omega) = \{h \in \mathcal{H}(X) \mid h : \omega\text{-preserving}\}, \quad \mathcal{H}(X; \omega\text{-reg}) = \{h \in \mathcal{H}(X) \mid h : \omega\text{-biregular}\}.$$

The group $\mathcal{H}(X)$ acts continuously on $\mathcal{M}(X)_w$ by $h \cdot \mu = h_*\mu$ ($h \in \mathcal{H}(X)$, $\mu \in \mathcal{M}(X)$). The orbit map at $\omega \in \mathcal{M}(X)$ is defined by $\pi_\omega : \mathcal{H}(X) \rightarrow \mathcal{M}(X)$, $\pi_\omega(h) = h_*\omega$. The subgroup $\mathcal{H}(X; \omega)$ coincides with the stabilizer of ω under this action.

Suppose M is a compact connected n -manifold. The von Neumann-Oxtoby-Ulam theorem [7] asserts that if $\mu, \nu \in \mathcal{M}_g^\partial(M)$ and $\mu(M) = \nu(M)$, then there exists $h \in \mathcal{H}_\partial(M)_0$ such that $h_*\mu = \nu$. A. Fathi [6] extended this theorem to a parametrized version.

Theorem 3.1. Suppose $\mu, \nu : P \rightarrow \mathcal{M}_g^\partial(M; \omega\text{-reg})_w$ are continuous maps with $\mu_p(M) = \nu_p(M)$ ($p \in P$). Then there exists a continuous map $h : P \rightarrow \mathcal{H}_\partial(M; \omega\text{-reg})_1$ such that $(h_p)_*\mu_p = \nu_p$ ($p \in P$) and if $p \in P$ and $\mu_p = \nu_p$, then $h_p = \text{id}_M$.

R. Berlanga [4] obtained a similar theorem in the case that M is noncompact.

3.3. Spaces of Radon measures with direct limit topology.

Suppose $A \in \mathcal{B}(X)$ and $\omega \in \mathcal{M}_g^A(X)$. Let $\mathcal{B}_\omega(X) = \{C \in \mathcal{B}(X) \mid \omega(\text{Fr}_X C) = 0\}$ and $\mathcal{F}_\omega(X) = \mathcal{F}(X) \cap \mathcal{B}_\omega(X)$. For $\mu, \nu \in \mathcal{M}_g^A(X, \omega\text{-cpt-reg})$ and $C \in \mathcal{B}(X)$ we define $(\mu - \nu)(C) \in \mathbb{R}$ by

$$(\mu - \nu)(C) = (\mu - \nu)(C \cap K), \text{ where } K \text{ is any compact subset of } X \text{ such that } \mu|_{X-K} = \nu|_{X-K}.$$

For the sake of notational simplicity, we put $\mathcal{M} = \mathcal{M}_g^A(X, \omega\text{-cpt-reg})$ (as a set). Let $C \in \mathcal{B}_\omega(X)$. One can see that the function $\mathcal{M}_w \times \mathcal{M}_w \rightarrow \mathbb{R} : (\mu, \nu) \mapsto (\mu - \nu)(C)$ is not continuous (if X is noncompact). This forces us to introduce the direct limit topology lim instead of the weak topology w (cf. [4, p 244]). For each $K \in \mathcal{K}(X)$ consider the subspace $\mathcal{M}_{K,w} = \left\{ \mu \in \mathcal{M}_w \mid \mu|_{X-K} = \omega|_{X-K} \right\}$ of $\mathcal{M}_w$. The family $\{\mathcal{M}_K\}_{K \in \mathcal{K}(X)}$ is a closed cover of $\mathcal{M}_w$ (cf. [4, Lemma 3.1]). The topology lim on $\mathcal{M}$ is the finest topology on $\mathcal{M}$ such that the inclusion $i_K : \mathcal{M}_{K,w} \subset \mathcal{M}$ is continuous for each $K \in \mathcal{K}(X)$. The space $\mathcal{M}$ equipped with this topology is denoted by $\mathcal{M}_{\text{lim}} = \mathcal{M}_g^A(X, \omega\text{-cpt-reg})_{\text{lim}}$. Each $\mathcal{M}_{K,w}$ is a closed subspace of $\mathcal{M}_{\text{lim}}$ and a map $f : \mathcal{M}_{\text{lim}} \rightarrow Z$ is continuous iff the composition $f i_K$ is continuous for each $K \in \mathcal{K}(X)$.

Lemma 3.1. (1) Suppose $\mu, \nu : P \rightarrow \mathcal{M}_g^A(X, \omega\text{-cpt-reg})_{\text{lim}}$ are continuous maps and $C \in \mathcal{B}_\omega(X)$. Then the map $P \rightarrow \mathbb{R} : p \mapsto (\mu_p - \nu_p)(C)$ is continuous.

(2) Suppose $F \in \mathcal{F}_\omega(X)$ and F is regular closed (i.e., $F = \text{cl } U$ for some $U \in \mathcal{O}(X)$). Then the restriction map $r : \mathcal{M}_g^A(X, \omega\text{-cpt-reg})_{\text{lim}} \rightarrow \mathcal{M}_g^{A \cap F}(F, \omega|_F\text{-cpt-reg})_{\text{lim}} : r(\mu) = \mu|_F$ is continuous.

Definition 3.3. Suppose G is a subgroup of $\mathcal{H}(X)$. Consider the following condition $(*)$ on a map $h : P \rightarrow G$.

- $(*)_0$ h is continuous.
- $(*)_1$ For any $p \in P$ there exists an open neighborhood U of p in P and $K \in \mathcal{K}(X)$ such that $h(U) \subset \mathcal{H}_{X-K}(X)$.
- $(*)_2$ There exists a locally compact T_2 space Q and continuous maps $f : P \rightarrow Q$, $g : Q \rightarrow G$ such that $h = gf$.

Since G is a topological group, if $h, k : P \rightarrow G$ satisfy the condition $(*)$, then the inverse $h^{-1} : P \rightarrow G : (h^{-1})_p = (h_p)^{-1}$ and the composition $kh : P \rightarrow G : (kh)_p = k_p h_p$ satisfy the same condition.

Lemma 3.2. Suppose $\mu, \nu : P \rightarrow \mathcal{M}_g^A(X, \omega\text{-cpt-reg})_{\text{lim}}$ are continuous maps and $h : P \rightarrow \mathcal{H}_A^c(X, \omega\text{-reg})$ satisfies the condition $(*)$.

- (1) For any $C \in \mathcal{B}_\omega(X)$ the map $\varphi : P \rightarrow \mathbb{R} : \varphi(p) = ((h_p)_* \mu_p - \nu_p)(C)$ is continuous.
- (2) The map $\psi : P \rightarrow \mathcal{M}_g^A(X, \omega\text{-cpt-reg})_{\text{lim}} : \psi(p) = (h_p)_* \mu_p$ is continuous.

4. MASS FLOW HOMOMORPHISM TOWARD ENDS

Suppose X is a connected, locally connected, locally compact separable, metrizable space and $\mu \in \mathcal{M}(X)$. Let $\mathcal{S}_b = \mathcal{S}_b(X) = \mathcal{S}(X) \cap \mathcal{B}(X)$.

Definition 4.1. For $h \in \mathcal{H}_E(X, \mu)$ we define a function $J_h = J_h^\mu : \mathcal{S}_b \rightarrow \mathbb{R}$ as follows: Since $\bar{h}|_E = id$, for $C \in \mathcal{S}_b$ it follows that $E_C = E_{h(C)}$ and that $C \Delta h(C) = (C - h(C)) \cup (h(C) - C)$ is relatively compact in X (Lemma 2.1 (2)(ii)). Thus $\mu(C - h(C)), \mu(h(C) - C) < \infty$ and we can set $J_h(C) = \mu(C - h(C)) - \mu(h(C) - C)$.

Lemma 4.1. Let $C, D \in \mathcal{S}_b$.

- (1) (i) If $D \subset C \cap h(C)$ and $cl_X(C - D)$ is compact, then $J_h(C) = \mu(C - D) - \mu(h(C) - D)$.
- (ii) If $L \in \mathcal{K}(X)$ and $C \cup L = h(C) \cup L$, then $J_h(C) = \mu(C \cap L) - \mu(h(C) \cap L)$.
- (2) If $cl_X(C \Delta D)$ is compact (i.e. $E_C = E_D$), then $J_h(C) = J_h(D)$.
- (3) If $C \cap D = \emptyset$, then $J_h(C \cup D) = J_h(C) + J_h(D)$.
- (4) If $\mu(C) < \infty$, then $J_h(C) = 0$.
- (5) $J_h(X) = 0$.

This lemma suggests the next definition of the mass flow homomorphism J .

Definition 4.2.

- (1) $V_\mu = V_\mu(X) = \{a : \mathcal{S}_b \rightarrow \mathbb{R} \mid (*_1), (*_2), (*_3), (*_4)\}$
 - $(*)_1$ If $C, D \in \mathcal{S}_b$ and $cl_X(C \Delta D)$ is compact (i.e., $E_C = E_D$), then $a(C) = a(D)$.
 - $(*)_2$ If $C, D \in \mathcal{S}_b$ and $C \cap D = \emptyset$, then $a(C \cup D) = a(C) + a(D)$.
 - $(*)_3$ If $C \in \mathcal{S}_b$ and $\mu(C) < \infty$, then $a(C) = 0$.
 - $(*)_4$ $a(X) = 0$
- (2) $J : \mathcal{H}_E(X, \mu) \rightarrow V_\mu : h \mapsto J_h$.

For $a, b \in V_\mu$ and $\alpha, \beta \in \mathbb{R}$, we define $\alpha a + \beta b \in V_\mu$ by $(\alpha a + \beta b)(C) = \alpha a(C) + \beta b(C)$ ($C \in \mathcal{S}_b$). Then V_μ is a real vector space under these addition and scalar product. We equip V_μ the product topology, that is, the topology induced by the projections $\pi_C : V_\mu \rightarrow \mathbb{R} : \pi_C(a) = a(C)$ ($C \in \mathcal{S}_b$). Thus, a map $f : Y \rightarrow V_\mu$ is continuous iff $\pi_C f : Y \rightarrow \mathbb{R}$ is continuous for each $C \in \mathcal{S}_b$. With this topology V_μ is a topological vector space.

Lemma 4.2. J is a continuous group homomorphism.

5. DEFORMATION OF MEASURES BY ENGULFING ISOTOPY

Throughout this section we suppose M^n is a connected separable metrizable PL n -manifold, d is any metric on the end compactification $\bar{M}$, and $\omega \in \mathcal{M}_g^\partial(M)$. As a consistency condition between the PL-structure of M and the measure ω , we assume that

$\mathcal{H}_g^{\text{PL}}(M) \subset \mathcal{H}(M; \omega\text{-reg})$. It follows that $\omega(K) = 0$ for any subpolyhedron K of M with $\dim K \leq n - 1$.

5.1. Deformation of measures by engulfing isotopy.

Consider a decomposition $M = L \cup_S N$ such that

- (i) L and N are connected PL n -submanifolds of M with $S = L \cap N = \text{Fr}_M L = \text{Fr}_M N$,
- (ii) S is a compact proper PL $(n - 1)$ -submanifold of M .

Lemma 5.1. *There exists a continuous map $f : (-\infty, \infty) \rightarrow \mathcal{H}_g^{\text{PL},c}(M)_1^*$ such that*

- (i) (a) $f_0 = \text{id}$, (b) $f_s(L) \subsetneq f_t(L)$ ($s < t$),
 (c) *there exists a subpolyhedron F of M such that $\dim F \leq n - 1$, $\partial M \subset F$, and for any $K \in \mathcal{K}(M - F)$ there exist $-\infty < s < t < \infty$ with $K \subset f_t(L) - f_s(L)$,*
- (ii) f satisfies the condition $(*)$,
- (iii) $\{f_t\}_{-\infty < t < \infty}$ is equi-continuous with respect to $d|_M$.

This engulfing isotopy f_t can be used to deform measures. Let $h_t = f_t^{-1}$. Then, for any $\mu \in \mathcal{M}_g^\partial(M, \omega\text{-reg})$ the function $(-\infty, \infty) \rightarrow (-\mu(L), \mu(N)) : t \mapsto ((h_t)_*\mu - \mu)(L)$ is a monotonically increasing homeomorphism. Consider the map

$$\lambda : \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{\lim}^2 \times \mathbb{R} \rightarrow \mathbb{R} : \lambda(\mu, \nu, t) = ((h_t)_*\mu - \nu)(L).$$

By Lemma 3.2 (1) λ is continuous. Since $((h_t)_*\mu - \nu)(L) = ((h_t)_*\mu - \mu)(L) + (\mu - \nu)(L)$, for any $\mu, \nu \in \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})$, the function $\lambda(\mu, \nu, *) : \mathbb{R} \rightarrow (-\mu(L), \mu(N)) + (\mu - \nu)(L)$ is a monotonically increasing homeomorphism. When $\mu(M) < \infty$, we have that $a \in (-\mu(L), \mu(N)) + (\mu - \nu)(L)$ iff $0 < a + \nu(L) < \mu(M)$. We need the inverse of the above homeomorphism.

Definition 5.1. We define a map $t : \mathcal{V} \rightarrow \mathbb{R}$ as follows:

- (1) $\mathcal{V} = \mathcal{V}_g^\partial(M, \omega\text{-cpt-reg})_{\lim}$
 $= \left\{ (\mu, \nu, a) \in \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{\lim}^2 \times \mathbb{R} \mid a \in (-\mu(L), \mu(N)) + (\mu - \nu)(L) \right\}$
- (2) $t : \mathcal{V} \rightarrow \mathbb{R} : t(\mu, \nu, a) = \lambda(\mu, \nu, *)^{-1}(a)$ (i.e., $t = t(\mu, \nu, a)$ iff $a = \lambda(\mu, \nu, t)$)

Then (i) $t : \mathcal{V} \rightarrow \mathbb{R}$ is continuous, and (ii) $(\mu - \nu)(L) = a$ iff $t(\mu, \nu, a) = 0$.

Lemma 5.2. *The map $H : \mathcal{V} \rightarrow \mathcal{H}_g^{\text{PL},c}(M)_1^*$, $H_{(\mu, \nu, a)} = h_{t(\mu, \nu, a)}$ has the following properties:*

- (i) H satisfies the condition $(*)$.
- (ii) $(H_{(\mu, \nu, a)})_*\mu : \mathcal{V} \rightarrow \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{\lim} : (\mu, \nu, a) \mapsto (H(\mu, \nu, a))_*\mu$ is continuous.
- (iii) (a) $((H_{(\mu, \nu, a)})_*\mu - \nu)(L) = a$, (b) $(\mu - \nu)(L) = a$ iff $H_{(\mu, \nu, a)} = \text{id}$.
- (iv) $\{H_{(\mu, \nu, a)}^{-1}\}_{(\mu, \nu, a) \in \mathcal{V}}$ is equi-continuous with respect to $d|_M$.

Lemma 5.3. Suppose $\mu, \nu : P \rightarrow \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{\text{lim}}$ and $a : P \rightarrow \mathbb{R}$ are continuous maps such that $a_p \in (-\mu_p(L), \mu_p(N)) + (\mu_p - \nu_p)(L)$ ($p \in P$). Then the map $h : P \rightarrow \mathcal{H}_\partial^{\text{PL},c}(M, \omega\text{-reg})_1$, $h_p = H_{(\mu_p, \nu_p, a_p)}$ has the following properties:

- (i) h satisfies the condition $(*)$,
- (ii) The map $h_*\mu : P \rightarrow \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{\text{lim}} : p \mapsto (h_p)_*\mu_p$ is continuous.
- (iii) (a) $((h_p)_*\mu_p - \nu_p)(L) = a_p$, (b) $(\mu_p - \nu_p)(L) = a_p$ iff $h_p = \text{id}_M$.
- (iv) $\{h_p^{-1}\}_p$ is equi-continuous with respect to $d|_M$.

5.2. Fundamental deformation lemma.

Consider a decomposition $M = N \cup A$, $A = A_1 \cup \dots \cup A_m$, such that

- (i) N is a connected PL n -submanifold of M such that $\text{Fr}_M N$ is a compact PL $(n-1)$ -submanifold of M .
- (ii) $A_1, \dots, A_m \in \mathcal{C}(\text{cl}(M - N))$.

Since $\text{Fr}_M N$ is assumed to be compact, we have $E_M = E_N \cup E_{A_1} \cup \dots \cup E_{A_m}$.

Suppose $\mu, \nu : P \rightarrow \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{\text{lim}}$ and $a(i) : P \rightarrow \mathbb{R}$ ($i = 1, \dots, m$) are continuous maps which satisfies the following conditions: for any $p \in P$

- (#)₁ $a_p(i) > -\nu_p(A_i) = -\mu_p(A_i) + (\mu_p - \nu_p)(A_i)$ ($i = 1, \dots, m$),
- (#)₂ $\sum_{i=1}^m a_p(i) < (\mu_p - \nu_p)(M) + \nu_p(N)$.

Lemma 5.4. There exists a map $\varphi : P \rightarrow \mathcal{H}_\partial^{\text{PL},c}(M)_1^*$ such that

- (1) φ satisfies the condition $(*)$,
- (2) $\varphi_*\mu : P \rightarrow \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{\text{lim}}$ is continuous,
- (3) $((\varphi_p)_*\mu_p - \nu_p)(A_i) = a_p(i)$ ($p \in P, i = 1, \dots, m$),
- (4) $\{\varphi_p^{-1}\}_p$ is equi-continuous with respect to $d|_M$,
- (5) (a) For any $p \in P$, “if $(\mu_p - \nu_p)(A_i) = a_p(i)$ ($i = 1, \dots, m$), then $\varphi_p = \text{id}$ ”,
(b) for any $i \in \{1, \dots, m\}$, “if $(\mu_p - \nu_p)(A_i) = a_p(i)$ ($p \in P$), then $\varphi_p|_{A_i} = \text{id}|_{A_i}$ ($p \in P$)”,
- (6) if $p \in P$ and $\sum_{i=1}^m a_p(i) = (\mu_p - \nu_p)(M)$, then $((\varphi_p)_*\mu_p - \nu_p)(N) = 0$.

The condition $(\#)$ on $a(i)$ is necessary to achieve the condition (3). In the sequel we write as $(\mu - \nu)(A_i) = a(i)$ or $\varphi|_{A_i} = \text{id}|_{A_i}$ instead of $(\mu_p - \nu_p)(A_i) = a_p(i)$ ($p \in P$) or $\varphi_p|_{A_i} = \text{id}|_{A_i}$ ($p \in P$) (we regard them as the identity of functions in $p \in P$).

6. REALIZATION OF MASS FLOW TOWARD ENDS

6.1. Topological manifold-case.

Main Theorem 1.1 is a consequence of the following realization theorem.

Theorem 6.1. *Suppose M^n is a noncompact connected separable metrizable n -manifold, $\omega \in \mathcal{M}_g^\partial(M)$, $\mu, \nu : P \rightarrow \mathcal{M}_g^\partial(M, \omega\text{-cpt-reg})_{lim}$ and $a : P \rightarrow V_\omega$ are continuous maps with $(\mu_p - \nu_p)(M) = 0$ ($p \in P$). Then there exists a continuous map $h : P \rightarrow \mathcal{H}_\partial(M, \omega\text{-reg})_1$ such that*

- (1) $(h_p)_* \mu_p = \nu_p$ ($p \in P$),
- (2) if $p \in P$ and $\mu_p = \nu_p$, then $h_p \in \mathcal{H}_\partial(M, \mu_p)_1$ and $J_{h_p}^{\mu_p} = a_p$,
- (3) if $p \in P$, $\mu_p = \nu_p$ and $a_p = 0$, then $h_p = id_M$.

According to the usual strategy, the proof of Theorem 6.1 can be reduced to the PL-manifold case by the next mapping theorem. We use the following notations: $I = [0, 1]$, I^n is the n -hold product of I and $I_1 = \{(t, 1/2, \dots, 1/2, 1) \in I^n \mid t \in [1/3, 2/3]\}$. m denotes the Lebesgue measure on I^n .

Proposition 6.1. ([4, Proposition 4.2]) *There exists a compact 0-dimensional subset $E \subset \partial I^n$ ($E \subset I_1$ if $n \geq 2$) and a continuous proper surjection $\pi : I^n - E \rightarrow M$ which satisfies the following conditions:*

- (i) $U \equiv \pi(\text{Int } I^n)$ is an open dense subset of $\text{Int } M$ and $\pi|_{\text{Int } I^n} : \text{Int } I^n \rightarrow U$ is a homeomorphism.
- (ii) $F \equiv \pi(\partial I^n - E) = M - U$ and $\omega(F) = 0$.
- (iii) (Since I^n is the end compactification of $I^n - E$, the map π has the natural extension $\bar{\pi} : I^n \rightarrow \bar{M}$.) The restriction $\bar{\pi}|_E : E \rightarrow E_M$ is a homeomorphism.
- (v) $\tilde{\omega} = \pi^* \omega$ is $m|_{I^n - E}$ -biregular.

6.2. PL-manifold case.

By Proposition 6.1 we may assume that M^n is a noncompact connected PL n -manifold, $\omega \in \mathcal{M}_g^\partial(M)$ and $\mathcal{H}_\partial^{\text{PL}}(M) \subset \mathcal{H}(M; \omega\text{-reg})$. Under this assumption Theorem 6.1 is proved in a series of lemmas. By $\mathcal{N}(E_M)$ we denote the set of PL n -submanifolds of M of the form $A = cl(M - N)$, where N is a compact, connected PL n -submanifold of M such that each $C \in \mathcal{C}(A)$ is noncompact. Let d be a fixed metric on $\bar{M}$. For any neighborhood U of E_M in $\bar{M}$ and any $\varepsilon > 0$ there exists $A \in \mathcal{N}(E_M)$ such that $A \subset U$ and $\text{diam}_d C < \varepsilon$ ($C \in \mathcal{C}(A)$).

The next statement follows from Lemma 5.4.

Lemma 6.1. *Suppose $A, B \in \mathcal{N}(E_M)$, $B \subset \text{Int}_M A$, $L = cl(M - A)$, $N = cl(M - B)$, and $\mathcal{C}(A) = \{A_1, \dots, A_m\}$. We assume that $N_i = cl(A_i - B)$ is connected ($i = 1, \dots, m$) and $(\mu_p - \nu_p)(A_i) = a_p(A_i)$ ($p \in P$, $i = 1, \dots, m$). Then there exists a continuous map $\varphi : P \rightarrow \mathcal{H}_{\partial UL}^{\text{PL}, c}(M)_1^*$ such that*

- (1) φ satisfies the condition (*),
- (2) $\varphi_p(A_i) = A_i$ ($p \in P$, $i = 1, \dots, m$),

- (3) (a) $((\varphi_p)_*\mu_p)(N_i) = \nu_p(N_i)$ ($p \in P, i = 1, \dots, m$),
 (b) $(\varphi_p*\mu_p - \nu_p)(B_j) = a_p(B_j)$ for each $B_j \in \mathcal{C}(B)$,
- (4) $\{\varphi_p^{-1}\}_{p \in P}$ is equi-continuous with respect to $d|_M$,
- (5) (a) for any $A_i \in \mathcal{C}(A)$ and $p \in P$ "if $(\mu_p - \nu_p)(B_j) = a_p(B_j)$ for each $B_j \in \mathcal{C}(B \cap A_i)$, then $\varphi_p|_{A_i} = id_{A_i}$ "
 (b) for any $B_j \in \mathcal{C}(B)$ "if $(\mu_p - \nu_p)(B_j) = a_p(B_j)$ ($p \in P$), then $\varphi_p|_{B_j} = id_{B_j}$ ($p \in P$)"

Let $\mu^0 = \mu, \nu^0 = \nu$, and $B^0 = M$. By the assumption we have $(\mu^0 - \nu^0)(B^0) = 0$. By the repeated application of Lemma 6.1 we obtain the following sequence of maps.

Lemma 6.2. For $k = 1, 2, \dots$ there exist

$$(k)_A : A^k \in \mathcal{N}(E), \varphi^k : P \rightarrow \mathcal{H}_{\partial \cup N^{k-1}}^{\text{PL},c}(M)_1^*, \mu^k : P \rightarrow \mathcal{M}_g^\partial(M; \omega\text{-cpt-reg})_{\text{lim}}$$

such that

- (0) φ^k and μ^k are continuous,
- (1) (a) $A^k \subset \text{Int}_M B^{k-1}, N^{k-1} \equiv cl(M - B^{k-1})$,
 (b) $L_j^k \equiv cl(B_j^{k-1} - A^k)$ is connected ($B_j^{k-1} \in \mathcal{C}(B^{k-1})$),
 (c) $\text{diam } A_i^k \leq \frac{1}{2^k}, \text{diam } (\psi_p^{k-1} \dots \psi_p^1)^{-1}(A_i^k) \leq \frac{1}{2^k}$ ($A_i^k \in \mathcal{C}(A^k)$),
- (2) $\varphi_p^k(B_j^{k-1}) = B_j^{k-1}$ ($B_j^{k-1} \in \mathcal{C}(B^{k-1})$),
- (3) (a) $\mu_p^k \equiv (\varphi_p^k)_*\mu_p^{k-1}$, (b) $\mu_p^k(L_j^k) = \nu_p^{k-1}(L_j^k)$, $(\mu_p^k - \nu_p^{k-1})(A_i^k) = a_p(A_i^k)$ ($A_i^k \in \mathcal{C}(A^k)$),
- (4) $\{(\varphi_p^k)^{-1}\}_p$ is equi-continuous with respect to $d|_M$,
- (5) (a) for any $B_j^{k-1} \in \mathcal{C}(B^{k-1})$ and any $p \in P$
 "if $(\mu_p^{k-1} - \nu_p^{k-1})(A_i^k) = a_p(A_i^k)$ ($A_i^k \in \mathcal{C}(A^k \cap B_j^{k-1})$), then $\varphi_p^k|_{B_j^{k-1}} = id_{B_j^{k-1}}$ ",
 (b) for any $A_i^k \in \mathcal{C}(A^k)$
 "if $(\mu_p^{k-1} - \nu_p^{k-1})(A_i^k) = a_p(A_i^k)$ ($p \in P$), then $\varphi_p^k|_{A_i^k} = id_{A_i^k}$ ($p \in P$)",

$$(k)_B : B^k \in \mathcal{N}(E), \psi^k : P \rightarrow \mathcal{H}_{\partial \cup L^k}^{\text{PL},c}(M)_1^*, \nu^k : P \rightarrow \mathcal{M}_g^\partial(M; \omega\text{-cpt-reg})_{\text{lim}}$$

such that

- (0) ψ^k and ν^k are continuous,
- (1) (a) $B^k \subset \text{Int}_M A^k, L^k \equiv cl(M - A^k)$,
 (b) $N_i^k = cl(A_i^k - B^k)$ is connected ($A_i^k \in \mathcal{C}(A^k)$)
 (c) $\text{diam } B_j^k \leq \frac{1}{2^k}, \text{diam } (\varphi_p^k \dots \varphi_p^1)^{-1}(B_j^k) \leq \frac{1}{2^k}$ ($B_j^k \in \mathcal{C}(B^k)$),
- (2) $\psi_p^k(A_i^k) = A_i^k$ ($A_i^k \in \mathcal{C}(A^k)$),
- (3) (a) $\nu_p^k \equiv (\psi_p^k)_*\nu_p^{k-1}$, (b) $\nu_p^k(N_i^k) = \mu_p^k(N_i^k)$, $(\mu_p^k - \nu_p^k)(B_j^k) = a(B_j^k)$ ($B_j^k \in \mathcal{C}(B^k)$),

- (4) $\left\{(\psi_p^k)^{-1}\right\}_p$ is equi-continuous with respect to $d|_M$,
- (5) (a) for any $A_i^k \in \mathcal{C}(A^k)$ and any $p \in P$
 “if $(\mu_p^k - \nu_p^{k-1})(B_j^k) = a_p(B_j^k)$ ($B_j^k \in \mathcal{C}(B^k \cap A_i^k)$), then $\psi_p^k|_{A_i^k} = id_{A_i^k}$ ”,
 (b) for any $B_j^k \in \mathcal{C}(B^k)$
 “if $(\mu_p^k - \nu_p^{k-1})(B_j^k) = a_p(B_j^k)$ ($p \in P$), then $\psi_p^k|_{B_j^k} = id_{B_j^k}$ ($p \in P$)”.

The next assertions follow from the conditions $(k)_A$ (0) $\sim$ (5) and $(k)_B$ (0) $\sim$ (5).

Lemma 6.3.

- (1) (i) For any $p \in P$ the sequence $\varphi_p^k \cdots \varphi_p^1$ ($k = 1, 2, \dots$) converges $d|_M$ -uniformly to some φ_p in $\mathcal{H}_\partial^{\text{PL}}(M)_1$.
 (ii) The map $\varphi : P \rightarrow \mathcal{H}_\partial^{\text{PL}}(M)_1 : p \mapsto \varphi_p$ is continuous.
 (iii) $\varphi_p^{-1}|_{N^k} = (\varphi_p^k \cdots \varphi_p^1)^{-1}|_{N^k}$ and $((\varphi_p)_* \mu_p)|_{N^k} = \mu_p^k|_{N^k}$ ($k = 1, 2, \dots$).
- (2) (i) For any $p \in P$ the sequence $\psi_p^k \cdots \psi_p^1$ ($k = 1, 2, \dots$) converges $d|_M$ -uniformly to some ψ_p in $\mathcal{H}_\partial^{\text{PL}}(M)_1$.
 (ii) The map $\psi : P \rightarrow \mathcal{H}_\partial^{\text{PL}}(M)_1 : p \mapsto \psi_p$ is continuous.
 (iii) $\psi_p^{-1}|_{L^k} = (\psi_p^{k-1} \cdots \psi_p^1)^{-1}|_{L^k}$ and $((\psi_p)_* \nu_p)|_{L^k} = \nu_p^{k-1}|_{L^k}$ ($k = 1, 2, \dots$).
- (3) For any $C = L_j^k \in \mathcal{C}(cl(B^{k-1} - A^k))$ and $N_i^k \in \mathcal{C}(cl(A^k - B^k))$ we have $((\varphi_p)_* \mu_p)(C) = ((\psi_p)_* \nu_p)(C)$ ($p \in P$).

The next lemma follows from Theorem 3.1.

Lemma 6.4. There exists a continuous map $\chi : P \rightarrow \mathcal{H}_\partial(M, \omega\text{-reg})_1$ such that

- (i) $(\chi\varphi)_* \mu = \psi_* \nu$
 (ii) $\chi(C) = C$ for any $C = L_j^k \in \mathcal{C}(cl(B^{k-1} - A^k))$ and $N_i^k \in \mathcal{C}(cl(A^k - B^k))$
 (iii) if $p \in P$ and $(\varphi_p)_* \mu_p = (\psi_p)_* \nu_p$, then $\chi_p = id_M$.

Proof of Theorem 6.1. The required map $h : P \rightarrow \mathcal{H}_\partial(M, \omega\text{-reg})_1$ is defined by $h_p = \psi_p^{-1} \chi_p \varphi_p$ ($p \in P$). This completes the proof of Theorem 6.1 and Theorem 1.1. $\square$

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