

The Picard group, the figure-eight knot group and Jørgensen groups

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0. Introduction.

In this paper we will state that the Picard group G_P and the figure-eight knot group G_F are two-generator groups and Jørgensen groups. Furthermore we will describe a complete set of relations for G_P as a two-generator group. The detail will appear elsewhere.

1. The Picard group.

DEFINITION 1.1. The group

$$G_P := \left\{ \frac{az + b}{cz + d} \mid a, b, c, d \in \mathbf{Z} + i\mathbf{Z}, ad - bc = 1 \right\}$$

is the *Picard group*.

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THEOREM A (Magnus [7]) *The Picard group G_P is generated by the following four Möbius transformations S_m, T_m, U_m and V_m with corresponding matrices*

$$S_m = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad T_m = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad U_m = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad V_m = \begin{pmatrix} i & -1 \\ 0 & -i \end{pmatrix}.$$

THEOREM B (Johnson-Weiss [3]) *The Picard group G_P is generated by the following three matrices:*

$$B_j = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad C_j = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \quad S_j = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

See Johnson-Kellerhals-Ratcliffe-Tschantz [1] and Johnson-Weiss [2] for more information about the Picard group and Coxeter groups.

2. Jørgensen groups.

THEOREM C (Jørgensen [4]). *If $\langle A, B \rangle$ is a non-elementary discrete subgroup of Möb, then*

$$J(A, B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

The lower bound 1 is best possible.

DEFINITION 2.1. Let A and B be Möbius transformations. The *Jørgensen number* $J(A, B)$ is

$$J(A, B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|.$$

DEFINITION 2.2. A non-elementary two-generator discrete subgroup G of Möb is a *Jørgensen group* if G has generators A and B with $J(A, B) = 1$.

THEOREM D (Jørgensen-Kiikka [5]). *Let $\langle A, B \rangle$ be a non-elementary discrete group with $J(A, B) = 1$, that is, a Jørgensen group. Then A is elliptic of order at least seven or A is parabolic.*

Here we only consider the case where A is parabolic, that is, Jørgensen groups of parabolic type. Namely, we consider two-generator groups $G_{ik,\sigma} = \langle A, B_{ik,\sigma} \rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{ik,\sigma} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where $k \in \mathbf{R}$ and $\sigma \in \mathbf{C} \setminus \{0\}$.

Let C be the following cylinder: $C = \{(\sigma, ik) \mid |\sigma| = 1, k \in \mathbf{R}\}$.

THEOREM E (Sato [9]). *Every Jørgensen group of type $G_{ik,\sigma}$ lies on the cylinder C .*

By Theorem E we consider two-generator groups $G_{\mu,\sigma} = \langle A, B_{\mu,\sigma} \rangle$ with $\mu = ik$ ($k \in \mathbf{R}$) and $\sigma = -ie^{i\theta}$ ($0 \leq \theta < 2\pi$). For simplicity we set $B_{k,\theta} := B_{ik,\sigma}$ and $G_{k,\theta} = \langle A, B_{k,\theta} \rangle$ for $\sigma = -ie^{i\theta}$.

We can see that it suffices to consider the case of $(0 \leq \theta \leq \pi/2)$ and $k \geq 0$.

THEOREM F (Jørgensen-Lascurain-Pignataro [6], Sato [9], Sato-Yamada [12]).

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{k,\theta} = \begin{pmatrix} ke^{i\theta} & ie^{-i\theta}(k^2e^{2i\theta} - 1) \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}$$

and let $G_{k,\theta} = \langle A, B_{k,\theta} \rangle$ be the group generated by A and $B_{k,\theta}$, where $k \in \mathbf{R}$ and $\sigma \in \mathbf{C} \setminus \{0\}$. Then

- (i) $G_{1/2,\pi/2}$ is a Jørgensen group.
- (ii) $G_{\sqrt{3}/2,\pi/6}$ is a Jørgensen group.

See Sato [9,10] for Jørgensen groups of parabolic type.

3. Theorems.

In this section we will state main theorems. We can prove the theorems by using Poincaré's polyhedron theorem (cf. Maskit [8]).

THEOREM 1 (Sato [9,11]) (i) *The Picard group G_P is conjugate to $G_{1/2,\pi/2}$, that is, $G_P = RG_{1/2,\pi/2}R^{-1}$, where*

$$R = \begin{pmatrix} 1 & i/2 \\ 0 & 1 \end{pmatrix}$$

(ii) *The following relations form a complete set of relations for G_P :*

$$(B^{-1}ABA^2BAB^{-1}A^2B^{-1}ABA^2BAB^{-1}AB)^2 = 1$$

$$(B^{-1}ABA^2BAB^{-1}A^2B^{-1}ABA)^2 = 1$$

$$(AB^{-1}ABA^2BAB^{-1}A^2B^{-1}ABA^2BAB^{-1}AB)^2 = 1$$

$$(AB^{-1}ABA^2BAB^{-1}A^2B^{-1}ABA)^2 = 1$$

$$(B^{-1}ABA)^3 = 1$$

$$(AB^{-1}ABA)^2 = 1$$

$$(AB^{-1}ABA^2B^{-1}ABA^2BAB^{-1}A^2B^{-1}ABA^2BAB^{-1}AB)^2 = 1$$

$$(AB^{-1}ABA^2B^{-1}ABA^2BAB^{-1}A^2B^{-1}ABA)^3 = 1,$$

where $B = RB_{1/2,\pi/2}R^{-1}$.

COROLLARY. *The Picard group is a two-generator group and a Jørgensen group.*

THEOREM 2 (Sato [9,11]). *The figure-eight knot group G_F is conjugate to $G_{\sqrt{3}/2,\pi/6}$, that is, $G_F = RG_{\sqrt{3}/2,\pi/6}R^{-1}$, where*

$$R = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

(ii) *The following relation forms a complete set of relations for G_F :*

$$ABA^{-1}B^{-1}A = BA^{-1}B^{-1}ABA,$$

where $B = RB_{\sqrt{3}/2, \pi/6}R^{-1}$.

COROLLARY. *The figure-eight knot group is a two-generator group and a Jørgensen group.*

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