

Optimal non-projective ternary linear codes

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Abstract

We prove the existence of a $[406, 6, 270]_3$ code and the nonexistence of linear codes with parameters $[458, 6, 304]_3$, $[467, 6, 310]_3$, $[471, 6, 313]_3$, $[522, 6, 347]_3$. These yield that $n_3(6, d) = g_3(6, d)$ for $268 \leq d \leq 270$, $n_3(6, d) = g_3(6, d) + 1$ for $d \in \{280 - 282, 304 - 306, 313 - 315, 347, 348\}$, $n_3(6, d) = g_3(6, d)$ or $g_3(6, d) + 1$ for $298 \leq d \leq 301$ and $n_3(6, d) = g_3(6, d) + 1$ or $g_3(6, d) + 2$ for $310 \leq d \leq 312$, where $n_q(k, d)$ denotes the minimum length n for which an $[n, k, d]_q$ code exists and $g_q(k, d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil$.

1. Introduction

Let $V(n, q)$ denote the vector space of n -tuples over $\text{GF}(q)$, the Galois field of order q . A q -ary linear code $\mathcal{C}$ of length n and dimension k is a k -dimensional subspace of $V(n, q)$. The Hamming distance $d(\mathbf{x}, \mathbf{y})$ between two vectors $\mathbf{x}, \mathbf{y} \in V(n, q)$ is the number of nonzero coordinate positions in $\mathbf{x} - \mathbf{y}$. Now the minimum distance of a linear code $\mathcal{C}$ is defined by $d(\mathcal{C}) = \min\{d(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in \mathcal{C}, \mathbf{x} \neq \mathbf{y}\}$ which is equal to the minimum weight of $\mathcal{C}$ defined by $wt(\mathcal{C}) = \min\{wt(\mathbf{x}) \mid \mathbf{x} \in \mathcal{C}, \mathbf{x} \neq \mathbf{0}\}$, where $\mathbf{0}$ is the all-0-vector and $wt(\mathbf{x}) = d(\mathbf{x}, \mathbf{0})$ is the weight of $\mathbf{x}$. A q -ary linear code of length n , dimension k and minimum distance d is referred to as an $[n, k, d]_q$ code. The weight distribution of $\mathcal{C}$ is the

list of numbers A_i which is the number of codewords of $\mathcal{C}$ with weight i . A $k \times n$ matrix having as rows the vectors of a basis of $\mathcal{C}$ is called a generator matrix of $\mathcal{C}$.

A fundamental problem in coding theory is to find $n_q(k, d)$, the minimum length n for which an $[n, k, d]_q$ code exists ([13]). An $[n, k, d]_q$ code is called *optimal* if $n = n_q(k, d)$. There is a natural lower bound on $n_q(k, d)$, the so-called Griesmer bound ([8],[25]):

$$n_q(k, d) \geq g_q(k, d) = \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x . The values of $n_q(k, d)$ are determined for all d only for some small values of q and k . For ternary linear codes, $n_3(k, d)$ is known for $k \leq 5$ for all d . As for the case $k = 6$, the value of $n_3(6, d)$ is unknown for many integers d ([1],[4],[5],[9],[10],[17],[20],[22]). See [2] or [24] for the updated table of $n_q(k, d)$ for some small q . A linear code $\mathcal{C}$ with a generator matrix G is called *projective* if any two columns of G are independent, equivalently, if the dual code of $\mathcal{C}$ has the minimum distance > 2 .

We concentrate ourselves to find optimal ternary linear codes of dimension 6 with the minimum distance $d > 243$, which are necessarily non-projective. For $d \geq 244$, it is only known ([20]) that $n_3(6, d) = g_3(6, d) + 1$ for $349 \leq d \leq 351$ and that $n_3(6, d) = g_3(6, d)$ for $d \geq 352$. The existence of an $[n_1, k, d_1]_q$ code and an $[n_2, k, d_2]_q$ code trivially implies the existence of an $[n_1 + n_2, k, d_1 + d_2]_3$ code. For example, one can get a $[372, 6, 246]_3$ code from a $[56, 6, 36]_3$ code and a $[316, 6, 210]_3$ code. Similarly one can get $[g_3(6, d), 6, d]_3$ codes for $d \in \{244 - 252, 271 - 279, 322 - 330, 334 - 336\}$, $[g_3(6, d) + 1, 6, d]_3$ codes for $d \in \{253 - 270, 331 - 333, 337 - 351\}$ and $[g_3(6, d) + 2, 6, d]_3$ codes for $280 \leq d \leq 315$ from the known $n_3(6, d)$ table. We also have $[g_3(6, d), 6, d]_3$ codes for $316 \leq d \leq 321$ by Theorem 2.1 in [13] and a $[474, 6, 315]_3$ code by Theorem 4.5 in [12] from a $[158, 5, 105]_3$ code. On the other hand, the nonexistence of $[n, 5, d]_3$ codes for $(n, d) \in \{(143, 94), (144, 95), (145, 96), (147, 97), (148, 98), (149, 99)\}$ implies $n_3(6, d) \geq g_3(6, d) + 1$ for $280 \leq d \leq 297$, for the residual code (see [13]) of each $[g_3(6, d), 6, d]_3$ code with respect to a codeword with weight d can not exist. Hence we obtain the following.

Theorem 1.1.

- (1) $n_3(6, d) = g_3(6, d)$ for $d \in \{244 - 252, 271 - 279, 316 - 330, 334 - 336\}$ and for $d \geq 352$.
- (2) $n_3(6, d) = g_3(6, d) + 1$ for $349 \leq d \leq 351$.
- (3) $n_3(6, d) = g_3(6, d)$ or $g_3(6, d) + 1$ for $d \in \{253 - 270, 313 - 315, 331 - 333, 337 - 348\}$.
- (4) $n_3(6, d) = g_3(6, d) + 1$ or $g_3(6, d) + 2$ for $280 \leq d \leq 297$.
- (5) $g_3(6, d) \leq n_3(6, d) \leq g_3(6, d) + 2$ for $298 \leq d \leq 312$.

We improve Theorem 1.1 for $d \in \{268 - 270, 280 - 282, 298 - 301, 304 - 306, 310 - 315, 347, 348\}$ as follows.

- Theorem 1.2.** (1) $n_3(6, d) = g_3(6, d)$ for $268 \leq d \leq 270$.
 (2) $n_3(6, d) = g_3(6, d) + 1$ for $d \in \{280 - 282, 304 - 306, 313 - 315, 347, 348\}$.
 (3) $n_3(6, d) = g_3(6, d)$ or $g_3(6, d) + 1$ for $298 \leq d \leq 301$.
 (4) $n_3(6, d) = g_3(6, d) + 1$ or $g_3(6, d) + 2$ for $310 \leq d \leq 312$.

To prove Theorem 1.2, we need to show the following theorems.

Theorem 1.3. *There exist a $[406, 6, 270]_3$ code.*

Theorem 1.4. *There exists no $[g_3(6, d), 6, d]_3$ code for $d = 304, 310, 313, 347$.*

We prove Theorem 1.4 in Section 4 and Theorems 1.3 and 1.2 in Section 5.

2. Preliminaries

We denote by $\text{PG}(r, q)$ the projective geometry of dimension r over $\text{GF}(q)$. A j -flat is a projective subspace of dimension j in $\text{PG}(r, q)$. 0-flats, 1-flats, 2-flats, 3-flats, $(r - 2)$ -flats and $(r - 1)$ -flats are called *points*, *lines*, *planes*, *solids*, *secundums* and *hyperplanes* respectively. We denote by $\mathcal{F}_j$ the set of j -flats of $\text{PG}(r, q)$ and denote by θ_j the number of points in a j -flat, i.e.

$$\theta_j = (q^{j+1} - 1)/(q - 1).$$

Let $\mathcal{C}$ be an $[n, k, d]_q$ code which does not have any coordinate position in which all the codewords have a zero entry. The columns of a generator matrix of $\mathcal{C}$ can be considered as a multiset of n points in $\Sigma = \text{PG}(k - 1, q)$ denoted also by $\mathcal{C}$. We see linear codes from this geometrical point of view. An i -point is a point of Σ which has multiplicity i in $\mathcal{C}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{C}$ and let C_i be the set of i -points in Σ , $0 \leq i \leq \gamma_0$. For any subset S of Σ we define the *multiplicity of S with respect to $\mathcal{C}$* , denoted by $m_{\mathcal{C}}(S)$, as

$$m_{\mathcal{C}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|,$$

where $|T|$ denotes the number of points in T for a subset T of Σ . When the code is *projective*, i.e. when $\gamma_0 = 1$, the multiset $\mathcal{C}$ forms an n -set in Σ and the above $m_{\mathcal{C}}(S)$ is equal to $|\mathcal{C} \cap S|$. A line l with $t = m_{\mathcal{C}}(l)$ is called a t -line. A t -plane, a t -solid and so on are defined similarly. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ such that

$$\begin{aligned} n &= m_{\mathcal{C}}(\Sigma), \\ n - d &= \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}. \end{aligned}$$

Conversely such a partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ as above gives an $[n, k, d]_q$ code in the natural manner. For an m -flat Π in Σ we define

$$\gamma_j(\Pi) = \max\{m_C(\Delta) \mid \Delta \subset \Pi, \Delta \in \mathcal{F}_j\}, \quad 0 \leq j \leq m.$$

We denote simply by γ_j instead of $\gamma_j(\Sigma)$. Clearly we have $\gamma_{k-2} = n - d$, $\gamma_{k-1} = n$.

Lemma 2.1 ([22]). (1) Let Π be an $(s-1)$ -flat in Σ , $2 \leq s \leq k-1$, with $m_C(\Pi) = w$. For any $(s-2)$ -flat δ in Π , we have

$$m_C(\delta) \leq \gamma_{s-1} - \frac{n-w}{\theta_{k-s}-1}.$$

In particular for $0 \leq j \leq k-3$,

$$\gamma_j \leq \gamma_{j+1} - \frac{n-\gamma_{j+1}}{\theta_{k-2-j}-1}.$$

(2) Let δ_1 and δ_2 be distinct t -flats in a fixed $(t+1)$ -flat Δ in Σ , $1 \leq t \leq k-2$. Then

$$m_C(\delta_1) + m_C(\delta_2) \geq m_C(\Delta) - (q-1)\gamma_t + q \cdot m_C(\delta_1 \cap \delta_2).$$

When $\mathcal{C}$ attains the Griesmer bound, γ_j 's are uniquely determined as follows.

Lemma 2.2 ([19]). Let $\mathcal{C}$ be an $[n, k, d]_q$ code attaining the Griesmer bound. Then it holds that

$$\gamma_j = \sum_{u=0}^j \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \quad \text{for } 0 \leq j \leq k-1.$$

By Lemma 2.2 every $[n, k, d]_q$ code attaining the Griesmer bound is projective if $d \leq q^{k-1}$. Denote by a_i the number of hyperplanes Π in Σ with $m_C(\Pi) = i$ and by λ_s the number of s -points in Σ . Note that we have $\lambda_2 = \lambda_0 + n - \theta_{k-1}$ when $\gamma_0 = 2$. The list of a_i 's is called the *spectrum* of $\mathcal{C}$. Simple counting arguments yield the following.

$$\begin{aligned} \text{Lemma 2.3. (1)} \quad & \sum_{i=0}^{\gamma_{k-2}} a_i = \theta_{k-1}. & (2) \quad & \sum_{i=1}^{\gamma_{k-2}} i a_i = n \theta_{k-2}. \\ (3) \quad & \sum_{i=2}^{\gamma_{k-2}} i(i-1) a_i = n(n-1) \theta_{k-3} + q^{k-2} \sum_{s=2}^{\gamma_0} s(s-1) \lambda_s. \end{aligned}$$

Lemma 2.4 ([22]). Let Π be an i -hyperplane through a t -secundum δ with $t = \gamma_{k-3}(\Pi)$. Then

- (1) $t \leq \gamma_{k-2} - \frac{n-i}{q} = \frac{i + q\gamma_{k-2} - n}{q}$.
- (2) $a_i = 0$ if an $[i, k-1, d_0]_q$ code with $d_0 \geq i - \left\lfloor \frac{i + q\gamma_{k-2} - n}{q} \right\rfloor$ does not exist, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .
- (3) $t = \left\lfloor \frac{i + q\gamma_{k-2} - n}{q} \right\rfloor$ if an $[i, k-1, d_1]_q$ code with $d_1 \geq i - \left\lfloor \frac{i + q\gamma_{k-2} - n}{q} \right\rfloor + 1$ does not exist.
- (4) Let c_j be the number of j -hyperplanes through δ other than Π . Then the following equality holds:
- $$\sum_j (\gamma_{k-2} - j)c_j = i + q\gamma_{k-2} - n - qt. \quad (2.1)$$
- (5) For a γ_{k-2} -hyperplane Π_0 with spectrum $(\tau_0, \dots, \tau_{\gamma_3})$, $\tau_t > 0$ holds if $i + q\gamma_{k-2} - n - qt < q$.

The code obtained by deleting the same coordinate from each codeword of $\mathcal{C}$ is called a *punctured code* of $\mathcal{C}$. If there exists an $[n+1, k, d+1]_q$ code $\mathcal{C}'$ which gives $\mathcal{C}$ as a punctured code, $\mathcal{C}$ is called *extendable* (to $\mathcal{C}'$) and $\mathcal{C}'$ is an *extension* of $\mathcal{C}$.

Let $\mathcal{C}$ be an $[n, k, d]_q$ code with $k \geq 3$, $\gcd(q, d) = 1$. Define

$$\Phi_0 = \frac{1}{q-1} \sum_{q|i, i \neq 0} A_i, \quad \Phi_1 = \frac{1}{q-1} \sum_{i \not\equiv 0, d \pmod{q}} A_i,$$

where the notation $x|y$ means that x is a divisor of y . The pair (Φ_0, Φ_1) is called the *diversity* of $\mathcal{C}$ ([21]).

Theorem 2.5 ([14]). *Let $\mathcal{C}$ be an $[n, k, d]_q$ code with diversity (Φ_0, Φ_1) , $\gcd(q, d) = 1$, $k \geq 3$. Then $\mathcal{C}$ is extendable if $\Phi_1 = 0$.*

See [23] for the extendability of ternary linear codes in detail. Note that $a_i = A_{n-i}/(q-1)$ for $0 \leq i \leq \gamma_{k-2}$. Hence the above diversity is given as

$$\Phi_0 = \sum_{i \equiv n \pmod{3}} a_i, \quad \Phi_1 = \sum_{i \not\equiv n, n-d \pmod{3}} a_i.$$

The following is known as the Ward's divisibility theorem.

Theorem 2.6 ([26]). *Let $\mathcal{C}$ be an $[n, k, d]_p$ code, p a prime, attaining the Griesmer bound. If $p^e | d$, then p^e is a divisor of all nonzero weights of $\mathcal{C}$.*

3. The spectra of some ternary linear codes of dimension $k \leq 5$

We supply the results about the possibilities of spectra for some ternary linear codes of dimension $k \leq 5$ which we need to prove Theorem 1.4 in the next section.

An f -set F in $\text{PG}(r, q)$ satisfying

$$m = \min\{|F \cap \pi| \mid \pi \in \mathcal{F}_{r-1}\}$$

is called an $\{f, m; r, q\}$ -minihyper. When an $[n, k, d]_q$ code is projective (i.e. $\gamma_0 = 1$), the set of 0-points C_0 forms a $\{\theta_{k-1} - n, \theta_{k-2} - (n - d); k - 1, q\}$ -minihyper, where $\theta_j = (q^{j+1} - 1)/(q - 1)$. The following lemma can be obtained from the classification of some minihypers by Hamada [11].

Lemma 3.1. (1) *The spectrum of a $[80, 5, 53]_3$ code is $(a_0, a_{26}, a_{27}) = (1, 40, 80)$.*

(2) *The spectrum of a $[81, 5, 54]_3$ code is $(a_0, a_{27}) = (1, 120)$.*

(3) *The spectrum of a $[104, 5, 69]_3$ code is $(a_{26}, a_{32}, a_{35}) = (4, 13, 104)$.*

(4) *The spectrum of a $[107, 5, 71]_3$ code is $(a_{26}, a_{27}, a_{35}, a_{36}) = (1, 3, 39, 78)$.*

(5) *The spectrum of a $[108, 5, 72]_3$ code is $(a_{27}, a_{36}) = (4, 117)$.*

(6) *The spectrum of a $[113, 5, 75]_3$ code is $(a_{32}, a_{35}, a_{38}) = (1, 24, 96)$.*

(7) *The spectrum of a $[116, 5, 77]_3$ code is $(a_{35}, a_{36}, a_{38}, a_{39}) = (4, 9, 36, 72)$.*

(8) *The spectrum of a $[117, 5, 78]_3$ code is $(a_{36}, a_{39}) = (13, 108)$.*

Since a $[\theta_{k-1} - e, k, q^{k-1} - e]_3$ code ($0 \leq e \leq 2$) is projective, the set of 0-points C_0 consists of e points. Hence the following lemma follows.

Lemma 3.2. Assume $k \geq 3$ and put $u = \theta_{k-2}$.

(1) *The spectrum of a $[\theta_{k-1} - 2, k, q^{k-1} - 2]_3$ code is*

$$(a_{u-2}, a_{u-1}, a_u) = (\theta_{k-3}, (\theta_{k-1} - \theta_{k-3})/2, (\theta_{k-1} - \theta_{k-3})/2).$$

(2) *The spectrum of a $[\theta_{k-1} - 1, k, q^{k-1} - 1]_3$ code is $(a_{u-1}, a_u) = (\theta_{k-2}, q^{k-1})$.*

(3) *The spectrum of a $[\theta_{k-1}, k, q^{k-1}]_3$ code is $a_u = \theta_{k-1}$.*

The following lemma relies upon the classification of some optimal ternary linear codes of small length by van Eupen and Lisoněk [7].

Lemma 3.3 ([7]). (1) *The spectrum of a $[8, 3, 5]_3$ code is $(a_0, a_2, a_3) = (1, 4, 8)$.*

(2) *The spectrum of a $[9, 3, 6]_3$ code is $(a_0, a_3) = (1, 12)$.*

(3) *The spectrum of a $[14, 3, 9]_3$ code is either $(a_4, a_5) = (9, 4)$, $(a_2, a_5) = (3, 10)$ or $(a_3, a_4, a_5) = (3, 3, 7)$.*

(4) *The spectrum of a $[18, 3, 12]_3$ code is $(a_0, a_6) = (1, 12)$ or $(a_3, a_6) = (2, 11)$.*

- (5) The spectrum of a $[20, 3, 13]_3$ code satisfies $a_i = 0$ for all $i \notin \{2, 3, 4, 5, 6, 7\}$.
- (6) The spectrum of a $[10, 4, 6]_3$ code is $(a_1, a_4) = (10, 30)$.
- (7) The spectrum of a $[19, 4, 12]_3$ code is $(a_1, a_4, a_7) = (1, 9, 30)$.
- (8) The spectrum of a $[27, 4, 35]_3$ code is $(a_0, a_9) = (1, 39)$.
- (9) The spectrum of a $[32, 4, 21]_3$ code is $(a_8, a_{13}) = (8, 32)$.
- (10) The spectrum of a $[35, 4, 23]_3$ code is $(a_8, a_9, a_{11}, a_{12}) = (1, 3, 12, 24)$.
- (11) The spectrum of a $[36, 4, 24]_3$ code is $(a_9, a_{12}) = (4, 36)$.

Lemma 3.4. The spectrum of a $[41, 4, 27]_3$ code satisfies $a_i = 0$ for all $i \notin \{11, 12, 13, 14\}$.

Lemma 3.5. (1) The spectrum of a $[52, 4, 34]_3$ code satisfies

$$a_i = 0 \text{ for all } i \notin \{0, 7, 8, 9, 16, 17, 18\}.$$

(2) The spectrum of a $[53, 4, 35]_3$ code is one of the following:

- (a) $(a_0, a_{17}, a_{18}) = (1, 13, 26)$, (b) $(a_8, a_9, a_{17}, a_{18}) = (1, 1, 12, 26)$, (c) $(a_9, a_{17}, a_{18}) = (2, 13, 25)$.

Lemma 3.6. The spectrum of a $[59, 4, 39]_3$ code satisfies $a_i = 0$ for all $i \notin \{8, 11, 14, 17, 20\}$.

Lemma 3.7. The spectrum of a $[122, 5, 81]_3$ code satisfies $a_i = 0$ for all $i \notin \{38, 39, 40, 41\}$.

The following lemma is due to Landjev [18].

Lemma 3.8 ([18]). (1) The spectrum of a $[50, 4, 33]_3$ code is one of the following:

- (a) $(a_8, a_{14}, a_{17}) = (2, 4, 34)$, (b) $(a_{11}, a_{14}, a_{17}) = (2, 6, 32)$, (c) $(a_{14}, a_{17}) = (11, 30)$.
- (2) Every $[49, 4, 32]_3$ code is extendable, so $a_i = 0$ for all $i \notin \{7, 8, 10, 11, 13, 14, 16, 17\}$.

Lemma 3.9. The spectrum of a $[154, 5, 102]_3$ code satisfies $a_i = 0$ for all $i \notin \{25, 46, 49, 52\}$.

Lemma 3.10. (1) The spectrum of a $[158, 5, 105]_3$ code is $(a_{26}, a_{50}, a_{53}) = (2, 13, 106)$.

(2) Every $[157, 5, 104]_3$ code is extendable.

We omit the proof of Lemmas 3.1–3.10 here.

Lemma 3.11. (1) The spectrum of a $[176, 5, 117]_3$ code is either

- (a) $(a_{32}, a_{50}, a_{59}) = (1, 8, 112)$ or
- (b) $(a_{41}, a_{50}, a_{59}) = (a, 11 - 2a, 110 + a)$ for some a with $0 \leq a \leq 5$.
- (2) Every $[175, 5, 116]_3$ code is extendable.

Proof. (1) See [20].

(2) Let $\mathcal{C}$ be a $[175, 5, 116]_3$ code. Then γ_3 -solid has no j -solid for $j < 8$ by Lemma 3.6, so $a_i = 0$ for all $i < 22$ by Lemma 2.1. Hence, by Lemma 2.4, we have

$$a_i = 0 \text{ for all } i \notin \{31, 32, 40, 41, 49, 50, 58, 59\},$$

which implies that $\mathcal{C}$ is extendable by Theorem 2.5. $\square$

4. Proof of Theorem 1.4

Theorem 4.1. *There exists no $[458, 6, 304]_3$ code.*

Proof. Let $\mathcal{C}$ be a $[458, 6, 304]_3$ code. Then a γ_4 -hyperplane has no j -solid for $j < 25$ by Lemma 3.9, so $a_i = 0$ for all $i < 71$ by Lemma 2.1. Hence

$$a_i = 0 \text{ for all } i \notin \{80, 81, 104, 107, 108, 113, 116, 117, 119-122, 134, 135, 136, 152, 153, 154\}.$$

by Lemma 2.4. Now, let Π be a 104-hyperplane. Then the spectrum of Π is $(\tau_{26}, \tau_{32}, \tau_{35}) = (4, 13, 104)$ by Lemma 3.1(3), which contradicts Lemma 3.9 (a γ_4 -hyperplane has no j -solid for $j = 26, 32, 35$). Hence $a_{104} = 0$. Similarly, we get $a_{107} = a_{108} = a_{113} = a_{122} = 0$ by Lemmas 3.1(4)(5)(6), 3.7, 3.9. Hence

$$a_i = 0 \text{ for all } i \notin \{80, 81, 116, 117, 119-121, 134-136, 152-154\}.$$

Next, let Π_0 be a 154-hyperplane. Since (2.1) with $i = 154$ has no solution for $t = 25$ and for $t = 49$, the spectrum of Π_0 satisfies $a_i = 0$ for all $i \notin \{46, 52\}$ by Lemma 3.9. Let Δ be a 52-solid in Π_0 . Applying Lemma 2.4 to Π_0 , (2.1) with $i = 52$ has no solution for $t = 0, 7, 8, 9, 17$. Hence the spectrum of Δ satisfies $a_i = 0$ for all $i \notin \{16, 18\}$ by Lemma 3.5(1). Let δ be a 16-plane in Δ . Applying Lemma 2.4 to Δ , (2.1) with $i = 16$ has no solution for $t = 0, 1, 2, 3, 5$. Hence the spectrum of δ satisfies $a_i = 0$ for all $i \notin \{4, 6\}$. But there exists no $[16, 3, 10]_3$ code with such spectrum (see [7]), a contradiction. This completes the proof. $\square$

Theorem 4.2. *There exists no $[467, 6, 310]_3$ code.*

Proof. Let $\mathcal{C}$ be a $[467, 6, 310]_3$ code. Then a γ_4 -hyperplane has no j -solid for $j < 25$ by Lemma 3.10, so $a_i = 0$ for all $i < 71$ by Lemma 2.1. Hence

$$a_i = 0 \text{ for all } i \notin \{74, 80, 81, 104, 107, 108, 113, 116, 117, 119-122, 146, 152-157\}$$

by Lemma 2.4. Let Π be a 108-hyperplane. Then the spectrum of Π is $(\tau_{27}, \tau_{36}) = (4, 117)$ by Lemma 3.1(5), which contradicts Lemma 3.10 (a γ_4 -hyperplane has no 27- nor 36-solid).

Hence $a_{108} = 0$. Similarly, we get $a_{81} = a_{113} = a_{116} = a_{117} = a_{119} = a_{120} = a_{121} = a_{122} = 0$ by Lemmas 3.1(2)(6)(7)(8), 3.2, 3.7, 3.10. Hence

$$a_i = 0 \text{ for all } i \notin \{74, 80, 104, 107, 146, 152 - 157\}.$$

Suppose $a_{80} > 0$ and let Π be a 80-hyperplane. Setting $(i, t) = (80, 27)$, (2.1) has no solution since $c_{157} = 0$ (a 157-hyperplane has no 27-solid), which contradicts the spectrum of Π (Lemma 3.1(1)). Hence $a_{80} = 0$. Similarly we get $a_{104} = a_{107} = 0$ by Lemmas 2.1, 2.4, 3.1(3)(4). Hence

$$a_i = 0 \text{ for all } i \notin \{74, 146, 152 - 157\}.$$

Now, let Π_0 be a 157-hyperplane with the spectrum $(\tau_{25}, \tau_{26}, \dots, \tau_{53})$. Then $\tau_{25} + \tau_{26} = 2$ by Lemma 3.10. Since all the solutions of (2.1) for $i = 157$ are $(c_{74}, c_{154}, c_{157}) = (1, 1, 1)$ or $(c_{74}, c_{155}, c_{156}) = (1, 1, 1)$ for $t = 25$; $(c_{74}, c_{157}) = (1, 2)$ for $t = 26$, and so on, we obtain

$$a_{74} \geq \tau_{25} + \tau_{26} = 2.$$

On the other hand, it holds that $a_{74} \leq 1$ by Lemma 2.1, a contradiction. This completes the proof. $\square$

Theorem 4.3. *There exists no $[471, 6, 313]_3$ code.*

Proof. Let $\mathcal{C}$ be a $[471, 6, 313]_3$ code. Then a γ_4 -hyperplane has no j -solid for $j < 26$ by Lemma 3.10(1), so $a_i = 0$ for all $i < 75$ by Lemma 2.1. Hence

$$a_i = 0 \text{ for all } i \notin \{81, 108, 117, 120, 121, 156 - 158\}$$

by Lemma 2.4. Now, let Π be a 158-hyperplane. Then the spectrum of Π is $(\tau_{26}, \tau_{50}, \tau_{53}) = (2, 13, 106)$ by Lemma 3.10(1), but (2.1) has no solution for $(i, t) = (158, 50)$, a contradiction. This completes the proof. $\square$

Theorem 4.4. *There exists no $[522, 6, 347]_3$ code.*

Proof. Let $\mathcal{C}$ be a $[522, 6, 347]_3$ code. Then a γ_4 -hyperplane has no j -solid for $j < 31$ by Lemma 3.11, so $a_i = 0$ for all $i < 90$ by Lemma 2.1. Hence

$$a_i = 0 \text{ for all } i \notin \{90, 91, 108, 117 - 122, 162, 171, 172, 174, 175\},$$

by Lemma 2.4. Let Π be a γ_4 -hyperplane. Then (2.1) for $i = 175$ has no solution for $t = 49, 50$, which contradicts that the spectrum of Π satisfies $\tau_{49} + \tau_{50} > 0$ by Lemma 3.11. This completes the proof. $\square$

5. Proof of Theorem 1.2

A linear code $\mathcal{C}$ is w -weight if $\mathcal{C}$ has exactly w non-zero weights i with $A_i > 0$. The method finding another code (called *projective dual* in [16]) from a given 2-weight code was first found by van Eupen and Hill [6], see also [3]. We consider the projective dual of a 3-weight code with $\gamma_0 = 2$. Recall that λ_i stands for the number of i -points in $\Sigma = \text{PG}(k-1, q)$ defined from $\mathcal{C}$. Considering $(n-d-2m)$ -hyperplanes, $(n-d-m)$ -hyperplanes and $(n-d)$ -hyperplanes of Σ as 2-points, 1-points and 0-points respectively in the dual space Σ^* of Σ , we obtain the following lemma.

Lemma 5.1. *Let $\mathcal{C}$ be a 3-weight $[n, k, d]_q$ code with $q = p^h$, p prime, $\gamma_0 = 2$, whose spectrum is $(a_{n-d-2m}, a_{n-d-m}, a_{n-d}) = (\alpha, \beta, \theta_{k-1} - \alpha - \beta)$, where $m = p^r$ for some $1 \leq r < h(k-2)$ satisfying $m|d$ and $\lambda_i > 0$ ($0 \leq i \leq 2$). Then there exists a 3-weight $[n^*, k, d^*]_q$ code $\mathcal{C}^*$ with $n^* = 2\alpha + \beta$, $d^* = 2\alpha + \beta - nt + \frac{d}{m}\theta_{k-2}$ whose spectrum is $(a_{n^*-d^*-2t}, a_{n^*-d^*-t}, a_{n^*-d^*}) = (\lambda_2, \lambda_1, \lambda_0)$, where $t = p^{h(k-2)-r}$.*

Proof of Theorem 1.3. Let $\mathcal{C}$ be a $[14, 6, 6]_3$ code with a generator matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}.$$

Then the spectrum of $\mathcal{C}$ is $(a_2, a_5, a_8) = (93, 220, 51)$ and we have $(\lambda_2, \lambda_1, \lambda_0) = (1, 12, 351)$. Applying Lemma 5.1 we get a $[406, 6, 270]_3$ code $\mathcal{C}^*$ with the spectrum $(a_{82}, a_{109}, a_{136}) = (1, 12, 351)$. $\square$

Lemma 5.2 ([15]). *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be $[n_1, k, d_1]_q$ and $[n_2, k-1, d_2]_q$ codes respectively and assume that $\mathcal{C}_1$ contains a codeword of weight at least $d_1 + d_2$. Then there exists an $[n_1 + n_2, k, d_1 + d_2]_q$ code.*

Applying Lemma 5.2 to a $[406, 6, 270]_3$ code as $\mathcal{C}_1$ and $[20, 5, 12]_3$, $[47, 5, 30]_3$, $[49, 5, 31]_3$, $[55, 5, 36]_3$ codes as $\mathcal{C}_2$, we get $[426, 6, 282]_3$, $[453, 6, 300]_3$, $[455, 6, 301]_3$, $[461, 6, 306]_3$ codes respectively. Hence Theorem 1.2 follows from Theorems 1.1 and 1.4.

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