

CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS

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ABSTRACT. The object of the present paper is to drive some properties of certain class $K_{n,p}(A, B)$ of multivalent analytic functions in the open unit disk E .

1. Introduction

Let A_p be the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (1.1)$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in A_p$ is said to be p -valently starlike functions of order α of it satisfies the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p, z \in E).$$

We denote by $S_p^*(\alpha)$.

On the other hand, a function $f \in A_p$ is said to be p -valently close-to-convex functions of order α if it satisfies the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > \alpha \quad (0 \leq \alpha < p, z \in E),$$

for some starlike function $g(z)$. We denote by $C_p(\alpha)$.

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For $f \in A_p$ given by (1.1), the generalized Bernardi integral operator F_c is defined by

$$\begin{aligned} F_c(z) &= \frac{c+p}{z^c} \int_0^z f(t)t^{c-1}dt \\ &= z^p + \sum_{k=1}^{\infty} \frac{c+p}{c+p+k} a_{p+k} z^{p+k} \quad (c+p > 0, z \in E). \end{aligned} \quad (1.2)$$

For an analytic function g , defined in E by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$$

and Flett [3] defined the multiplier transform I^η for a real number η by

$$I^\eta g(z) = \sum_{k=0}^{\infty} (p+k+1)^{-\eta} b_{p+k} z^{p+k} \quad (z \in E).$$

Clearly, the function $I^\eta g$ is analytic in E and

$$I^\eta(I^\mu g(z)) = I^{\eta+\mu} g(z)$$

for all real number η and μ .

For any integer n , J. Patel and P. Sahoo [5] also defined the operator D^n , for an analytic function f given by (1.1), by

$$\begin{aligned} D^n f(z) &= z^p + \sum_{k=1}^{\infty} \left(\frac{p+k+1}{1+p} \right)^{-n} a_{p+k} z^{p+k} \\ &= f(z) * z^{p-1} \left[z + \sum_{k=1}^{\infty} \left(\frac{k+1+p}{1+p} \right)^{-n} z^{k+1} \right] \quad (z \in E) \end{aligned} \quad (1.3)$$

where $*$ stands for the Hadamard product or convolution.

It follows from (1.3) that

$$z(D^n f(z))' = (p+1)D^{n-1} f(z) - D^n f(z). \quad (1.4)$$

We also have

$$D^0 f(z) = f(z) \quad \text{and} \quad D^{-1} f(z) = \frac{zf'(z) + f(z)}{p+1}.$$

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If f and g are analytic functions in E , then we say that f is subordinate to g written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w analytic in E , with $w(0) = 0$, $|w(z)| < 1$ for $z \in E$, such that $f(z) = g(w(z))$, for $z \in E$. If g is univalent then $f \prec g$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$.

Making use of the operator notation D^n , we introduce a subclass of A_p as follows:

Definition 1.1. For any integer n and $-1 \leq B < A \leq 1$, a function $f \in A_p$ is said to be in the class $K_{n,p}(A, B)$ if

$$\frac{z(D^n f(z))'}{z^p} \prec \frac{p(1 + Az)}{1 + Bz} \quad (1.5)$$

where $\prec$ denotes subordination.

For convenience, we write

$$K_{n,p} \left(1 - \frac{2\alpha}{p}, -1 \right) = K_{n,p}(\alpha),$$

where $K_{n,p}(\alpha)$ denote the class of function $f \in A_p$ satisfying the inequality

$$\operatorname{Re} \left\{ \frac{z(D^n f(z))'}{z^p} \right\} > \alpha \quad (0 \leq \alpha < p, z \in E).$$

We also note that $K_{0,p}(\alpha) \equiv C_p(\alpha)$ is the class of p -valently close-to-convex functions of order α .

In this present paper, we derive some properties of certain class $K_{n,p}(A, B)$ by using the differential subordination.

2. Preliminaries and Main Results

In our present investigation of the general class $K_{n,p}(A, B)$, we shall require the following lemmas.

Lemma 1 [4]. If the function $p(z) = 1 + c_1z + c_2z^2 + \dots$ is analytic in E , $h(z)$ is convex in E with $h(0) = 1$, and γ is complex number such that $\operatorname{Re} \gamma > 0$. Then the Briot-Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z)$$

implies

$$p(z) \prec q(z) = \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} h(t) dt \prec h(z) \quad (z \in E)$$

and $q(z)$ is the best dominant.

For complex number a, b and $c \neq 0, -1, -2, \dots$, the hypergeometric series

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \dots \quad (2.1)$$

represents an analytic function in E . It is well known by [1] that

Lemma 2. Let a, b and c be real $c \neq 0, -1, -2, \dots$ and $c > b > 0$. Then

$$\begin{aligned} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; z), \\ {}_2F_1(a, b; c; z) &= (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \end{aligned} \quad (2.2)$$

and

$${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z). \quad (2.3)$$

Lemma 3 [6]. Let $\phi(z)$ be convex and $g(z)$ is starlike in E . Then for F analytic in E with $F(0) = 1$, $\frac{\phi * Fg}{\phi * g}(E)$ is contained in the convex hull of $F(E)$.

Lemma 4 [2]. Let $\phi(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ and $\phi(z) \prec \frac{1 + Az}{1 + Bz}$. Then

$$|c_k| \leq (A - B).$$

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Theorem 1. Let n be any integer and $-1 \leq B < A \leq 1$. If $f \in K_{n,p}(A, B)$, then

$$\frac{z(D^{n+1}f(z))'}{z^p} \prec q(z) \prec \frac{p(1+Az)}{1+Bz} \quad (z \in E), \quad (2.4)$$

where

$$q(z) = \begin{cases} {}_2F_1(1, p+1; p+2; -Bz) \\ \quad + \frac{p+1}{p+2} Az {}_2F_1(1, p+2; p+3; -Bz), & B \neq 0 \\ 1 + \frac{p+1}{p+2} Az, & B = 0 \end{cases} \quad (2.5)$$

and $q(z)$ is the best dominant of (2.4). Furthermore, $f \in K_{n+1,p}(\rho(p, A, B))$, where

$$\rho(p, A, B) = \begin{cases} p {}_2F_1(1, p+1; p+2; B) \\ \quad - \frac{p(p+1)}{p+2} A {}_2F_1(1, p+2; p+3; B), & B \neq 0 \\ 1 - \frac{p+1}{p+2} A, & B = 0. \end{cases} \quad (2.6)$$

Proof. Let

$$p(z) = \frac{z(D^{n+1}f(z))'}{pz^p} \quad (2.7)$$

where $p(z)$ is analytic function with $p(0) = 1$.

Using the identity (1.4) in (2.7) and differentiating the resulting equation, we get

$$\frac{z(D^n f(z))'}{pz^p} = p(z) + \frac{zp'(z)}{p+1} \prec \frac{1+Az}{1+Bz} (\equiv h(z)). \quad (2.8)$$

Thus, by using Lemma 1 (for $\gamma = p+1$), we deduce that

$$\begin{aligned} p(z) &\prec (p+1)z^{-(p+1)} \int_0^z \frac{t^p(1+At)}{1+Bt} dt (\equiv q(z)) \\ &= (p+1) \int_0^1 \frac{s^p(1+Asz)}{1+Bsz} ds \\ &= (p+1) \int_0^1 \frac{s^p}{1+Bsz} ds + (p+1)Az \int_0^1 \frac{s^{p+1}}{1+Bsz} ds. \end{aligned} \quad (2.9)$$

By using (2.2) in (2.9), we obtain

$$p(z) \prec q(z) = \begin{cases} {}_2F_1(1, p+1; p+2; -Bz) \\ \quad + \frac{p+1}{p+2} Az {}_2F_1(1, p+2; p+3; -Bz), & B \neq 0 \\ 1 + \frac{p+1}{p+2} Az, & B = 0. \end{cases}$$

Thus, this proves (2.5).

Now, we show that

$$\operatorname{Re} q(z) \geq q(-r) \quad (|z| = r < 1). \quad (2.10)$$

Since $-1 \leq B < A \leq 1$, the function $(1 + Az)/(1 + Bz)$ is convex(univalent) in E and

$$\operatorname{Re} \left(\frac{1 + Az}{1 + Bz} \right) \geq \frac{1 - Ar}{1 - Br} > 0 \quad (|z| = r < 1).$$

Setting

$$g(s, z) = \frac{1 + Asz}{1 + Bsz} \quad (0 \leq s \leq 1, z \in E)$$

and $d\mu(s) = (p+1)s^p ds$, which is a positive measure on $[0, 1]$, we obtain from (2.9) that

$$q(z) = \int_0^1 g(s, z) d\mu(s) \quad (z \in E).$$

Therefore, we have

$$\operatorname{Re} q(z) = \int_0^1 \operatorname{Re} g(s, z) d\mu(s) \geq \int_0^1 \frac{1 - Asr}{1 - Bsr} d\mu(s)$$

which proves the inequality (2.10).

Now, using (2.10) in (2.9) and letting $r \rightarrow 1^-$, we obtain

$$\operatorname{Re} \left\{ \frac{z(D^{n+1}f(z))'}{z^p} \right\} > \rho(p, A, B),$$

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where

$$\rho(p, A, B) = \begin{cases} p {}_2F_1(1, p+1; p+2; B) \\ \quad - \frac{p(p+1)}{p+2} A {}_2F_1(1, p+2; p+3; B), & B \neq 0 \\ p - \frac{p(p+1)}{p+2} A, & B = 0. \end{cases}$$

This proves the assertion of Theorem 1. The result is best possible because of the best dominant property of $q(z)$.

Putting $A = 1 - \frac{2\alpha}{p}$ and $B = -1$ in Theorem 1, we have the following:

Corollary 1. For any integer n and $0 \leq \alpha < p$, we have

$$K_{n,p}(\alpha) \subset K_{n+1,p}(\rho(p, \alpha)),$$

where

$$\rho(p, \alpha) = p {}_2F_1(1, p+1; p+2; -1) - \frac{p(p+1)}{p+2} (1-2\alpha) {}_2F_1(1, p+2; p+3; -1). \quad (2.11)$$

The result is best possible.

Taking $p = 1$ in Corollary 1, we have the following:

Corollary 2. For any integer n and $0 \leq \alpha < 1$, we have

$$K_n(\delta) \subset K_{n+1}(\delta(\alpha))$$

where

$$\delta(\alpha) = 1 + 4(1-2\alpha) \sum_{k=1}^{\infty} \frac{1}{k+2} (-1)^k. \quad (2.12)$$

Theorem 2. For any integer n and $0 \leq \alpha < p$, if $f(z) \in K_{n+1,p}(\alpha)$ then $f \in K_{n,p}(\alpha)$ for $|z| < R(p)$, where $R(p) = \frac{-1 + \sqrt{1 + (p+1)^2}}{p+1}$.

The result is best possible.

Proof. Since $f(z) \in K_{n+1,p}(\alpha)$, we have

$$\frac{z(D^{n+1}f(z))'}{z^p} = \alpha + (p-\alpha)w(z), \quad (0 \leq \alpha < p), \quad (2.13)$$

where $w(z) = 1 + w_1 z + w_2 z^2 + \dots$ is analytic and has a positive real part in E . Making use of the logarithmic differentiation and using identity (1.4) in (2.13), we get

$$\frac{z(D^n f(z))'}{z^p} - \alpha = (p - \alpha) \left[w(z) + \frac{zw'(z)}{p+1} \right]. \quad (2.14)$$

Now, using the well-known by [5],

$$\frac{|zw'(z)|}{\operatorname{Re} w(z)} \leq \frac{2r}{1-r^2} \quad \text{and} \quad \operatorname{Re} w(z) \geq \frac{1-r}{1+r} \quad (|z| = r < 1),$$

in (2.14). We get

$$\begin{aligned} \operatorname{Re} \left\{ \frac{z(D^n f(z))'}{z^p} - \alpha \right\} &= (p - \alpha) \operatorname{Re} w(z) \left\{ 1 + \frac{1}{p+1} \frac{\operatorname{Re} zw'(z)}{\operatorname{Re} w(z)} \right\} \\ &\geq (p - \alpha) \operatorname{Re} w(z) \left\{ 1 - \frac{1}{p+1} \frac{|zw'(z)|}{\operatorname{Re} w(z)} \right\} \\ &\geq (p - \alpha) \frac{1-r}{1+r} \left\{ 1 - \frac{1}{p+1} \frac{2r}{1-r^2} \right\}. \end{aligned}$$

It is easily seen that the right-hand side of the above expression is positive if $|z| < R(p) = \frac{-1 + \sqrt{1 + (p+1)^2}}{p+1}$. Hence $f \in K_{n,p}(\alpha)$ for $|z| < R(p)$.

To show that the bound $R(p)$ is best possible, we consider the function $f \in A_p$ defined by

$$\frac{z(D^{n+1} f(z))'}{z^p} = \alpha + (p - \alpha) \frac{1-z}{1+z} \quad (z \in E).$$

Noting that

$$\begin{aligned} \frac{z(D^n f(z))'}{z^p} - \alpha &= (p - \alpha) \cdot \frac{1-z}{1+z} \left\{ 1 + \frac{1}{p+1} \frac{-2z}{(p+1)(1-z^2)} \right\} \\ &= (p - \alpha) \cdot \frac{1-z}{1+z} \left\{ \frac{(p+1) - (p+1)z^2 - 2z}{(p+1) - (p+1)z^2} \right\} \\ &= 0 \end{aligned}$$

for $z = \frac{-1 + \sqrt{1 + (p+1)^2}}{p+1}$, we complete the proof of Theorem 2.

Putting $n = -1$, $p = 1$ and $0 \leq \alpha < 1$ in Theorem 2, we have the following:

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Corollary 3. If $\operatorname{Re} f'(z) > \alpha$, then $\operatorname{Re} \{zf''(z) + 2f'(z)\} > \alpha$ for $|z| < \frac{-1 + \sqrt{5}}{2}$.

Theorem 3. (a) If $f \in K_{n,p}(A, B)$, then the function F_c defined by (1.2) belongs to $K_{n,p}(A, B)$.

(b) $f \in K_{n,p}(A, B)$ implies that $F_c \in K_{n,p}(\eta(p, c, A, B))$ where

$$\eta(p, c, A, B) = \begin{cases} p_2 F_1(1, p+c; p+c+1; B) \\ -\frac{p(p+c)}{p+c+1} A_2 F_1(1, p+c+1; p+c+2; B), & B \neq 0 \\ p - \frac{p(p+c)}{p+c+1} A, & B = 0. \end{cases}$$

Proof. Let

$$\phi(z) = \frac{z(D^n F_c(z))'}{pz^p}, \quad (2.15)$$

where $\phi(z)$ is analytic function with $\phi(0) = 1$. Using the identity

$$z(D^n F_c(z))' = (p+c)D^n f(z) - cD^n F_c(z) \quad (2.16)$$

in (2.15) and differentiating the resulting equation, we get

$$\frac{z(D^n f(z))'}{pz^p} = \phi(z) + \frac{z\phi'(z)}{p+c}.$$

Since $f \in K_{n,p}(A, B)$,

$$\phi(z) + \frac{z\phi'(z)}{p+c} \prec \frac{1+Az}{1+Bz}.$$

By Lemma 1, we obtain $F_c(z) \in K_{n,p}(A, B)$. We deduce that

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (2.17)$$

where $q(z)$ is given (2.5) and $q(z)$ is best dominant of (2.17).

This proves the (a) part of theorem. Proceeding as in Theorem 3, the (b) part follows.

Putting $A = 1 - \frac{2\alpha}{p}$ and $B = -1$ in Theorem 2, we have the following:

Corollary 4. *If $f \in K_{n,p}(A, B)$ for $0 \leq \alpha < p$, then $F_c \in K_{n,p}\mathcal{H}(p, c, \alpha)$ where*

$$\begin{aligned} \mathcal{H}(p, c, \alpha) = & p \cdot {}_2F_1(1, p+c; p+c+1; -1) \\ & - \frac{p+c}{p+c+1} (p-2\alpha) {}_2F_1(1, p+c; p+c+1; -1). \end{aligned}$$

Setting $c = p = 1$ in Theorem 3, we get the following result.

Corollary 4. *If $f \in K_{n,p}(\alpha)$ for $0 \leq \alpha < 1$, then the function*

$$G(z) = \frac{2}{z} \int_0^z f(t) dt$$

belongs to the class $K_n(\delta(\alpha))$, where $\delta(\alpha)$ is given by (2.12).

Theorem 4. *For any integer n and $0 \leq \alpha < p$ and $c > -p$, if $F_c \in K_{n,p}(\alpha)$ then the function f defined by (1.1) belongs to $K_{n,p}(\alpha)$ for $|z| < R(p, c) = \frac{-1 + \sqrt{1 + (p+c)^2}}{p+c}$. The result is best possible.*

Proof. Since $F_c \in K_{n,p}(\alpha)$, we write

$$\frac{z(D^n F_c)'}{z^p} = \alpha + (p - \alpha)w(z), \quad (2.18)$$

where $w(z)$ is analytic, $w(0) = 1$ and $\operatorname{Re} w(z) > 0$ in E . Using (2.16) in (2.18) and differentiating the resulting equation, we obtain

$$\operatorname{Re} \left\{ \frac{z(D^n f(z))'}{z^p} - \alpha \right\} = (p - \alpha) \operatorname{Re} \left\{ w(z) + \frac{zw'(z)}{p+c} \right\}. \quad (2.19)$$

Now, by following the line of proof of Theorem 2, we get the assertion of Theorem 4.

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Theorem 5. Let $f \in K_{n,p}(A, B)$ and $\phi(z) \in A_p$ convex in E . Then

$$(f * \phi(z))(z) \in K_{n,p}(A, B).$$

Proof. Since $f(z) \in K_{n,p}(A, B)$,

$$\frac{z(D^n f(z))'}{pz^p} \prec \frac{1 + Az}{1 + Bz}.$$

Now

$$\begin{aligned} \frac{z(D^n(f * \phi)(z))'}{pz^p * \phi(z)} &= \frac{\phi(z) * z(D^n f)'}{\phi(z) * pz^p} \\ &= \frac{\phi(z) * \frac{z(D^n f(z))'}{pz^p} pz^p}{\phi(z) * pz^p}. \end{aligned} \quad (2.20)$$

Then applying Lemma 3, we deduce that

$$\frac{\phi(z) * \frac{z(D^n f(z))'}{pz^p} pz^p}{\phi(z) * pz^p} \prec \frac{1 + Az}{1 + Bz}.$$

Hence $(f * \phi(z))(z) \in K_{n,p}(A, B)$.

Theorem 6. Let a function $f(z)$ defined by (1.1) be in the class $K_{n,p}(A, B)$. Then

$$|a_{p+k}| \leq \frac{p(A - B)(p + k + 1)^n}{(1 + p)^n(p + k)} \quad \text{for } k = 1, 2, \dots \quad (2.21)$$

The result is sharp.

Proof. Since $f(z) \in K_{n,p}(A, B)$, we have

$$\frac{z(D^n f(z))'}{pz^p} \equiv \phi(z) \quad \text{and} \quad \phi(z) \prec \frac{1 + Az}{1 + Bz}.$$

Hence

$$z(D^n f(z))' = pz^p \phi(z) \quad \text{and} \quad \phi(z) = 1 + \sum_{k=1}^{\infty} c_k z^k. \quad (2.22)$$

From (2.22), we have

$$\begin{aligned}
 z(D^n f(z))' &= z \left(z^p + \sum_{k=1}^{\infty} \left(\frac{1+p}{p+k+1} \right)^n a_{p+k} z^{p+k} \right)' \\
 &= pz^p + \sum_{k=1}^{\infty} \left(\frac{1+p}{p+k+1} \right)^n (p+k) a_{p+k} z^{p+k} \\
 &= pz^p \left(1 + \sum_{k=1}^{\infty} c_k z^k \right).
 \end{aligned}$$

Therefore

$$\left(\frac{1+p}{p+k+1} \right)^n (p+k) a_{p+k} = pc_k. \quad (2.23)$$

By using Lemma 4 in (2.23),

$$\frac{\left(\frac{1+p}{p+k+1} \right)^n (p+k) |a_{p+k}|}{p} = |c_k| \leq A - B.$$

Hence

$$|a_{p+k}| \leq \frac{p(A-B)(p+k+1)^n}{(1+p)^n(p+k)}.$$

The equality sign in (2.21) holds for the function f given by

$$(D^n f(z))' = \frac{pz^{p-1} + p(A-B-1)z^p}{1-z}. \quad (2.24)$$

Hence

$$\frac{z(D^n f(z))'}{pz^p} = \frac{1 + (A-B-1)z}{1-z} \prec \frac{1 + Az}{1 + Bz} \text{ for } k = 1, 2, \dots.$$

The function $f(z)$ defined in (2.24) has the power series representation in E ,

$$f(z) = z^p + \sum_{k=1}^{\infty} \frac{p(A-B)(p+k+1)^n}{(1+p)^n(p+k)} z^{p+k}.$$

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