

Algebraic Models of Organizations

Toyo University, Takao ASAHII

(東洋大学 旭 貴朗)

1. Introduction

H.A.Simon[3] has suggested a framework that permits a comparison of the economist's theory of the firm and the theory of organizational equilibrium. In his paper, the former theory is defined as "F-theory" and the latter as "O-theory". He obtained a result that the F-theory solution, in the case of perfect competition, is identical with the particular O-theory solution that is optimal to the entrepreneur.

In this paper we will induce a corresponding result in an algebraic framework on contrast to Simon's analytical framework. Our Framework is more general in the meanings that Simon's is one of special cases of ours.

2. Models of Participants

Definition 1 Behavior Selection Model

A five-tuple (U, Y, M, g, Ψ) is called a "behavior selection model", if the following conditions hold.

1) U , Y , and M are sets. They are called an inducement set, a contribution set and an alternative set, respectively. $M \subset U \times Y$ is

supposed.

2) g is a function from M to the set of real numbers. It is called an objective function. The pair (M, g) is called decision problem.

3) Ψ is a symbol of a particular operator which defines a subset $\Psi(M, g) \subset M$ for a given decision problem (M, g) . Ψ is called a decision principle. Details will be defined in Definition 3.

$$\begin{array}{c} \Psi \\ (M, g) \rightarrow \Psi(M, g) \subset M \end{array}$$

Figure 1 decision principle

In this paper, the inducement set U and the contribution set Y are fixed. So a behavior selection model will be denoted briefly by $S = \Psi(M, g)$.

Assumption 1

1) Objective function

The reverse function $g^{-1}(r): U \rightarrow Y$ are "one to one" and "onto" for all real number r .

2) Alternative sets with parameter p

There exists a class of functions $\{M(p): U \rightarrow Y\}$ such that $M = \bigcup M(p)$ and $(\forall p)(\forall q)(p \neq q \rightarrow M(p) \cap M(q) = \emptyset)$.

Definition 2 Operators (Sat, Max)

For any decision problem (M, g) , we can define the following two subsets of M .

$$m \in \text{Sat}(M, g) \leftrightarrow g(m) \geq 0,$$

$$m \in \text{Max}(M, g) \leftrightarrow (\forall m')(g(m) \geq g(m')).$$

Definition 3 Decision Principles

- 1) If $\Psi(M, g) = \text{Sat}(M, g)$, Ψ is called a "satisfactory principle".
- 2) If $\Psi(M, g) = \text{Max}(M, g)$, Ψ is called an "optimazing principle".
- 3) If $\Psi(M, g) = \bigcup \text{Max}(M(p), g)$, Ψ is called an "optimizing principle with parameter p".

Lemma 1 Properties of Operators

For any decision problem (M, g) , the followings hold.

- 1) $\text{Sat}(M, g) = \bigcup \text{Sat}(M(p), g)$, where $\bigcup$ is a union in terms of parameters p .
- 2) If $\text{Sat}(M, g) \neq \phi$, then $\text{Max}(\text{Sat}(M, g), g) = \text{Max}(M, g)$ where ϕ is a symbol of empty set.

3. Complete Competetion

Definition 4 (q, r) -completeness

- 1) If there exists an parameter q such that

$$\text{Max}(M(p), g) = \begin{cases} M(q) & \text{if } p=q \\ \phi & \text{if } p \neq q \end{cases}$$

holds, an objective function g is called "complete".

- 2) If g is complete, then there exists an real number r such that $M(q) = g^{-1}(r)$. Hence the objective function g can be called " (q, r) -complete".

Definition 5 Restricted Set

For any set $S \subset U \times Y$, a "restricted set" $S[g(m)=r] \subset S$ is defined as follows.

$$S[g(m)=r] = \{m \in S \mid g(m)=r\}.$$

Lemma 2 Property of Completeness

Let an objective function g be (q,r) -complete and $r \geq 0$.

Then ,

$$S(p)[g(m)=r]=\begin{cases} M(q) & \text{if } p=q \\ \phi & \text{if } p \neq q \end{cases}.$$

holds for $S(p)=\text{Sat}(M(p),g)$.

Proposition 3

Let an objective function g be (q,r) -complete and $r \geq 0$.

And let $S=\text{Sat}(M,g)$ and $S^{\wedge}=\bigcup \text{Max}(M(p),g)$. Then $S[g(m)=r]=S^{\wedge}=M(q)$.

4. Models of Organizations

Definition 6 Organization $(R,\{S_i\})$

An "organization with n participants" is defined by a pair $(R,\{S_i\})$ such that

$$S_i = \Psi_i(M_i, g_i) \quad (i=1,2,\dots,n)$$

$$U = U_1 \times \dots \times U_n$$

$$Y = Y_1 \times \dots \times Y_n$$

$$R \subset U \times Y$$

where S_i , R and $S = R \cap (S_1 \times S_2 \times \dots \times S_n)$ is called a "participant", an "organizational restriction" and an "organizational behavior", respectively.

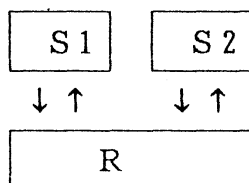


Figure 2 Structure of an organization

Assumption 2

- 1) We assume $\text{proj1}(R)=M1$ and $\text{proj2}(R)=M2$ where $\text{proj1}:U \times Y \rightarrow U1 \times Y1$ is a projection function.
- 2) From now on, we will focus on the case $n=2$, i.e., two-participants organizations.

Definition 7 Internal Model

If the alternative set of participant 1 satisfies

$$M1 = \{m1 \mid \exists m2 \in S2, (m1, m2) \in R\},$$

then participant 1 is called to "have an internal model of participant 2".

Proposition 4

Let $(R, \{S1, S2\})$ be an organization.

If participant 1 have a internal model of participant 2, then

$$S \neq \phi \leftrightarrow S1 \neq \phi.$$

5. Comparison of Organizations

Definition 8 Optimum Set of Organization

For any organizational behavior S of any organization $(R, \{S1, S2\})$, the following set

$$\text{OPT1}(S) = \text{Max}(\text{proj1}(S), g1)$$

is called an "optimum set" of the organization $(R, \{S1, S2\})$.

Proposition 5

Assume that $\Psi1 = \text{Sat}$ in an organization $(R, \{S1, S2\})$.

If $S \neq \phi$, then

$$\text{OPT1}(R \cap (S1 \times S2)) = \text{OPT1}(R \cap (M1 \times S2)).$$

Let us consider two organizations such that organizational restriction R and objective functions g_i are common. It means that decision principles Ψ_i and alternative sets M_i might be different. The optimum sets of those organizations below will be compared. Those organizations correspond to "0-theory" and "F-theory", respectively.

Organization 1

$$S_1 = \text{Sat}(M_1, g_1), S_2 = \text{Sat}(M_2, g_2), S = R \cap (S_1 \times S_2).$$

Organization 2

$$S^1 = \text{Max}(M_1^1, g_1), S^2 = \bigcup \text{Max}(M_2(p), g_2), S^* = R \cap (S^1 \times S^2).$$

$$\text{where } M_1^1 = \{m_1 \mid \exists m_2 \in S_2, (m_1, m_2) \in R\}$$

(i.e. internal model of participant 2)

$$\bigcup M_2(p) = M_2$$

Theorem 6

Assume that the common objective function g_2 is (q, r) -complete in the above two organizations. If $S[g_2(m_2)=r] \neq \emptyset$, then $\text{OPT}_1(S[g_2(m_2)=r]) = S^1$.

6. Discussion

Theorem 6 is the main result of this paper. The set $S[g_2(m_2)=r]$ in the left hand side is an organizational behavior under the condition that the utility value of the participant 2 is restricted to a fixed r . Then, the optimum set

$$\text{OPT}_1(S[g_2(m_2)=r])$$

of Organization 1 corresponds to the optimum solution to entrepreneur in "0-theory". On the other hand, the participant 1

of Organization 2 have an internal model of participant 2. Then $S1^*$ corresponds to the optimum solution of "F-theory" in the case of complete competition. Theorem 6 shows that both are identical.

As mentioned in Introduction, our framework is algebraic and including Simon's framework as a special case. Indeed, we can get the complete competition condition ($\phi(u)=qu$) with additional properties. Let us assume that $U = Y =$ (the set of positive real numbers) and that $M(p) = \{ (u,y) \mid y = pu \}$. Then $M = \bigcup M(p) = U \times Y$. And let us define $g:M \rightarrow$ (real numbers) by

$$g(u,y) = \phi(u) - y$$

such that $\phi(u)$ is an utility function of inducement u and that $\phi(u)$ is differentiable. If $g(0,0) = 0$, then we have $\phi(u)=qu$ from (q,r) -completeness of g . Therefore $S1^*$ of Organization 2 is the optimum solution of the entrepreneur.

References

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- [3] H.A.Simon, Models of Man, John Wiley & Sons (1957)
- [4] T.Asahi, "A Comparison of Administrative Systems and Economic Systems : Reformulation of Simon's framework", in Proceedings of 15th. Systems Symposium, SICE (in Japanese) (1989)
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Appendices

Proof of Lemma 1

1) Since

$$\begin{aligned}
 m &\in \bigcup \text{Sat}(M(p), g) \\
 &\leftrightarrow \exists p, m \in \text{Sat}(M(p), g) \\
 &\leftrightarrow \exists p, m \in M(p), g(m) \geq 0 \\
 &\leftrightarrow m \in \bigcup M(p) = M, g(m) \geq 0 \\
 &\leftrightarrow m \in \text{Sat}(M, g)
 \end{aligned}$$

hold, we have $\bigcup \text{Sat}(M(p), g) = \text{Sat}(M, g)$.

2) Since $\text{Sat}(M, g) \neq \emptyset$, $\exists m^* \in M, g(m^*) \geq 0$. Then

$$m \in \text{Max}(M, g) \rightarrow g(m) \geq g(m^*) \geq 0 \rightarrow m \in \text{Sat}(M, g).$$

That is, $\text{Max}(M, g) \subset \text{Sat}(M, g)$. Then we have

$$\text{Max}(\text{Sat}(M, g), g) = \text{Sat}(M, g). \quad \blacksquare$$

Proof of Lemma 2

Let an objective function g be (q, r) -complete.

From Definition 4, we have $M(q) = g^{-1}(r)$. Then the condition $r \geq 0$ implies $S(p)[g(m)=r] = M(p) \cap M(q)$. Indeed,

$$\begin{aligned}
 m &\in S(p)[g(m)=r] \\
 &\leftrightarrow m \in S(p) = \text{Sat}(M(p), g), g(m)=r \\
 &\leftrightarrow m \in M(p), g(m)=r \geq 0 \\
 &\leftrightarrow m \in M(p) \cap g^{-1}(r) = M(p) \cap M(q).
 \end{aligned}$$

Hence Assumption 1 (2) implies the result which is to be proved.

■

Proof of Proposition 3

Firstly, let an objective function g be (q,r) -complete.

From Definition 4, we have

$$\text{Max}(M(p), g) = \begin{cases} M(q) & \text{if } p=q \\ \phi & \text{if } p \neq q \end{cases}$$

Then $S^{\wedge} = \bigcup \text{Max}(M(p), g) = M(q)$.

Secondly, let $S(p) = \text{Sat}(M(p), g)$. Lemma 1 (1) implies

$$S = \text{Sat}(M, g) = \bigcup \text{Sat}(M(p), g) = \bigcup S(p).$$

Then we will have $S[g(m)=r] = \bigcup (S(p)[g(m)=r])$. Indeed,

$$\begin{aligned} m \in S[g(m)=r] \\ \leftrightarrow m \in S, g(m)=r \\ \leftrightarrow \exists p, m \in S(p), g(m)=r \\ \leftrightarrow \exists p, m \in S(p)[g(m)=r] \\ \leftrightarrow m \in \bigcup S(p)[g(m)=r]. \end{aligned}$$

Hence Lemma 2 implies $S[g(m)=r] = M(q)$. ■

Proof of Proposition 4

(nessesity)

It is trivial since $S = R \cap (S_1 \times S_2) \neq \phi \rightarrow S_1 \neq \phi$.

(sufficiency)

$$\begin{aligned} m_1 \in S_1 \\ \rightarrow m_1 \in M_1 = \{m_1 \mid \exists m_2 \in S_2, (m_1, m_2) \in R\} \\ \rightarrow \exists m_2 \in S_2, (m_1, m_2) \in R \\ \rightarrow (m_1, m_2) \in R \cap (S_1 \times S_2) = S. \quad \blacksquare \end{aligned}$$

Proof of Proposition 5

Let $S = R \cap (S1 \times S2)$ and $S^* = R \cap (M1 \times S2)$.

Then, $\text{proj1}(S) = \text{Sat}(\text{proj1}(S^*), g1)$. Indeed,

$$m1 \in \text{proj1}(S)$$

$$\leftrightarrow \exists m2, (m1, m2) \in S$$

$$\leftrightarrow \exists m2 \in S2, (m1, m2) \in R, m1 \in S1$$

$$\leftrightarrow m1 \in \text{proj1}(S^*), g1(m1) \geq 0$$

$$\leftrightarrow m1 \in \text{Sat}(\text{proj1}(S^*), g1) .$$

Since $S \neq \emptyset$ from assumption, we have

$\text{Sat}(\text{proj1}(S^*), g1) = \text{proj1}(S) \neq \emptyset$. Therefore Lemma 1 (2) implies

$$\text{OPT1}(S) = \text{Max}(\text{proj1}(S), g1) = \text{Max}(\text{Sat}(\text{proj1}(S^*), g1), g1)$$

$$= \text{Max}(\text{proj1}(S^*), g1) = \text{OPT1}(S^*) . \quad \blacksquare$$

Proof of Theorem 6

Firstly, from the definition of organizational restriction, we have

$$S[g2(m2)=r] = R \cap (S1 \times (S2[g2(m2)=r])) .$$

And from assumption $S[g2(m2)=r] \neq \emptyset$, we have $S2[g2(m2)=r] \neq \emptyset$.

Then $r = g2(m2) \geq 0$.

Secondly, Proposition 5 implies

$$\text{OPT1}(S[g2(m2)=r]) = \text{OPT1}(R \cap (M1 \times (S2[g2(m2)=r]))) .$$

Thirdly, from the definition of participant 2 in Organization 2, we have

$$S1^{\wedge} = \text{OPT1}(R \cap (M1 \times S2^{\wedge})) .$$

On the other hand, since $r \geq 0$ and $g2$ is (q, r) -complete, Proposition 3 implies

$$S2^{\wedge} = S2[g2(m2)=r] .$$

Therefore $\text{OPT1}(S[g2(m2)=r]) = S1^{\wedge}$. $\blacksquare$