

Nonlinear Operators and Fixed Point Theorems in Hilbert Spaces

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Abstract. In this article, we first consider new classes of nonlinear mappings containing the class of firmly nonexpansive mappings which can be deduced from an equilibrium problem in a Hilbert space. Further, we deal with fixed point theorems and ergodic theorems for these nonlinear mappings.

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . Then a mapping $T : C \rightarrow H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that if C is a bounded closed convex subset of H and $T : C \rightarrow C$ is nonexpansive, then the set $F(T)$ of fixed points of T is nonempty. Further, from Baillon [1] we know the first nonlinear ergodic theorem in a Hilbert space: Let C be a nonempty bounded closed convex subset of H and let $T : C \rightarrow C$ be nonexpansive. Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping T is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \quad \forall x, y \in C;$$

see, for instance, Goebel and Kirk [8]. It is also known that a firmly nonexpansive mapping T can be deduced from an equilibrium problem in a Hilbert space as follows: Let C be a nonempty closed convex subset of H and let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying the following conditions:

- (A1) $f(x, x) = 0$, $\forall x \in C$;
- (A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$, $\forall x, y \in C$;
- (A3) $\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$, $\forall x, y, z \in C$;
- (A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

We know the following lemma; see, for instance, [2] and [7].

Lemma 1.1. *Let C be a nonempty closed convex subset of H and let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3) and (A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Further, if $T_r x = \{z \in C : f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C\}$ for all $x \in H$, then the following hold:

- (1) T_r is single-valued;
- (2) T_r is firmly nonexpansive, i.e.,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \quad \forall x, y \in H.$$

Recently, Kohsaka and Takahashi [12] introduced the following nonlinear mapping: Let E be a smooth, strictly convex and reflexive Banach space, let J be the duality mapping of E and let C be a nonempty closed convex subset of E . Then, a mapping $S : C \rightarrow E$ is said to be nonspreading if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \leq \phi(Sx, y) + \phi(Sy, x), \quad \forall x, y \in C,$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$ for all $x, y \in E$. They considered such a mapping to study the resolvents of a maximal monotone operator in the Banach space. In the case when E is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. So, a nonspreading mapping S in a Hilbert space is defined as follows:

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|Sy - x\|^2, \quad \forall x, y \in C.$$

On the other hand, Takahashi [29] found another new nonlinear mapping called a hybrid mapping which is also deduced from a firmly nonexpansive mapping.

In this paper, we first discuss new classes of nonlinear mappings containing the class of firmly nonexpansive mappings which can be deduced from a firmly nonexpansive mapping in a Hilbert space. Further, we deal with fixed point theorems and ergodic theorems for these nonlinear mappings.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. In a Hilbert space, it is known that for all $x, y \in H$ and $\alpha \in \mathbb{R}$,

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha(1 - \alpha) \|x - y\|^2; \quad (2.1)$$

see, for instance, [28]. Further, in a Hilbert space, we have that for all $x, y, z, w \in H$,

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2. \quad (2.2)$$

Let C be a nonempty subset of H and let T be a mapping of C into H . We denote by $F(T)$ the set of all fixed points of T , that is, $F(T) = \{z \in C : Tz = z\}$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. The following lemma is in [20].

Lemma 2.1. *Let C be a nonempty closed convex subset of H and let $f : C \rightarrow (-\infty, \infty]$ be a proper convex lower semicontinuous function such that $f(z_m) \rightarrow \infty$ as $\|z_m\| \rightarrow \infty$. Then there exists an element $z_0 \in C$ such that*

$$f(z_0) = \min\{f(z) : z \in C\}.$$

Let $\mathbb{N}$ be the set of positive integers and let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞ ; see [20] for more details.

3 Nonlinear Mappings

Let H be a Hilbert space. Let C be a nonempty closed convex subset of H and let T be a mapping of C into H . Then, from [29], we have the following equality:

$$\|Tx - Ty\|^2 = \|x - y - (Tx - Ty)\|^2 - \|x - y\|^2 + 2\langle x - y, Tx - Ty \rangle \quad (3.1)$$

for all $x, y \in C$. We have also from (2.2) that

$$2\langle x - y, Tx - Ty \rangle = \|x - Ty\|^2 + \|y - Tx\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2. \quad (3.2)$$

Further, we have that

$$\|x - y - (Tx - Ty)\|^2 = \|x - Tx\|^2 + \|y - Ty\|^2 - 2\langle x - Tx, y - Ty \rangle. \quad (3.3)$$

If $T : C \rightarrow H$ is firmly nonexpansive, then

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \quad \forall x, y \in C.$$

So, we have from (3.1) that for all $x, y \in C$,

$$\begin{aligned} 2\|Tx - Ty\|^2 &\leq 2\langle x - y, Tx - Ty \rangle \\ &= \|Tx - Ty\|^2 - \|x - y - (Tx - Ty)\|^2 + \|x - y\|^2 \\ &\leq \|Tx - Ty\|^2 + \|x - y\|^2. \end{aligned}$$

Then, we have $\|Tx - Ty\|^2 \leq \|x - y\|^2$ and hence $\|Tx - Ty\| \leq \|x - y\|$. Such a mapping is nonexpansive. We know that $T : C \rightarrow H$ is nonexpansive if and only if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C;$$

see [29]. Thus, we can get new classes of nonlinear operators which contain the class of firmly nonexpansive mappings in a Hilbert space. For example, Kohsaka and Takahashi [12] obtained a *nonspreading mapping*. Let $T : C \rightarrow H$ be a firmly nonexpansive mapping. Then, we have that for all $x, y \in C$,

$$2\|Tx - Ty\|^2 \leq 2\langle x - y, Tx - Ty \rangle.$$

From (3.2), we obtain

$$\begin{aligned} 2\|Tx - Ty\|^2 &\leq \|x - Ty\|^2 + \|y - Tx\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2 \\ &\leq \|x - Ty\|^2 + \|y - Tx\|^2. \end{aligned}$$

So, we have

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2.$$

This is a nonspreading mapping. From Iemoto and Takahashi [10], we know the following lemma.

Lemma 3.1. *Let C be a nonempty closed convex subset of H . Then a mapping $S : C \rightarrow H$ is nonspreading if and only if*

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + 2\langle x - Sx, y - Sy \rangle$$

for all $x, y \in C$.

Further, from a firmly nonexpansive mapping, i.e.,

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \quad \forall x, y \in C,$$

we have

$$\|Tx - Ty\|^2 \leq 2\langle Tx - Ty, x - y \rangle, \quad \forall x, y \in C.$$

Such a mapping $T : C \rightarrow H$ is called $\frac{1}{2}$ -inverse strongly monotone. Takahashi [29] also defined the following *hybrid mapping*, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2, \quad \forall x, y \in C.$$

From Takahashi [29], we know the following lemma.

Lemma 3.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Then a mapping $T : C \rightarrow H$ is hybrid if and only if*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C.$$

So, a hybrid mapping $T : C \rightarrow H$ is different from a nonspreading mapping. Further, we define a new nonlinear operator from a firmly nonexpansive mapping. We have that for any $x, y \in C$,

$$\begin{aligned} 2\|Tx - Ty\|^2 &\leq 2\langle x - y, Tx - Ty \rangle \\ \iff \|Tx - Ty\|^2 + \|Tx\|^2 + \|Ty\|^2 - 2\langle Tx, Ty \rangle &\leq 2\langle x - y, Tx - Ty \rangle \\ \implies \|Tx - Ty\|^2 - 2\langle Tx, Ty \rangle &\leq 2\langle x - y, Tx - Ty \rangle \\ \iff \|Tx - Ty\|^2 &\leq 2\langle Tx, Ty \rangle + 2\langle x - y, Tx - Ty \rangle. \end{aligned}$$

So, we can define a new mapping called a metric mapping, i.e.,

$$\|Tx - Ty\|^2 \leq 2\langle Tx, Ty \rangle + 2\langle x - y, Tx - Ty \rangle, \quad \forall x, y \in C.$$

Let $T : C \rightarrow H$ be a nonexpansive mapping and put $A = I - T$. Then, we have from [28] that A is $1/2$ -inverse strongly monotone, i.e.,

$$\frac{1}{2} \|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay \rangle, \quad \forall x, y \in C.$$

Let $T : C \rightarrow H$ be a nonspreading mapping and put $A = I - T$. Then, we have from Lemma 3.1 and (3.1) that for any $x, y \in C$,

$$\begin{aligned} \|Ax - Ay\|^2 &= \|x - y - (Ax - Ay)\|^2 - \|x - y\|^2 + 2\langle x - y, Ax - Ay \rangle \\ &= \|Tx - Ty\|^2 - \|x - y\|^2 + 2\langle x - y, Ax - Ay \rangle \\ &\leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle - \|x - y\|^2 + 2\langle x - y, Ax - Ay \rangle \\ &= 2\langle Ax, Ay \rangle + 2\langle x - y, Ax - Ay \rangle. \end{aligned}$$

This implies that A is a metric mapping.

4 Generalized Fixed Point Theorem and its Applications

In this section, we obtain a generalized fixed point theorem in a Hilbert space. Before proving the theorem, we can obtain the following lemma from Lemma 2.1.

Lemma 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H , let $\{x_n\}$ be a bounded sequence in H and let μ be a Banach limit. If $g : C \rightarrow \mathbb{R}$ is defined by*

$$g(z) = \mu_n \|x_n - z\|^2, \quad \forall z \in C,$$

then there exists a unique $z_0 \in C$ such that

$$g(z_0) = \min\{g(z) : z \in C\}.$$

Using Lemma 4.1, we can prove the following generalized fixed point theorem [31].

Theorem 4.2. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself. Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded and*

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2, \quad \forall y \in C$$

for some Banach limit μ . Then, T has a fixed point in C .

Proof. Using a Banach limit μ on l^∞ , we can define $g : C \rightarrow \mathbb{R}$ as follows:

$$g(z) = \mu_n \|T^n x - z\|^2, \quad \forall z \in C.$$

From Lemma 4.1, there exists a unique $z_0 \in C$ such that

$$g(z_0) = \min\{g(z) : z \in C\}.$$

So, we have

$$g(Tz_0) = \mu_n \|T^n x - Tz_0\|^2 \leq \mu_n \|T^n x - z_0\|^2 = g(z_0).$$

Since Tz_0 is in C and $z_0 \in C$ is a unique element such that

$$g(z_0) = \min\{g(z) : z \in C\},$$

we have $Tz_0 = z_0$. This completes the proof. $\square$

Using Theorem 4.2, we can obtain some fixed point theorems. The following is the well-known fixed point theorem for nonexpansive mappings in a Hilbert space; see, for instance, [28].

Theorem 4.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be a nonexpansive mapping, i.e.,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C .

Proof. Let μ be a Banach limit on l^∞ . For any $n \in \mathbb{N}$ and $y \in C$, we have

$$\|T^{n+1}x - Ty\|^2 \leq \|T^n x - y\|^2.$$

So, we have

$$\mu_n \|T^n x - Ty\|^2 = \mu_n \|T^{n+1}x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2$$

for all $y \in C$. By Theorem 4.2, T has a fixed point in C . $\square$

The following is a fixed point theorem for nonspreading mappings in a Hilbert space.

Theorem 4.4 ([12]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be a nonspreading mapping, i.e.,*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C .

Proof. Let μ be a Banach limit on l^∞ . For any $n \in \mathbb{N}$ and $y \in C$, we have

$$2\|T^{n+1}x - Ty\|^2 \leq \|T^{n+1}x - y\|^2 + \|T^n x - Ty\|^2.$$

So, we have

$$\begin{aligned} 2\mu_n \|T^n x - Ty\|^2 &= 2\mu_n \|T^{n+1}x - Ty\|^2 \\ &\leq \mu_n \|T^{n+1}x - y\|^2 + \mu_n \|T^n x - Ty\|^2 \\ &= \mu_n \|T^n x - y\|^2 + \mu_n \|T^n x - Ty\|^2 \end{aligned}$$

and hence

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2.$$

By Theorem 4.2, T has a fixed point in C . $\square$

The following is a fixed point theorem for hybrid mappings in a Hilbert space.

Theorem 4.5 ([29]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be a hybrid mapping, i.e.,*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C .

Proof. Let μ be a Banach limit on l^∞ . We know from Lemma 3.2 that a mapping $T : C \rightarrow C$ is hybrid if and only if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

So, for any $n \in \mathbb{N}$ and $y \in C$, we have

$$3\|T^{n+1}x - Ty\|^2 \leq \|T^n x - y\|^2 + \|T^{n+1}x - y\|^2 + \|T^n x - Ty\|^2.$$

So, we have

$$\begin{aligned} 3\mu_n \|T^n x - Ty\|^2 &= 3\mu_n \|T^{n+1}x - Ty\|^2 \\ &\leq 2\mu_n \|T^n x - y\|^2 + \mu_n \|T^n x - Ty\|^2 \end{aligned}$$

and hence

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2.$$

By Theorem 4.2, T has a fixed point in C . □

We can also prove the following two fixed point theorems in a Hilbert space.

Theorem 4.6. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be a mapping such that*

$$2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C .

Proof. Let μ be a Banach limit on l^∞ . For any $n \in \mathbb{N}$ and $y \in C$, we have

$$2\|T^{n+1}x - Ty\|^2 \leq \|T^n x - y\|^2 + \|T^{n+1}x - y\|^2.$$

So, we have

$$\begin{aligned} 2\mu_n \|T^n x - Ty\|^2 &= 2\mu_n \|T^{n+1}x - Ty\|^2 \\ &\leq 2\mu_n \|T^n x - y\|^2 \end{aligned}$$

and hence

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2.$$

By Theorem 4.2, T has a fixed point in C . □

Theorem 4.7. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be a mapping such that*

$$3\|Tx - Ty\|^2 \leq 2\|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C .

Proof. Let μ be a Banach limit on l^∞ . For any $n \in \mathbb{N}$ and $y \in C$, we have

$$3\|T^{n+1}x - Ty\|^2 \leq 2\|T^{n+1}x - y\|^2 + \|T^n x - Ty\|^2.$$

So, we have

$$3\mu_n\|T^n x - Ty\|^2 \leq 2\mu_n\|T^n x - y\|^2 + \mu_n\|T^n x - Ty\|^2$$

and hence

$$\mu_n\|T^n x - Ty\|^2 \leq \mu_n\|T^n x - y\|^2.$$

By Theorem 4.2, T has a fixed point in C . □

We also know the following theorem by Ray [15].

Theorem 4.8. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Then, the following are equivalent:*

- (i) *Every nonexpansive mapping of C into itself has a fixed point in C ;*
- (ii) *C is bounded.*

Using Ray's theorem, we can prove the following theorem [29].

Theorem 4.9. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Then, the following are equivalent:*

- (i) *Every hybrid mapping of C into itself has a fixed point in C ;*
- (ii) *C is bounded.*

Proof. From Theorem 4.5, we know that (ii) implies (i). Let us show that (i) implies (ii). We know that every firmly nonexpansive mapping is a hybrid mapping. So, the class of hybrid mappings of C into itself contains the class of firmly nonexpansive mappings of C into itself. To show (i) $\implies$ (ii), it is sufficient to show that if every firmly nonexpansive mapping in C into itself has a fixed point in C , then every nonexpansive mapping of C into itself has a fixed point in C . Let T be a nonexpansive mapping of C into itself. Then, $S = \frac{1}{2}I + \frac{1}{2}T$ is a firmly nonexpansive mapping. Further, it is not difficult to show $F(T) = F(S)$. So, every firmly nonexpansive mapping in C into itself has a fixed point in C if and only if every nonexpansive mapping of C into itself has a fixed point in C . This completes the proof. □

Similarly, we have the following theorem.

Theorem 4.10. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Then, the following are equivalent:*

- (i) *Every nonspreading mapping of C into itself has a fixed point in C ;*
- (ii) *C is bounded.*

5 Nonlinear Erdodic Theorems

Baillon [1] proved the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space.

Theorem 5.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

We can also prove the following nonlinear ergodic theorem [31] for our nonlinear operators in a Hilbert space. Before proving it, we need, for example, the following result [31].

Lemma 5.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be a mapping such that*

$$2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$$

Then T is demiclosed, i.e., $x_n \rightarrow u$ and $x_n - Tx_n \rightarrow 0$ imply $u \in F(T)$.

Proof. Let $\{x_n\} \subset C$ be a sequence such that $x_n \rightarrow u$ and $x_n - Tx_n \rightarrow 0$ as $n \rightarrow \infty$. Then the sequences $\{x_n\}$ and $\{Tx_n\}$ are bounded. Suppose that $u \neq Tu$. From Opial's theorem [14], we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \|x_n - u\|^2 &< \liminf_{n \rightarrow \infty} \|x_n - Tu\|^2 \\ &= \liminf_{n \rightarrow \infty} \|x_n - Tx_n + Tx_n - Tu\|^2 \\ &= \liminf_{n \rightarrow \infty} \|Tx_n - Tu\|^2 \\ &\leq \liminf_{n \rightarrow \infty} \frac{1}{2} (\|x_n - u\|^2 + \|Tx_n - u\|^2) \\ &= \liminf_{n \rightarrow \infty} \frac{1}{2} (\|x_n - u\|^2 + \|Tx_n - x_n + x_n - u\|^2) \\ &= \liminf_{n \rightarrow \infty} \|x_n - u\|^2. \end{aligned}$$

This is a contradiction. Hence we get the conclusion. □

Theorem 5.3. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself such that $F(T)$ is nonempty. Suppose that T satisfies one of the following conditions:*

- (i) T is nonspreading;
- (ii) T is hybrid;
- (iii) $2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C;$
- (iv) $3\|Tx - Ty\|^2 \leq 2\|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$

Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

Proof. Let us prove the case of (iii) by using Lemma 5.2. We first show that $F(T)$ is closed and convex. It follows from Lemma 5.2 that $F(T)$ is closed. In fact, let $\{x_n\} \subset F(T)$ and $x_n \rightarrow z$. Then, we have $x_n \rightharpoonup z$ and $x_n - Tx_n = 0$. So, from Lemma 5.2 we have $z = Tz$. This implies that $F(T)$ is closed. Let us show that $F(T)$ is convex. Let $x, y \in F(T)$ and let $\alpha \in [0, 1]$. Put $z = \alpha x + (1 - \alpha)y$. Then, we have from (2.1) that

$$\begin{aligned} \|z - Tz\|^2 &= \|\alpha x + (1 - \alpha)y - Tz\|^2 \\ &= \alpha \|x - Tz\|^2 + (1 - \alpha) \|y - Tz\|^2 - \alpha(1 - \alpha) \|x - y\|^2 \\ &= \alpha \|Tx - Tz\|^2 + (1 - \alpha) \|Ty - Tz\|^2 - \alpha(1 - \alpha) \|x - y\|^2 \\ &\leq \alpha(1 - \alpha)^2 \|x - y\|^2 + (1 - \alpha)\alpha^2 \|x - y\|^2 - \alpha(1 - \alpha) \|x - y\|^2 \\ &= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1) \|x - y\|^2 \\ &= 0. \end{aligned}$$

So, we have $Tz = z$. This implies that $F(T)$ is convex. Let $x \in C$ and let P be the metric projection of H onto $F(T)$. Then, we have

$$\begin{aligned} \|PT^n x - T^n x\| &\leq \|PT^{n-1} x - T^n x\| \\ &= \|TPT^{n-1} x - T^n x\| \\ &\leq \|PT^{n-1} x - T^{n-1} x\|. \end{aligned}$$

This implies that $\{\|PT^n x - T^n x\|\}$ is nonincreasing. We also know that for any $v \in C$ and $u \in F(T)$,

$$\langle v - Pv, Pv - u \rangle \geq 0$$

and hence

$$\|v - Pv\|^2 \leq \langle v - Pv, v - u \rangle.$$

So, we get

$$\begin{aligned} \|Pv - u\|^2 &= \|Pv - v + v - u\|^2 \\ &= \|Pv - v\|^2 - 2\langle Pv - v, u - v \rangle + \|v - u\|^2 \\ &\leq \|v - u\|^2 - \|Pv - v\|^2. \end{aligned}$$

Let $m, n \in \mathbb{N}$ with $m \geq n$. Putting $v = T^m x$ and $u = PT^n x$, we have

$$\begin{aligned} \|PT^m x - PT^n x\|^2 &\leq \|T^m x - PT^n x\|^2 - \|PT^m x - T^m x\|^2 \\ &\leq \|T^n x - PT^n x\|^2 - \|PT^m x - T^m x\|^2. \end{aligned}$$

So, $\{PT^n x\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{PT^n x\}$ converges strongly to an element p of $F(T)$. Take $u \in F(T)$. Then we obtain that for any $n \in \mathbb{N}$,

$$\|S_n x - u\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k x - u\| \leq \|x - u\|.$$

So, $\{S_n x\}$ is bounded and hence there exists a weakly convergent subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$. If $S_{n_i} x \rightharpoonup v$, then we have $v \in F(T)$. In fact, for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$, we have that

$$\begin{aligned} 2\|T^{k+1}x - Ty\|^2 &\leq \|T^k x - y\|^2 + \|T^{k+1}x - y\|^2 \\ &= \|T^k x - Ty\|^2 + 2\langle T^k x - Ty, Ty - y \rangle + \|Ty - y\|^2 \\ &\quad + \|T^{k+1}x - Ty\|^2 + 2\langle T^{k+1}x - Ty, Ty - y \rangle + \|Ty - y\|^2. \end{aligned}$$

So, we obtain that

$$\begin{aligned} \|T^{k+1}x - Ty\|^2 &\leq \|T^k x - Ty\|^2 + 2\langle T^k x - Ty, Ty - y \rangle \\ &\quad + 2\langle T^{k+1}x - Ty, Ty - y \rangle + 2\|Ty - y\|^2. \end{aligned}$$

Summing these inequalities with respect to $k = 0, 1, \dots, n-1$, we have

$$\begin{aligned} \|T^n x - Ty\|^2 &\leq \|x - Ty\|^2 + 2\left\langle \sum_{k=0}^{n-1} T^k x - nTy, Ty - y \right\rangle \\ &\quad + 2\left\langle \sum_{k=0}^{n-1} T^{k+1}x - nTy, Ty - y \right\rangle + 2n\|Ty - y\|^2 \\ &= \|x - Ty\|^2 + 4\left\langle \sum_{k=0}^{n-1} T^k x - nTy, Ty - y \right\rangle \\ &\quad + 2\langle T^n x - x, Ty - y \rangle + 2n\|Ty - y\|^2. \end{aligned}$$

Deviding this inequality by n , we have

$$\begin{aligned} \frac{1}{n}\|T^n x - Ty\|^2 &\leq \frac{1}{n}\|x - Ty\|^2 + 4\langle S_n x - Ty, Ty - y \rangle \\ &\quad + \frac{2}{n}\langle T^n x - x, Ty - y \rangle + 2\|Ty - y\|^2, \end{aligned}$$

where $S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$. Replacing n by n_i and letting $n_i \rightarrow \infty$, we obtain from $S_{n_i} x \rightharpoonup v$ that

$$0 \leq 2\|Ty - y\|^2 + 4\langle v - Ty, Ty - y \rangle.$$

Putting $y = v$, we have $0 \leq 2\|Tv - v\|^2 + 4\langle v - Tv, Tv - v \rangle$ and hence

$$0 \leq \|Tv - v\|^2 + 2\langle v - Tv, Tv - v \rangle.$$

So, we have $0 \leq -\|Tv - v\|^2$ and hence $Tv = v$. To complete the proof of (iii), it is sufficient to show that if $S_{n_i} x \rightharpoonup v$ and $p = \lim_{n \rightarrow \infty} PT^n x$, then $v = p$. We have that for any $u \in F(T)$,

$$\langle T^k x - PT^k x, PT^k x - u \rangle \geq 0.$$

Since $\{\|T^k x - PT^k x\|\}$ is nonincreasing, we have

$$\begin{aligned} \langle u - p, T^k x - PT^k x \rangle &\leq \langle PT^k x - p, T^k x - PT^k x \rangle \\ &\leq \|PT^k x - p\| \cdot \|T^k x - PT^k x\| \\ &\leq \|PT^k x - p\| \cdot \|x - Px\|. \end{aligned}$$

Adding these inequalities from $k = 0$ to $k = n - 1$ and dividing n , we have

$$\langle u - p, S_n x - \frac{1}{n} \sum_{k=0}^{n-1} PT^k x \rangle \leq \frac{\|x - Px\|}{n} \sum_{k=0}^{n-1} \|PT^k x - p\|.$$

Since $S_n x \rightarrow v$ and $PT^k x \rightarrow p$, we have

$$\langle u - p, v - p \rangle \leq 0.$$

We know $v \in F(T)$. So, putting $u = v$, we have $\langle v - p, v - p \rangle \leq 0$ and hence $\|v - p\|^2 \leq 0$. So, we obtain $v = p$. This completes the proof of (iii). See [31] for the proofs of (1), (ii) and (iv). $\square$

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