

Convergence of space-time occupation measures of stochastic processes and its application to collisions

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Abstract

We introduce a new perspective on positive continuous additive functionals (PCAFs) of Markov processes, which we call *space-time occupation measures (STOMs)*. This notion provides a natural generalization of classical occupation times and occupation measures, and offers a unified framework for studying their convergence. We analyze STOMs via so-called smooth measures associated with PCAFs through the Revuz correspondence. We establish that if the underlying spaces, the processes living on them, their heat kernels, and the associated smooth measures converge, and if the corresponding potentials of these measures satisfy a uniform decay condition, then the associated PCAFs and STOMs also converge in suitable Gromov–Hausdorff-type topologies.

We then apply this framework to the analysis of collisions of independent stochastic processes. Specifically, by exploiting the STOM formulation, we introduce the notion of *collision measures*, which record both the collision sites and times of two processes, and prove general convergence theorems for these measures. The abstract results are further specialized to random walks on electrical networks via the theory of resistance metric spaces, leading to concrete scaling limits for collision measures of random walks on critical random graphs, such as critical Galton–Watson trees, critical Erdős–Rényi random graphs, and the uniform spanning tree.

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1 Introduction

Positive continuous additive functionals (PCAFs) play a crucial role in the analysis of Markov processes, appearing in various contexts, such as the Feynman–Kac formula, time-changed processes, and reflected processes. This paper is concerned with convergence of PCAFs, which has important applications in a wide range of problems.

- (i) Convergence of PCAFs directly implies convergence of the associated Feynman–Kac semi-groups (cf. [3]). As an application in the PDE context, Fan [30] constructed a discrete approximation for solutions of the heat equation with general Robin boundary conditions, by proving the convergence of the boundary local times of random walks in a certain Euclidean domain to those of the reflected diffusion.
- (ii) Convergence of PCAFs is a crucial step in deriving convergence of corresponding time-changed processes, such as Liouville Brownian motion and the Bouchaud trap model [28, 33, 66].
- (iii) Convergence of local times can be used to deduce convergence of cover times of random walks [7, 25].

A fundamental result in the theory of PCAFs is the Revuz correspondence [72], which establishes a one-to-one correspondence between PCAFs and so-called smooth measures. Recent studies [60, 65, 67] have shown that this correspondence is continuous, in the sense that if smooth measures converge, then the associated PCAFs also converge, with the mode of convergence specified suitably. These works, however, have focused on convergence of PCAFs of a fixed Markov process. In this paper, we establish convergence results for PCAFs of Markov processes living on spaces that vary along a sequence, where the convergence of the underlying spaces is treated within the framework of Gromov–Hausdorff-type topologies (see Section 3 below). This extension of the framework makes the theory applicable to a wider range of settings, such as discrete approximations of spaces and scaling limits of random graphs.

In this paper, we also introduce a new perspective on PCAFs, viewing them as random measures on space and time. Let $X = (X_t)_{t \geq 0}$ be a standard process on a suitable topological space S (see [18, 22] for the definition of standard processes, for example). For a PCAF $A = (A_t)_{t \geq 0}$ of X (see Definition 4.12 below), we define a random measure on $S \times [0, \infty)$ by

$$\Pi(\cdot) := \int_0^\infty \mathbf{1}_{\cdot}(X_t, t) dA_t, \quad (1.1)$$

where $\mathbf{1}_{\cdot}$ denotes the indicator function. We refer to Π as the *space-time occupation measure (STOM)* associated with A . This notion extends the classical occupation time and occupation measure in the following sense. If one takes $A_t = t$, $t \geq 0$, then, for any Borel set $E \subseteq S$ and $t \geq 0$,

$$\Pi(E \times [0, t]) = \int_0^t \mathbf{1}_E(X_s) ds.$$

Thus, fixing E and varying t recovers the occupation time, while fixing t and varying E recovers the occupation measure.

The introduction of STOMs is not merely a reformulation of PCAFs. It provides a new framework that we apply to the study of collisions of stochastic processes. As a particular class of STOMs, we introduce the notion of *collision measures* for two processes, which simultaneously record both the locations and the times of collisions. We establish convergence results for these measures, and present several concrete examples to illustrate their applications.

The contributions of this paper can be summarized as follows.

- (i) We prove that if processes, heat kernels, and smooth measures converge, then, under suitable assumptions on the latter two, the associated PCAFs and STOMs also converge.
- (ii) We introduce the STOM framework as a new tool for the study of collisions of stochastic processes. Within this framework, we establish convergence results for collision measures and apply them to several concrete examples.

In the following subsections, we elaborate on these contributions in more detail.

1.1 An overview of the framework

Here we present an overview of the framework underlying our main results, mainly focusing on the topologies used to discuss convergence of PCAFs and STOMs.

We first recall some notion concerning PCAFs and smooth measures, which play a central role in this work. For details, see Section 4.1. Let (S, d_S) be a boundedly-compact metric space (a *bcm space*, for short), that is, every bounded closed subset of S is compact. Let

$$X = ((X_t)_{t \geq 0}, (P^x)_{x \in S})$$

be a conservative standard process on S satisfying the duality and absolute continuity condition (Assumption 4.1). In particular, we assume that it admits a transition density (heat kernel) $p: (0, \infty) \times S \times S \rightarrow [0, \infty]$ with respect to a certain invariant reference measure of X . Let μ be a Radon measure on S , i.e., a Borel measure such that its restriction $\mu|_K$ to every compact subset K is finite. We define its α -potential by

$$\mathcal{R}_p^\alpha \mu(x) := \int_S r^\alpha(x, y) \mu(dy), \quad x \in S,$$

where r^α denotes the α -resolvent density associated with p , namely,

$$r^\alpha(x, y) := \int_0^\infty e^{-\alpha t} p(t, x, y) dt.$$

The measure μ is said to belong to the *local Kato class* if it does not charge any semipolar set (see Definition 4.3 below) and, for every compact subset $K \subseteq S$,

$$\|\mathcal{R}_p^\alpha(\mu|_K)\|_\infty = \sup_{x \in S} \int_K r^\alpha(x, y) \mu(dy) \xrightarrow{\alpha \rightarrow \infty} 0.$$

If μ is in the local Kato class, then it is a smooth measure of X , and by the Revuz correspondence there exists a unique PCAF $A = (A_t)_{t \geq 0}$ of X satisfying, for all non-negative Borel measurable f on S and all $\alpha > 0$,

$$E^x \left[\int_0^\infty e^{-\alpha t} f(X_t) dA_t \right] = \int_S r^\alpha(x, y) f(y) \mu(dy).$$

The associated STOM is defined by (1.1).

Spaces for PCAFs and STOMs. We next introduce the spaces used to describe convergence of PCAFs and STOMs. We write $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ for the space of non-decreasing continuous functions from $\mathbb{R}_{\geq 0}$ to itself, equipped with the compact-convergence topology, i.e., uniform convergence on compacts. The conservativeness of X ensures that each PCAF is finite at every finite time, so that convergence of PCAFs can be discussed within $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$. For the same reason, each STOM is a Radon measure on $S \times \mathbb{R}_{\geq 0}$, and one may consider convergence of STOMs under the vague topology. However, the vague topology is, in general, too weak to capture convergence of the associated PCAFs. Indeed, if Π is the STOM associated with a PCAF A , then

$$\Pi(S \times [0, t]) = A_t, \quad t \geq 0.$$

Thus, even if $\Pi_n \rightarrow \Pi$ vaguely, the above relation does not guarantee the convergence of the corresponding PCAFs when S is non-compact. To overcome this difficulty, we introduce a space $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$ containing all STOMs, and equip it with a Polish topology such that convergence in this space implies convergence of the associated PCAFs in $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$. (See Section 2.5 for details.) In the remainder of this introduction, we mainly focus on convergence of STOMs, as convergence of PCAFs in this setting follows from that of the corresponding STOMs. (Nevertheless, both PCAFs and STOMs will be treated in parallel in the main sections.)

Remark 1.1. We make a few comments on the conservativeness assumption.

- (a) Without conservativeness, a PCAF may blow up in finite time, in which case the space $C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ is no longer suitable for treating PCAFs. By instead considering the associated STOMs, one can still discuss convergence in a natural topology without assuming conservativeness. This demonstrates the generality of the STOM framework. See Section 5 (in particular, Remark 5.13) for further details.
- (b) Conservativeness is also, in a sense, a natural assumption when discussing convergence of stochastic processes in a general setting. In the presence of explosion, usual topologies such as the J_1 -Skorohod topology are overly sensitive to the behavior of paths near explosion times, and one must carefully choose an appropriate topology. See [35] for recent developments in this direction.

Law maps and their continuity. We write $D(\mathbb{R}_{\geq 0}, S)$ for the space of S -valued càdlàg functions on $\mathbb{R}_{\geq 0}$, equipped with the usual J_1 -Skorohod topology. Fix a Radon measure μ in the local Kato class, and denote the associated STOM by Π . For each $x \in S$, set

$$\mathcal{L}_X(x) := P^x(X \in \cdot), \quad \mathcal{L}_{(X, \Pi)}(x) := P^x((X, \Pi) \in \cdot),$$

which are probability measures on $D(\mathbb{R}_{\geq 0}, S)$ and on $D(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$, respectively. We regard these as maps from S , and refer to them as *law maps*.

For a wide class of processes (for example, Feller processes), one can verify that the distribution of X depends continuously on the starting point, that is, the map

$$\mathcal{L}_X: S \rightarrow \mathcal{P}(D(\mathbb{R}_{\geq 0}, S))$$

is continuous, where $\mathcal{P}(D(\mathbb{R}_{\geq 0}, S))$ denotes the space of probability measures on $D(\mathbb{R}_{\geq 0}, S)$ endowed with the weak topology. In such cases, Theorem 5.30 below shows that $\mathcal{L}_{(X, \Pi)}$ inherits the continuity of $\mathcal{L}_X$ under an additional joint continuity assumption on the heat kernel. This result serves as a cornerstone for our main theorems, as convergence of PCAFs and STOMs will be formulated in terms of convergence of the corresponding law maps. (See Remark 1.2 below for an explanation of this.) The proof is based on an approximation argument, which also forms the essential backbone of the proofs of our main results. A brief outline of this approximation method is given in the next subsection.

Remark 1.2. Distributional convergence of Markov processes is usually formulated for a fixed starting point (or a fixed sequence of starting points). However, in many cases the proof implicitly yields convergence whenever the starting points themselves converge. Indeed, a standard argument based on tightness and convergence of semigroups typically suffices. This mode of convergence is equivalent to compact convergence of their law maps, provided that the maps are continuous. Although this viewpoint may seem less conventional, it is particularly natural in the Gromov–Hausdorff-type setting, where processes are defined on different spaces and there may be no canonical fixed starting point. Accordingly, throughout this paper, convergence of processes and of their associated PCAFs/STOMs is formulated as convergence of the corresponding law maps.

1.2 Main results

Here we present the main results of this paper together with a sketch of their proofs.

For each $n \geq 1$, let (S_n, d_{S_n}) be a bcm space equipped with a distinguished element ρ_n , called the *root* of S_n . Let X_n be a conservative standard process on S_n satisfying the duality and absolute continuity condition (Assumption 4.1), and assume that X_n has a jointly continuous heat kernel p_n . We assume that the law map $\mathcal{L}_{X_n}$ is continuous. Fix a Radon measure μ_n on

S_n belonging to the local Kato class of X_n , and let Π_n be the associated STOM. As discussed in the previous subsection, the joint law map $\mathcal{L}_{(X_n, \Pi_n)}$ is also continuous.

To describe the limit objects, let $(S, d_S, \rho, X, p, \mu, \Pi)$ be a tuple satisfying the same conditions as $(S_n, d_{S_n}, \rho_n, X_n, p_n, \mu_n, \Pi_n)$. In particular, (S, d_S, ρ) is a rooted bcm space, X is a conservative standard process satisfying the same assumptions as X_n , p is its jointly continuous heat kernel, μ is a Radon measure in the local Kato class of X , and Π is the corresponding STOM.

The following theorem is one of the main results of this paper. To discuss convergence of metric spaces equipped with additional structures, such as measures and laws of Markov processes, we use Gromov–Hausdorff-type spaces $\mathfrak{M}_\bullet(\tau)$. Here, τ specifies both which structures are attached and how convergence of those structures is defined. For example, $\tau_{\mathcal{M}}$ treats Radon measures endowed with the vague topology, τ_{HK} treats heat kernels with the compact-convergence topology, and τ_{SP} treats law maps of Markov processes with a suitable topology induced by the J_1 -Skorohod topology. The notation $\tau_{\text{SP,STOM}}$ extends τ_{SP} so as to incorporate STOMs simultaneously, using the topology introduced in Section 1.1. See Section 3 below for details. For each $r > 0$ and $n \geq 1$, we write

$$S_n^{(r)} := \{x \in S_n \mid d_{S_n}(\rho_n, x) \leq r\},$$

and $\mu_n^{(r)}$ for the restriction of μ_n to $S_n^{(r)}$.

Theorem 1.3. *Assume that the following conditions are satisfied.*

(i) *It holds that*

$$(S_n, d_{S_n}, \rho_n, \mu_n, p_n, \mathcal{L}_{X_n}) \rightarrow (S, d_S, \rho, \mu, p, \mathcal{L}_X)$$

in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau_{\text{HK}} \times \tau_{\text{SP}})$.

(ii) *For each $r > 0$,*

$$\lim_{\alpha \rightarrow \infty} \limsup_{n \rightarrow \infty} \|\mathcal{R}_{p_n}^\alpha \mu_n^{(r)}\|_\infty = 0,$$

or equivalently,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{x \in S_n^{(r)}} \int_0^\delta \int_{S_n^{(r)}} p_n(t, x, y) \mu_n(dy) dt = 0.$$

Then

$$(S_n, d_{S_n}, \rho_n, \mu_n, p_n, \mathcal{L}_{(X_n, \Pi_n)}) \rightarrow (S, d_S, \rho, \mu, p, \mathcal{L}_{(X, \Pi)})$$

in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau_{\text{HK}} \times \tau_{\text{SP,STOM}})$.

We also obtain a corresponding result for random environments. That is, assume that the tuples $\mathcal{S}_n = (S_n, d_{S_n}, \rho_n, \mu_n)$ and $\mathcal{S} = (S, d_S, \rho, \mu)$ are random rooted-and-measured bcm spaces. For each realization of $\mathcal{S}_n$, we consider a process X_n with jointly continuous heat kernel p_n , satisfying the same conditions as in the deterministic case, a Radon measure μ_n in the local Kato class of X_n , and the associated STOM Π_n . Similarly, for each realization of $\mathcal{S}$, we have a process X with jointly continuous heat kernel p , a Radon measure μ in the local Kato class of X , and the associated STOM Π . (We also assume suitable measurability of these objects, so that we can discuss their laws in Gromov–Hausdorff-type spaces; see Section 6.2 for the precise setting.) Under a random analogue of the above assumptions, we establish distributional convergence of the law maps $\mathcal{L}_{(X_n, \Pi_n)}$. Below, $\mathbf{E}_n$ denotes the expectation with respect to the randomness of the environment, and $\xrightarrow{d}$ means convergence in distribution of random elements.

Theorem 1.4. *Assume that the following conditions are satisfied.*

(i) *It holds that*

$$(S_n, d_{S_n}, \rho_n, \mu_n, p_n, \mathcal{L}_{X_n}) \xrightarrow{d} (S, d_S, \rho, \mu, p, \mathcal{L}_X)$$

as random elements of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau_{\text{HK}} \times \tau_{\text{SP}})$.

(ii) For each $r > 0$,

$$\lim_{\alpha \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[\|\mathcal{R}_{p_n}^\alpha \mu_n^{(r)}\|_\infty \wedge 1 \right] = 0,$$

or equivalently,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[\left(\sup_{x \in S_n^{(r)}} \int_0^\delta \int_{S_n^{(r)}} p_n(t, x, y) \mu_n(dy) dt \right) \wedge 1 \right] = 0.$$

Then

$$(S_n, d_{S_n}, \rho_n, \mu_n, p_n, \mathcal{L}_{(X_n, \Pi_n)}) \xrightarrow{d} (S, d_S, \rho, \mu, p, \mathcal{L}_{(X, \Pi)})$$

as random elements of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau_{\text{HK}} \times \tau_{\text{SP, STOM}})$.

The above theorems follow from Theorems 6.6 and 6.14 (see also Remark 6.7), respectively, which are formulated in a slightly more general setting where the convergence of processes is assumed in a weaker topology. See Remark 1.5 for further discussions on this point.

Sketch of the proofs. The main proof idea is to approximate STOMs using heat kernels as mollifiers. For each $\delta > 0$, set

$$\Pi_n^{(\delta, *)}(dz dt) := \frac{1}{\delta} \int_\delta^{2\delta} p_n(u, X_n(t), z) du \mu_n(dz) dt.$$

From the convergence of μ_n , p_n , and $\mathcal{L}_{X_n}$, we deduce convergence of the law maps $\mathcal{L}_{(X_n, \Pi_n^{(\delta, *)})}$. Using moment estimates for differences of PCAFs established in [65], we bound the expected distance between $\Pi_n^{(\delta, *)}$ and Π_n by the potential norm $\|\mathcal{R}_{p_n}^\alpha \mu_n^{(r)}\|_\infty$. Thus, applying condition (ii) in Theorem 1.3 (or Theorem 1.4), we obtain that $\mathcal{L}_{(X_n, \Pi_n^{(\delta, *)})} \rightarrow \mathcal{L}_{(X_n, \Pi_n)}$ as $\delta \rightarrow 0$ uniformly in n , which yields the desired convergence.

Remark 1.5. Our fundamental assumption concerns the convergence of processes. To impose the weakest possible requirement, we employ the L^0 topology on $D(\mathbb{R}_{\geq 0}, S)$, induced by the metric

$$d_S^{L^0}(f, g) := \int_0^\infty e^{-t} (1 \wedge d_S(f_t, g_t)) dt, \quad f, g \in D(\mathbb{R}_{\geq 0}, S)$$

(see Section 2.4 for details of this topology). This topology is substantially weaker than the compact convergence, J_1 -Skorohod, or M_1 topologies, thereby broadening the scope of the framework. Accordingly, in our main results, Theorems 6.6 and 6.14, the convergence of the joint laws of processes and STOMs is formulated so that the process components converge with respect to the L^0 topology. Nevertheless, when convergence of processes holds under a stronger Polish topology, the same conclusions automatically extend to that topology by a simple tightness argument (see Appendix C). Thus the use of the L^0 topology does not restrict generality, while keeping the framework maximally flexible. Indeed, this level of generality is particularly important for applications to the study of collision, see Remark 7.14.

1.3 Application to the study of collisions

Let $(X_t^1)_{t \geq 0}$ and $(X_t^2)_{t \geq 0}$ be two (independent) stochastic processes taking values in a common state space S . Given a realization of such processes, we say that $t \in [0, \infty)$ is a *collision time* if $X_t^1 = X_t^2 = x$ for some $x \in S$, in which case we refer to x as the *collision site* (at time t). Based on the framework of STOMs, we introduce the notion of *collision measures* for a broad class of pairs of stochastic processes. These measures capture collision times and sites simultaneously, allowing us to treat collisions in a unified manner. By applying the main convergence theorems for STOMs, we obtain convergence results for the corresponding collision measures.

Background. The study of collisions of independent stochastic processes can be traced back to Pólya's classical work [70] on the recurrence of simple random walks (SRWs) on Euclidean lattices. On such lattices, recurrence is equivalent to the *infinite collision property (ICP)*, namely the property that two independent SRWs collide infinitely many times with probability one. However, this equivalence fails on general graphs [52], and the ICP has been extensively studied for various (random) graph models (cf. [13, 26, 37, 39, 77]). In the continuous setting, most studies have focused on the Hausdorff dimension of the set of collision times or collision sites (cf. [40, 51, 74]).

The scaling limit of collision times and sites has, to the best of our knowledge, barely been investigated in the existing literature. Even in a very simple case, the one-dimensional lattice $\mathbb{Z}$, no explicit results were known until the recent work by Nguyen [59]. He showed that the point process consisting of pairs of collision times and sites of two SRWs on $\mathbb{Z}$ converges, under Brownian scaling, to a non-trivial random measure. However, his approach essentially relies on the scaling limit of the partition function for a directed polymer model and thus does not extend easily to more general settings. Moreover, the limiting measure was characterized only via moment identities, leaving its pathwise connection with the collisions of the limiting Brownian motions unclear.

In contrast, our formulation provides a direct, pathwise definition of collision measures for a wide class of stochastic processes, together with a robust convergence theorem derived from the STOM framework.

Collision measure. Here we define collision measures. See Section 7.1 for details. Let S be a bcm space. For each $i \in \{1, 2\}$, let X^i be a conservative standard process on S satisfying the duality and absolute continuity condition (Assumption 4.1) with heat kernel p^i and reference measure m^i . We define the associated *product process* $\hat{X} = ((\hat{X}_t)_{t \geq 0}, (\hat{P}^{\mathbf{x}})_{\mathbf{x} \in S \times S})$ by

$$\begin{aligned}\hat{X}_t &:= (X_t^1, X_t^2), \quad t \geq 0, \\ \hat{P}^{\mathbf{x}} &:= P_1^{x_1} \otimes P_2^{x_2}, \quad \mathbf{x} = (x_1, x_2) \in S \times S.\end{aligned}$$

Then $\hat{X}$ is again a conservative standard process satisfying the duality and absolute continuity condition. In particular, the STOM framework applies to this product process.

Define the *diagonal map* $\text{diag}: S \rightarrow S \times S$ by $\text{diag}(x) := (x, x)$. The collision of X^1 and X^2 can be rephrased as the event that $\hat{X}$ hits the diagonal (cf. [74]). Specifically, for any $t \geq 0$ and any subset $B \subseteq S$,

$$X_t^1 = X_t^2 \in B \iff \hat{X}_t \in \text{diag}(B).$$

This observation naturally leads to the following definition of collision measures.

Fix a Radon measure μ on S . We write $\mu^{\text{diag}} := \mu \circ \text{diag}^{-1}$ for the pushforward of μ by diag . Assume that μ^{diag} is a smooth measure of $\hat{X}$, and let $\hat{\Pi}$ denote the associated STOM of $\hat{X}$. We then define a random measure Π on $S \times [0, \infty)$ by

$$\Pi := \hat{\Pi} \circ (\text{diag} \times \text{id}_{\mathbb{R}_{\geq 0}}), \tag{1.2}$$

where $\text{id}_{\mathbb{R}_{\geq 0}}$ denotes the identity map on $\mathbb{R}_{\geq 0} = [0, \infty)$. We refer to Π as the *collision measure* of X^1 and X^2 associated with μ . The measure $\mu(dx)$ plays the role of specifying the contribution of collisions at each site x ; we therefore refer to μ as the *weighting measure* (see (1.3)). In Remarks 7.4 and 7.5 below, we verify that the limiting random measure obtained by Nguyen [59] can be interpreted as a collision measure within our framework.

For example, when S is discrete,

$$\Pi(dx dt) = \frac{1}{m^1(\{x\}) m^2(\{x\})} \mathbf{1}_{\{X_t^1 = X_t^2 = x\}} \mu(dx) dt, \tag{1.3}$$

(see Proposition 7.8). This expression suggests that the natural choice of weighting measure μ is the pointwise product of the reference measures, that is,

$$\mu(\{x\}) = (m^1 \cdot m^2)(\{x\}) := m^1(\{x\})m^2(\{x\}), \quad x \in S. \quad (1.4)$$

Indeed, the collision measure Π associated with this weighting measure assigns a unit contribution to each collision at every site. Namely, we have

$$\Pi(dx dt) = \sum_{y \in S} \mathbf{1}_{\{X_t^1 = X_t^2 = y\}} \delta_y(dx) dt, \quad (1.5)$$

where δ_y denotes the Dirac measure at y . In particular, for each $x \in S$ and $t \geq 0$,

$$\Pi(\{x\} \times [0, t]) = \int_0^t \mathbf{1}_{\{X_s^1 = X_s^2 = x\}} ds,$$

which represents the total amount of time up to t during which X^1 and X^2 collide at x . We refer to Π in (1.5) as the *uniform collision measure*, and refer to the associated weighting measure $\mu = m^1 \cdot m^2$ as the *canonical weighting measure*.

Convergence of collision measures. Although the definition (1.2) is not in the direct form to which the continuous mapping theorem applies, an inverse version of the continuous mapping theorem (see Appendix B) can be employed to handle this situation. In particular, the convergence of collision measures can be reduced to that of the associated STOMs. Hence we do not restate the full setup and theorems here. Instead, we provide a brief outline for clarity. See Theorems 7.13 and 7.16 for precise formulations.

For each $n \geq 1$, suppose that X_n^1 and X_n^2 are independent conservative standard processes living on a common rooted bcm space (S_n, d_{S_n}, ρ_n) with heat kernels p_n^1 and p_n^2 , respectively, and let μ_n be a weighting measure. Write Π_n for the collision measure of X_n^1 and X_n^2 associated with μ_n . By Theorem 1.3, we deduce the following: if the rooted metric spaces (S_n, d_{S_n}, ρ_n) , the weighting measures μ_n , the heat kernels p_n^1 and p_n^2 , and the law maps $\mathcal{L}_{X_n^1}$ and $\mathcal{L}_{X_n^2}$ converge in a certain Gromov–Hausdorff type topology, and if

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{x_1, x_2 \in S_n^{(r)}} \int_0^\delta \int_{S_n^{(r)}} p_n^1(t, x_1, y) p_n^2(t, x_2, y) \mu_n(dy) dt = 0, \quad \forall r > 0, \quad (1.6)$$

then the joint law maps $\mathcal{L}_{(X_n^1, X_n^2, \Pi_n)}$ converge to the limiting one. An analogous result also holds for random spaces, after suitable modification of the assumptions, as in the case of Theorem 1.4.

Integration with the framework of resistance metric spaces. Collisions of independent stochastic processes are phenomena that typically occur in spaces of low spectral dimension. The framework of *resistance metric spaces* due to Kigami [49, 50] is particularly useful for analyzing such low-dimensional behavior, where point recurrence typically holds. It was originally developed for analysis on fractals, and recent studies have revealed that this framework is also highly effective in describing scaling limits of random walks and related objects on critical random graphs [28, 24, 61, 63, 64].

The key advantage of working with measured resistance metric spaces lies in the fact that each such space naturally carries a symmetric Markov process through the theory of Dirichlet forms [32], and the associated process admits a jointly continuous heat kernel. Moreover, convergence of measured resistance metric spaces in Gromov–Hausdorff-type topologies implies convergence of the corresponding processes and their heat kernels, under the so-called non-explosion condition (see Section 8). Consequently, once the convergence of the underlying spaces is established, the convergence of the associated collision measures is reduced essentially to verifying two ingredients:

- (A) the convergence of weighting measures,
- (B) the heat-kernel condition (1.6).

The latter can be checked by heat kernel estimates based on volume estimates.

Collisions of variable-speed random walks (VSRWs). In Section 9.3, we develop general convergence theorems (Theorems 9.10 and 9.13) that yields, from scaling limits of electrical networks, scaling limits of uniform collision measures of two independent variable-speed random walks (VSRWs) living on those networks, where a VSRW refers to a continuous-time random walk whose jump rate between two vertices is proportional to the network conductance (see Section 8.1 below). A key point is that, although the canonical weighting measure is defined as the squared invariant measure (1.4), the invariant measure for VSRWs is the counting measure. Hence, squaring does not alter it, and condition (A) automatically follows from the convergence of the underlying measured spaces. This also simplifies condition (1.6) via the symmetry and the Chapman–Kolmogorov equation to the following:

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{x_1, x_2 \in S_n^{(r)}} \int_0^\delta p_n(2t, x_1, x_2) dt = 0, \quad \forall r > 0.$$

(Note that $p_n^1 = p_n^2$ in this setting, since we consider collisions of independent and identically distributed (i.i.d.) random walks.) In Theorems 9.10 and 9.13 below, we provide simple sufficient conditions, expressed in terms of lower volume estimates, to verify the above condition (and its random version). The results apply to a broad range of critical random graphs, including critical Galton–Watson trees, the uniform spanning tree on $\mathbb{Z}^d$ with $d = 2, 3$, and on high-dimensional tori, the critical Erdős–Rényi graph, and the critical configuration model (see Example 9.14).

Collisions of constant-speed random walks (CSRWs). The situation for *constant-speed random walks* (CSRWs) is markedly subtler, where a CSRW refers to a continuous-time random walk with a constant jump rate equal to 1. Here, the invariant measure is the conductance measure (which is reduced to the degree measure under unit conductances), so the canonical weighting measure becomes the *squared conductance measure*, whose convergence is genuinely nontrivial and must be analyzed on a model-by-model basis. Under the assumption that the squared conductance measures converge under the same scaling as the invariant measures, Theorems 9.16 and 9.20 below provide sufficient conditions, again in terms of lower volume estimates, to verify condition (B). In Section 9.4.1, we apply this result to critical Galton–Watson trees conditioned on size whose offspring distribution has a finite $(3 + \varepsilon)$ -moment for some $\varepsilon > 0$, establishing the convergence of the squared conductance measures and, consequently, the scaling limit of uniform collision measures of two independent CSRWs on those trees.

CSRWs are closely related to discrete-time random walks, since they share the same jump chain and hence are expected to exhibit identical scaling behavior. In this sense, scaling limits of CSRW collision measures naturally indicate the corresponding limits for discrete-time random walks, although making this connection rigorous requires additional analysis because collision measures involve occupation-time quantities (see [64] for a bridging argument transferring the CSRW result to its discrete-time counterpart in the case of local times).

Outlook: persistence of inhomogeneities and future directions. The study of squared invariant measures appears to be new, and it provides an intriguing perspective for future research. For instance, in a version of one-dimensional Bouchaud trap models [14, 20, 31] where the tail of the trap distribution lies in a regime with finite mean but infinite second moment, the invariant measure converges to the Lebesgue measure, whereas the squared measure converges to a limit that is singular with respect to the Lebesgue measure; namely, the invariant measure of the Fontes–Isopi–Newman diffusion [31]. This indicates that, while each individual walk homogenizes to Brownian motion, the collision sites may still reflect microscopic irregularities of the environment, accumulating near deep traps. However, making this observation rigorous

remains open. Indeed, our main results do not apply to this setting, since the limiting singular measure is not smooth for the product process of the limiting processes, namely, two-dimensional Brownian motion, and thus the associated collision measure cannot be defined. Similar behavior is also expected in other settings, such as critical Galton–Watson trees with heavy-tailed offspring distributions or random graphs with strongly inhomogeneous degree sequences.

Such a phenomenon—where the underlying processes lose their inhomogeneity in the limit but their collisions retain it—offers a new viewpoint on interactions in disordered systems. In this sense, collisions provide a novel observable that can detect microscopic irregularities even when homogenization occurs at the process level. These observations suggest rich possibilities for exploring how local inhomogeneities may persist through scaling limits in multi-particle systems.

1.4 Organization of the paper and notational conventions

The remainder of the paper is organized as follows. Section 2 summarizes the topologies and their metrizations that will be used throughout. Section 3 develops a general Gromov–Hausdorff-type framework for metric spaces equipped with additional structures, which serves as the basis for formulating the convergence results of the paper. Section 4 reviews the notion of dual processes, smooth measures, PCAFs, and STOMs, together with the Revuz correspondence linking them. Section 5 introduces approximation schemes for PCAFs and STOMs and establishes continuity of their joint laws with the underlying processes. Section 6 presents the main convergence theorems for PCAFs and STOMs on both deterministic and random spaces. Section 7 applies these results to define collision measures and to prove their convergence. Section 8 and Section 9 further specialize the general theory to stochastic processes on resistance metric spaces, culminating in concrete scaling limit results for collision measures of variable-speed and constant-speed random walks on critical random graphs. Finally, several auxiliary results are collected in the appendices, including an inverse version of the continuous mapping theorem.

Below we list the notational conventions used throughout this paper.

- (1) We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$, the sets of integers, non-negative integers, and positive integers, respectively.
- (2) We write $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{R}_{> 0} := (0, \infty)$, and equip each of these spaces with the usual Euclidean metric. The space $[0, \infty]$ is the usual one-point compactification of $[0, \infty)$.
- (3) For $a, b \in \mathbb{R} \cup \{\pm\infty\}$, we write $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.
- (4) For a non-empty set S and $f: S \rightarrow [-\infty, \infty]$, we write $\|f\|_\infty := \sup\{|f(x)| \mid x \in S\}$.
- (5) We denote by Leb the Lebesgue measure on $\mathbb{R}^d$, where the dimension is clear from the context.
- (6) For a set S and a subset $S_0 \subseteq S$, $\mathbf{1}_{S_0}$ denotes the indicator function of S_0 , i.e., $\mathbf{1}_{S_0}(x) = 1$ for $x \in S_0$, and $\mathbf{1}_{S_0}(x) = 0$ for $x \in S \setminus S_0$.
- (7) For a topological space S , we write $\mathcal{B}(S)$ for the set of Borel subsets of S .
- (8) When we say that S is a metric space, we always assume that S is nonempty and that its associated metric is denoted by d_S , unless otherwise specified.
- (9) Given a metric space S , we write

$$\begin{aligned} B_S(x, r) &= B_{d_S}(x, r) := \{y \in S \mid d_S(x, y) < r\}, \\ D_S(x, r) &= D_{d_S}(x, r) := \{y \in S \mid d_S(x, y) \leq r\}. \end{aligned}$$

- (10) Given a topological space S and a subset A of S , we denote by $\text{cl}(A) = \text{cl}_S(A)$ the closure of A in S .

- (11) Let S be a topological space. Given a function $f: S \rightarrow \mathbb{R}$, we write $\text{supp}(f)$ for the (closed) support of f , i.e.,

$$\text{supp}(f) := \text{cl}(\{x \in S \mid f(x) \neq 0\}).$$

Given a Borel measure μ on S , we write $\text{supp}(\mu)$ for the (topological) support of μ , i.e., the set of $x \in S$ such that any open neighborhood U of x satisfies $\mu(U) > 0$.

- (12) For a set A , we denote the identity map from A to itself by id_A .
- (13) Given maps $f_i: X_i \rightarrow Y_i$, $i = 1, 2$, we define $f_1 \times f_2: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ by setting $(f_1 \times f_2)(x_1, x_2) := (f_1(x_1), f_2(x_2))$.
- (14) When X and Y are topological spaces, we always equip $X \times Y$ with the product topology. Moreover, if X and Y are metric spaces, then we always equip $X \times Y$ with the *max product metric* defined as follows:

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) := d_X(x_1, x_2) \vee d_Y(y_1, y_2).$$

- (15) We say that a map $f: X \rightarrow Y$ between topological spaces is a *topological embedding* if and only if it is a homeomorphism onto its image with the relative topology.
- (16) We say that a map $f: X \rightarrow Y$ between metric spaces is an *isometric embedding* (resp. *isometry*) if and only if it is distance-preserving (resp. distance-preserving and bijective).
- (17) A metric space is said to be boundedly compact if every closed and bounded subset is compact. We refer to such a space as a *bcm space*.

2 Preliminaries on topologies and metrizations

This paper deals with various objects such as measures, stochastic processes, and heat kernels. The aim of this section is to summarize the topologies on these objects and their metrizations, which will be used throughout the paper. This section is based on [62, Section 3].

2.1 The Hausdorff and Fell topologies

In this subsection, we recall the Hausdorff and Fell topologies, which are standard topologies on the collections of compact and closed subsets, respectively. For detailed properties of these topologies as metric spaces, see [62, Section 2.1], and for further topological background see [56, Appendix C]. In this subsection, we fix a *bcm* space (S, d_S) .

Definition 2.1. We define $\mathcal{C}(S)$ to be the set of closed subsets of S . We denote by $\mathcal{C}_c(S)$ the subset of $\mathcal{C}(S)$ consisting of compact subsets. (NB. Both sets include the empty set.)

We equip $\mathcal{C}_c(S)$ with the Hausdorff metric d_S^H . To recall it, we write, for each subset $A \subseteq S$ and $\varepsilon \geq 0$,

$$A^{(\varepsilon)} := \{x \in S \mid \exists y \in A \text{ such that } d_S(x, y) \leq \varepsilon\},$$

which is the (*closed*) ε -neighborhood of A in S . The Hausdorff metric d_S^H on $\mathcal{C}_c(S)$ is then defined by

$$d_S^H(A, B) := \inf \left\{ \varepsilon \geq 0 \mid A \subseteq B^{(\varepsilon)}, B \subseteq A^{(\varepsilon)} \right\},$$

where we set the infimum over the empty set to be ∞ . The function d_S^H is indeed a complete and separable (extended) metric on $\mathcal{C}_c(S)$ (see [76, Section 17.6]). (Note that the distance between the empty set and a non-empty set is always infinite.) We call the topology on $\mathcal{C}_c(S)$ induced by d_S^H the *Hausdorff topology* (also known as the *Vietoris topology*).

We now fix a distinguished element $\rho \in S$, which we call the *root* of S . To extend d_S^H to a metric on $\mathcal{C}(S)$, we write, for each $A \in \mathcal{C}(S)$ and $r > 0$,

$$A|_\rho^{(r)} := A \cap D_S(\rho, r).$$

When the root is clear from the context, we write $A^{(r)} := A|_\rho^{(r)}$. We then define, for each $A, B \in \mathcal{C}(S)$,

$$d_{S,\rho}^F(A, B) := \int_0^\infty e^{-r} (1 \wedge d_S^H(A|_\rho^{(r)}, B|_\rho^{(r)})) dr.$$

The function $d_{S,\rho}^F$ is a metric on $\mathcal{C}(S)$ ([62, Theorem 3.8]) and induces the Fell topology (see [56] for the Fell topology, for example). In particular, the topology is compact and only depends on the topology on S .

2.2 The weak and vague topologies

We use the weak and vague topologies to discuss convergence of probability measures and Radon measures, respectively. In this subsection, we recall the definitions and metrizations of these topologies. For background, see [17] for the weak topology and [43] for the vague topology. We fix a separable metric space (S, d_S) .

Definition 2.2. We define $\mathcal{M}(S)$ to be the set of Radon measures μ on S , that is, μ is a Borel measure on S such that $\mu(K) < \infty$ for every compact subset K . We denote by $\mathcal{M}_{\text{fin}}(S)$ (resp. $\mathcal{P}(S)$) the subset of $\mathcal{M}(S)$ consisting of finite Borel measures (resp. probability measures).

A commonly used metric on $\mathcal{M}_{\text{fin}}(S)$ is the *Prohorov metric*, given as follows: for $\mu, \nu \in \mathcal{M}_{\text{fin}}(S)$,

$$d_S^P(\mu, \nu) := \inf \left\{ \varepsilon > 0 \mid \mu(A) \leq \nu(A^{(\varepsilon)}) + \varepsilon, \nu(A) \leq \mu(A^{(\varepsilon)}) + \varepsilon \text{ for all Borel subsets } A \subseteq S \right\}.$$

The Prohorov metric d_S^P induces the weak topology on $\mathcal{M}_{\text{fin}}(S)$, that is, finite Borel measures μ_n converge to μ in $\mathcal{M}_{\text{fin}}(S)$ if and only if, for all bounded continuous function $f: S \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \int_S f(x) \mu_n(dx) = \int_S f(x) \mu(dx).$$

We also equip $\mathcal{P}(S)$ with the Prohorov metric.

The following lemma is useful for estimating the Prohorov distance between the laws of random elements defined on a common probability space.

Lemma 2.3 ([63, Lemma 6.6]). *Let X and Y be random elements of S defined on a common probability space with probability measure P . Suppose that on an event E , we have $d_S(X, Y) \leq \varepsilon$ almost surely. Then it holds that*

$$d_S^P(P(X \in \cdot), P(Y \in \cdot)) \leq P(E^c) + \varepsilon.$$

We now assume that (S, d_S) is boundedly compact and fix an element $\rho \in S$ as the root of S . We extend the Prohorov metric to $\mathcal{M}(S)$ in a way analogous to the metric $d_{S,\rho}^F$ for the Fell topology. We write, for each $\mu \in \mathcal{M}(S)$ and $r > 0$,

$$\mu|_\rho^{(r)}(\cdot) := \mu(D_S(\rho, r) \cap \cdot). \tag{2.1}$$

When the root is clear from the context, we write $\mu^{(r)} := \mu|_\rho^{(r)}$. We then define, for each $\mu, \nu \in \mathcal{M}(S)$,

$$d_{S,\rho}^V(\mu, \nu) := \int_0^\infty e^{-r} (1 \wedge d_S^P(\mu|_\rho^{(r)}, \nu|_\rho^{(r)})) dr. \tag{2.2}$$

The function $d_{S,\rho}^V$ is a complete and separable metric on $\mathcal{M}(S)$ and induces the vague topology, that is, Radon measures μ_n converge to μ with respect to $d_{S,\rho}^V$ if and only if, for all compactly-supported continuous function $f: S \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \int_S f(x) \mu_n(dx) = \int_S f(x) \mu(dx)$$

(see [62, Theorems 3.9 and 3.10]).

We introduce the smooth truncation of measures, which will be used later. For each measure $\mu \in \mathcal{M}(S)$ and $R > 0$, we define

$$\tilde{\mu}^{(R)}(dx) := \int_{R-1}^R \mathbf{1}_{S(r)}(x) dr \mu(dx). \quad (2.3)$$

In general, the restriction of measures in the form of (2.1) may fail to be continuous with respect to the vague convergence at some radii. In contrast, the above truncation is smooth in the following sense.

Lemma 2.4. *The map $\mathcal{M}(S) \times (1, \infty) \ni (\mu, R) \mapsto \tilde{\mu}^{(R)} \in \mathcal{M}_{\text{fin}}(S)$ is continuous.*

Proof. For each $R > 1$, we set

$$\chi^{(R)}(x) := \int_{R-1}^R \mathbf{1}_{S(r)}(x) dr, \quad x \in S. \quad (2.4)$$

It is straightforward to check that each $\chi^{(R)}$ is continuous. Indeed, if $x_n \rightarrow x$ in S , then $\mathbf{1}_{S(r)}(x_n) \rightarrow \mathbf{1}_{S(r)}(x)$ for all $r > 0$ except possibly for $r = d_S(\rho, x)$. Moreover, we readily verify that

$$\sup_{x \in S} |\chi^{(R)}(x) - \chi^{(R')}(x)| \leq 2|R - R'|.$$

Suppose that a sequence $(\mu_n, R_n)_{n \geq 1}$ converges to an element (μ, R) in $\mathcal{M}(S) \times (1, \infty)$. Fix a bounded and continuous function f on S . The above inequality implies that $\chi^{(R_n)} \rightarrow \chi^{(R)}$ uniformly on S . Using this, one readily verifies that

$$\lim_{n \rightarrow \infty} \int_S f(x) \tilde{\mu}_n^{(R_n)}(dx) = \int_S f(x) \tilde{\mu}^{(R)}(dx),$$

which yields the desired conclusion. (The above convergence also follows from Lemma 2.9 below.) $\square$

2.3 The compact-convergence topology with variable domains

Fix a bcm space (S, d_S) and a Polish space Ξ . We write $C(S, \Xi)$ for the space of continuous functions $f: S \rightarrow \Xi$, equipped with the compact-convergence topology. Recall that convergence in this topology is characterized as follows: $f_n \rightarrow f$ if and only if $f_n \rightarrow f$ uniformly on every compact subset of S . The purpose of this subsection is to introduce an extension of this topology, which enables us to handle continuous functions defined on possibly different domains. Such an extension is needed for discussing convergence of heat kernels of stochastic processes on different spaces, as will become clear in our main results in Section 6. For further details of this extended topology, see [62, Section 3.3].

We first introduce the spaces of interest. Recall from Definition 2.1 that $\mathcal{C}(S)$ denotes the collection of closed subsets of S , including the empty set.

Definition 2.5. We define

$$\widehat{C}(S, \Xi) := \bigcup_{F \in \mathcal{C}(S)} C(F, \Xi).$$

Note that $\widehat{C}(S, \Xi)$ contains the empty map $\emptyset_\Xi: \emptyset \rightarrow \Xi$. For a function f , we write $\text{dom}(f)$ for its domain. We then define $\widehat{C}_c(S, \Xi)$ to be the subset of $\widehat{C}(S, \Xi)$ consisting of those functions f whose domain $\text{dom}(f)$ is compact.

Below, we introduce metrics on $\widehat{C}_c(S, \Xi)$ and $\widehat{C}(S, \Xi)$. For each function f , we write $\mathbf{g}(f)$ for its graph, i.e.,

$$\mathbf{g}(f) := \{(x, f(x)) \in S \times \Xi \mid x \in \text{dom}(f)\}.$$

We then define

$$d_{S, \Xi}^{\widehat{C}_c}(f, g) := d_{S \times \Xi}^H(\mathbf{g}(f), \mathbf{g}(g)), \quad f, g \in \widehat{C}_c(S, \Xi).$$

Here, $d_{S \times \Xi}^H$ denotes the Hausdorff metric on $\mathcal{C}_c(S \times \Xi)$, the set of compact subsets of $S \times \Xi$. Fix an element $\rho \in S$ as the root of S . For each $f \in \widehat{C}(S, \Xi)$ and $r > 0$, we define $f|_\rho^{(r)} \in \widehat{C}_c(S, \Xi)$ by setting

$$f|_\rho^{(r)} := f|_{D_S(\rho, r)}.$$

When the root is clear from the context, we write $f^{(r)} := f|_\rho^{(r)}$. We then define, for each $f, g \in \widehat{C}(S, \Xi)$,

$$d_{(S, \rho), \Xi}^{\widehat{C}}(f, g) := \int_0^\infty e^{-r} (1 \wedge d_{S, \Xi}^{\widehat{C}_c}(f|_\rho^{(r)}, g|_\rho^{(r)})) dr. \quad (2.5)$$

The functions $d_{S, \Xi}^{\widehat{C}_c}$ and $d_{(S, \rho), \Xi}^{\widehat{C}}$ are metrics on $\widehat{C}_c(S, \Xi)$ and $\widehat{C}(S, \Xi)$, respectively, and they induce Polish topologies. (NB. Neither metric is necessarily complete.) The topology on $\widehat{C}(S, \Xi)$ induced by $d_{(S, \rho), \Xi}^{\widehat{C}}$ is independent of the choice of ρ . (See [62, Section 3.3] for these results).

Lemma 2.6 ([62, Theorems 3.30 and 3.36]). *Let $f, f_1, f_2, \dots$ be elements of $\widehat{C}_c(S, \Xi)$ (resp. $\widehat{C}(S, \Xi)$). Then the following statements are equivalent.*

- (i) *The functions f_n converge to f in $\widehat{C}_c(S, \Xi)$ (resp. $\widehat{C}(S, \Xi)$).*
- (ii) *The sets $\text{dom}(f_n) \rightarrow \text{dom}(f)$ in the Hausdorff (resp. Fell) topology as subsets of S , and, for any $x_n \in \text{dom}(f_n)$ and $x \in \text{dom}(f)$ such that $x_n \rightarrow x$ in S , it holds that $f_n(x_n) \rightarrow f(x)$ in Ξ .*
- (iii) *The sets $\text{dom}(f_n)$ converge to $\text{dom}(f)$ in the Hausdorff (resp. Fell) topology as subsets of S , and there exist functions $g_n, g \in C(S, \Xi)$ such that $g_n|_{\text{dom}(f_n)} = f_n$, $g|_{\text{dom}(f)} = f$ and $g_n \rightarrow g$ in the compact-convergence topology.*

By the above result, the inclusion map from $C(S, \Xi)$ into $\widehat{C}(S, \Xi)$ is a topological embedding, which shows that the topology on $C(S, \Xi)$ is a natural extension of the compact-convergence topology (see [62, Corollary 3.37]).

The following result extends the classical fact that uniform convergence preserves continuity of functions.

Proposition 2.7. *Let $(f_n)_{n \geq 1}$ be a sequence in $\widehat{C}(S, \Xi)$ and f be a function from a closed subset $\text{dom}(f) \subseteq S$ to Ξ . Assume that the following conditions are satisfied.*

- (i) *The domains $\text{dom}(f_n)$ converge to $\text{dom}(f)$ in the Fell topology.*
- (ii) *If elements $x_n \in \text{dom}(f_n)$ converge to $x \in \text{dom}(f)$, then $f_n(x_n) \rightarrow f(x)$ in Ξ .*

Then the function f is continuous and $f_n \rightarrow f$ in $\widehat{C}(S, \Xi)$.

Proof. By Lemma 2.6, it suffices to show that f is continuous. Suppose that a sequence $(x_k)_{k \geq 1} \subseteq \text{dom}(f)$ converges to some $x \in \text{dom}(f)$. By condition (i), for each k , there exist points $y_n \in \text{dom}(f_n)$ converging to x_k . For such y_n , condition (ii) yields $f_n(y_n) \rightarrow f(x_k)$ as $n \rightarrow \infty$. Hence we can choose a subsequence $(n_k)_{k \geq 1}$ and points $y_k \in \text{dom}(f_{n_k})$ such that

$$d_S(x_k, y_k) \vee d_\Xi(f(x_k), f_{n_k}(y_k)) \leq k^{-1}.$$

Since $x_k \rightarrow x$, it follows that $y_k \rightarrow x$. Again by condition (ii), we then have $f_{n_k}(y_k) \rightarrow f(x)$. Combining this with the above inequality gives

$$\limsup_{k \rightarrow \infty} d_\Xi(f(x_k), f(x)) \leq \limsup_{k \rightarrow \infty} \{d_\Xi(f(x_k), f_{n_k}(y_k)) + d_\Xi(f_{n_k}(y_k), f(x))\} = 0,$$

which shows that f is continuous. This completes the proof. $\square$

For later use, we prove the following two technical lemmas. The first lemma provides a quantitative estimate for the distance between two functions in $\widehat{C}(S, \Xi)$ that are close on a ball.

Lemma 2.8. *Let $f_1, f_2 \in \widehat{C}(S, \Xi)$ and $r > 0$ be such that $\text{dom}(f_1^{(r)}) = \text{dom}(f_2^{(r)}) =: K$. It then holds that*

$$d_{\widehat{C}(S, \rho), \Xi}^\widehat{C}(f_1, f_2) \leq e^{-r} + \sup_{x \in K} d_\Xi(f_1(x), f_2(x)).$$

Proof. For any $s \leq r$, we have that $\text{dom}(f_1^{(s)}) = \text{dom}(f_2^{(s)}) = K^{(s)}$. The Hausdorff distance between the graphs of two functions is always bounded above by the uniform distance between them. Thus, we have that

$$d_{S, \Xi}^{\widehat{C}_c}(f_1^{(s)}, f_2^{(s)}) \leq \sup_{x \in K^{(s)}} d_\Xi(f_1(x), f_2(x)) \leq \sup_{x \in K} d_\Xi(f_1(x), f_2(x)),$$

where the last inequality is a trivial one. It then follows that

$$\begin{aligned} d_{(S, \rho), \Xi}^\widehat{C}(f_1, f_2) &\leq \int_r^\infty e^{-s} ds + \sup_{x \in K} d_\Xi(f_1(x), f_2(x)) \cdot \int_0^r e^{-s} ds \\ &\leq e^{-r} + \sup_{x \in K} d_\Xi(f_1(x), f_2(x)), \end{aligned}$$

which completes the proof. $\square$

The second lemma concerns convergence of integrals for convergent functions and measures.

Lemma 2.9. *Let $\mu, \mu_1, \mu_2, \dots \in \mathcal{M}_{\text{fin}}(S)$ with $\mu_n \rightarrow \mu$ weakly, and let $f, f_1, f_2, \dots \in \widehat{C}(S, \mathbb{R})$ with $f_n \rightarrow f$ in $\widehat{C}(S, \mathbb{R})$. Suppose further that there exists a compact subset $K \subseteq S$ such that the support of every μ_n and of μ is contained in K . Then*

$$\lim_{n \rightarrow \infty} \int_{\text{dom}(f_n)} f_n(x) \mu_n(dx) = \int_{\text{dom}(f)} f(x) \mu(dx).$$

Proof. By Lemma 2.6(iii), the domains of f_n and f may be extended continuously to S so that $f_n \rightarrow f$ in the compact-convergence topology. Let $g: S \rightarrow [0, 1]$ be a compactly-supported continuous function such that $g \equiv 1$ on K . Since the support of μ_n is contained in K , it holds that

$$\int_{\text{dom}(f_n)} f_n(x) \mu_n(dx) = \int_{\text{dom}(f_n) \cap K} f_n(x) \mu_n(dx) = \int_S f_n(x) g(x) \mu_n(dx),$$

and, similarly

$$\int_{\text{dom}(f)} f(x) \mu(dx) = \int_S f(x) g(x) \mu(dx).$$

Hence

$$\begin{aligned}
 & \left| \int_{\text{dom}(f_n)} f_n(x) \mu_n(dx) - \int_{\text{dom}(f)} f(x) \mu(dx) \right| \\
 & \leq \int_S |f_n(x) - f(x)| g(x) \mu_n(dx) + \left| \int_S f(x) g(x) \mu_n(dx) - \int_S f(x) g(x) \mu(dx) \right| \\
 & \leq \mu_n(K) \sup_{x \in K} |f_n(x) - f(x)| + \left| \int_S f(x) g(x) \mu_n(dx) - \int_S f(x) g(x) \mu(dx) \right|.
 \end{aligned}$$

The first term tends to 0 by the uniform convergence $f_n \rightarrow f$ on K , and the second by the vague convergence $\mu_n \rightarrow \mu$. This proves the claim. $\square$

2.4 Topologies on the space of càdlàg functions

In studying convergence of sample paths of stochastic processes, a standard choice of topology is the J_1 -Skorohod topology (see, for example, [17]). As already mentioned in Remark 1.5, in this paper, we also employ another topology, which we call the L^0 topology. The purpose of this subsection is to introduce these topologies. Throughout this subsection, we fix a bcm space (S, d_S) .

We write $D(\mathbb{R}_{\geq 0}, S)$ for the set of càdlàg functions $\xi: \mathbb{R}_{\geq 0} \rightarrow S$, and equip $D(\mathbb{R}_{\geq 0}, S)$ with the Skorohod metric $d_S^{J_1}$. To recall its definition, let $\Lambda_t^{J_1}$ denote the set of continuous bijections $\lambda: [0, t] \rightarrow [0, t]$. For càdlàg functions $\xi, \eta: [0, t] \rightarrow S$, set

$$d_S^{J_1, t}(\xi, \eta) := \inf_{\lambda \in \Lambda_t^{J_1}} \left\{ \sup_{0 \leq s \leq t} |\lambda(s) - s| \vee \sup_{0 \leq s \leq t} d_S(\eta(s), \xi \circ \lambda(s)) \right\}.$$

We then define the Skorohod metric by

$$d_S^{J_1}(\eta, \xi) := \int_0^\infty e^{-t} (1 \wedge d_S^{J_1, t}(\xi|_{[0, t]}, \eta|_{[0, t]})) dt, \quad \xi, \eta \in D(\mathbb{R}_{\geq 0}, S). \quad (2.6)$$

The topology induced by $d_S^{J_1}$ is Polish and is called the J_1 -Skorohod topology; see [78, Theorem 2.6]. (NB. The metric $d_S^{J_1}$ itself is not complete.)

We write $L^0(\mathbb{R}_{\geq 0}, S)$ for the set of Borel measurable functions $\xi: \mathbb{R}_{\geq 0} \rightarrow S$. We adopt the convention that any functions ξ and η in $L^0(\mathbb{R}_{\geq 0}, S)$ are identified whenever $\xi(t) = \eta(t)$ for Lebesgue-almost every $t \in \mathbb{R}_{\geq 0}$. We define a metric on $L^0(\mathbb{R}_{\geq 0}, S)$ as follows: for each $\xi, \eta \in L^0(\mathbb{R}_{\geq 0}, S)$,

$$d_S^{L^0}(\xi, \eta) := \int_0^\infty e^{-t} (1 \wedge d_S(\xi(t), \eta(t))) dt. \quad (2.7)$$

By [62, Proposition 8.32], $d_S^{L^0}$ is a complete and separable metric on $L^0(\mathbb{R}_{\geq 0}, S)$. We refer to the topology on $L^0(\mathbb{R}_{\geq 0}, S)$ induced by $d_S^{L^0}$ as the L^0 topology, the name being derived from the L^0 space endowed with the topology of convergence in measure (cf. [19]).

By [44, Lemma 5.2], convergence with respect to $d_S^{L^0}$ coincides with convergence in probability (with respect to the probability measure $e^{-t} dt$). The following is immediate from the characterization of convergence in probability via almost-sure convergence.

Lemma 2.10 ([44, Lemma 5.2]). *Let $\xi, \xi_1, \xi_2, \dots$ be elements of $L^0(\mathbb{R}_{\geq 0}, S)$. Then $\xi_n \rightarrow \xi$ in the L^0 topology if and only if the following holds:*

(L^0) *for any subsequence $(n_k)_{k \geq 1}$, there exists a further subsequence $(n_{k(l)})_{l \geq 1}$ such that, for Lebesgue-almost every $t \in \mathbb{R}_{\geq 0}$, $\xi_{n_{k(l)}}(t) \rightarrow \xi(t)$.*

As a consequence, the L^0 topology is independent of the choice of the metric d_S and depends only on the topology on S .

The following lemma is an immediate consequence of Prohorov's theorem, which states that tightness of probability measures is equivalent to their relative compactness in the weak topology (see [17, Section 5]). In what follows we only use one direction, namely, that relative compactness implies tightness.

Lemma 2.11. *Let $(\xi_n)_{n \geq 1}$ be a sequence converging to a function ξ in $L^0(\mathbb{R}_{\geq 0}, S)$. Fix $T > 0$. Then, for any $\varepsilon > 0$, there exists a compact subset $K \subseteq S$ such that*

$$\sup_{n \geq 1} \text{Leb}(\{t \in [0, T] \mid \xi_n(t) \notin K\}) < \varepsilon,$$

where we denote by Leb the Lebesgue measure.

Since $D(\mathbb{R}_{\geq 0}, S)$ can be viewed as a subset of $L^0(\mathbb{R}_{\geq 0}, S)$, the L^0 topology is also defined on $D(\mathbb{R}_{\geq 0}, S)$ as a relative topology. Thus, we have two topologies on $D(\mathbb{R}_{\geq 0}, S)$, the J_1 -Skorohod topology and the L^0 topology. Henceforth, to distinguish these two topological structures, we write

(D1) $D_{J_1}(\mathbb{R}_{\geq 0}, S)$ for $D(\mathbb{R}_{\geq 0}, S)$ endowed with the J_1 -Skorohod topology,

(D2) $D_{L^0}(\mathbb{R}_{\geq 0}, S)$ for $D(\mathbb{R}_{\geq 0}, S)$ endowed with the L^0 topology.

Below, we verify that the two spaces coincide as measurable spaces.

Proposition 2.12. *The Borel σ -algebras on $D(\mathbb{R}_{\geq 0}, S)$ generated by the J_1 -Skorohod and L^0 topologies are the same.*

Proof. We first note that the Borel σ -algebra generated by the J_1 -Skorohod topology coincides with the σ -algebra generated by the evaluation maps $\pi_t: D(\mathbb{R}_{\geq 0}, S) \rightarrow S$, $t \geq 0$, defined by $\pi_t(f) := f(t)$ (see [78, Lemma 2.7]). By Lemma 2.10, the J_1 -Skorohod topology is finer than the L^0 topology. Thus, it suffices to show that each π_t is measurable with respect to the L^0 topology. Fix $t \geq 0$. For each $x \in S$, we define $h_{x,t}: D(\mathbb{R}_{\geq 0}, S) \rightarrow \mathbb{R}_{\geq 0}$ by setting

$$h_{x,t}(\xi) := 1 \wedge d_S(x, \xi(t)).$$

We approximate $h_{x,t}$ by functions $h_{x,t}^{(n)}: D(\mathbb{R}_{\geq 0}, S) \rightarrow \mathbb{R}_{\geq 0}$, $n \in \mathbb{N}$, given by

$$h_{x,t}^{(n)}(\xi) := n \int_t^{t+n^{-1}} 1 \wedge d_S(x, \xi(s)) ds, \quad \xi \in D(\mathbb{R}_{\geq 0}, S).$$

By the right-continuity of ξ and the dominated convergence theorem, we have $h_{x,t}^{(n)}(\xi) \rightarrow h_{x,t}(\xi)$ as $n \rightarrow \infty$ for each $\xi \in D(\mathbb{R}_{\geq 0}, S)$. Moreover, it is not difficult to see that $h_{x,t}^{(n)}$ is continuous with respect to the L^0 topology. Thus, $h_{x,t}$ is measurable with respect to the L^0 topology. Fix an open subset U of S . Since S is second countable, we can find countable subsets $\{x_n\}_{n \geq 1}$ of U and $\{\varepsilon_n\}_{n \geq 1}$ of $[0, 1)$ such that $U = \bigcup_{n \geq 1} B_S(x_n, \varepsilon_n)$. We then have that

$$(\pi_t)^{-1}(U) = \bigcup_{n \geq 1} \pi_t^{-1}(B_S(x_n, \varepsilon_n)) = \bigcup_{n \geq 1} h_{x_n,t}^{-1}([0, \varepsilon_n)).$$

Therefore, π_t is measurable with respect to the L^0 topology. $\square$

One might expect that it suffices to work within $D_{L^0}(\mathbb{R}_{\geq 0}, S)$ when discussing convergence of càdlàg functions in the L^0 topology. However, since $D_{L^0}(\mathbb{R}_{\geq 0}, S)$ is not Polish, it is more convenient to carry out probabilistic arguments in the larger space $L^0(\mathbb{R}_{\geq 0}, S)$. The following result ensures that, for stochastic processes with càdlàg paths, weak convergence in $L^0(\mathbb{R}_{\geq 0}, S)$ remains valid when restricted to $D_{L^0}(\mathbb{R}_{\geq 0}, S)$.

Proposition 2.13. *The set $D(\mathbb{R}_{\geq 0}, S)$ is a Borel subset of $L^0(\mathbb{R}_{\geq 0}, S)$.*

This result is readily verified by the Lusin–Souslin theorem, which we recall below. The theorem will also be used in later arguments.

Lemma 2.14 ([45, Corollary 15.2]). *Let Ξ_1 and Ξ_2 be Polish spaces. Fix a Borel subset B of Ξ_1 . If a map $f: B \rightarrow \Xi_2$ is Borel measurable and injective, then the image $f(B)$ is a Borel subset of Ξ_2 .*

Proof of Proposition 2.13. Define a map $F: D_{J_1}(\mathbb{R}_{\geq 0}, S) \rightarrow L^0(\mathbb{R}_{\geq 0}, S)$ by setting $F(\xi) := \xi$. Then F is injective and continuous. Since both spaces $D_{J_1}(\mathbb{R}_{\geq 0}, S)$ and $L^0(\mathbb{R}_{\geq 0}, S)$ are Polish, the above lemma implies that the image of F , namely $D(\mathbb{R}_{\geq 0}, S)$, is a Borel subset of $L^0(\mathbb{R}_{\geq 0}, S)$. This completes the proof. $\square$

The following lemma, which concerns different notions of a topology on the space of product processes, will be used in Section 7.

Lemma 2.15. *The following map is a homeomorphism:*

$$D_{L^0}(\mathbb{R}_{\geq 0}, S \times S) \ni \xi = (\xi_1, \xi_2) \mapsto (\xi_1, \xi_2) \in D_{L^0}(\mathbb{R}_{\geq 0}, S) \times D_{L^0}(\mathbb{R}_{\geq 0}, S).$$

Proof. This follows directly from Lemma 2.10. $\square$

2.5 The spaces for PCAFs and STOMs

Fix a bcm space S . Recall from Section 1 (in particular, from (1.1)) that a space-time occupation measure (STOM) associated with a PCAF is a random measure on $S \times \mathbb{R}_{\geq 0}$. As noted in Section 1.1, the vague topology is a natural choice for discussing convergence of STOMs, but vague convergence of STOMs does not recover the local uniform convergence of the corresponding PCAFs. The aim of this subsection is to introduce spaces for PCAFs and STOMs and define a suitable topology.

We first introduce a space that will be used to discuss the convergence of PCAFs in our main results.

Definition 2.16. We define $C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ to be the collection of functions $f \in C(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ that are non-decreasing. We equip this set with the following metric: for each $\varphi_1, \varphi_2 \in C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$,

$$d_{C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})}(\varphi_1, \varphi_2) := \sum_{n=1}^{\infty} 2^{-n} \left(1 \wedge \sup_{0 \leq t \leq n} |\varphi_1(t) - \varphi_2(t)| \right). \quad (2.8)$$

In particular, the topology on $C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ is the compact-convergence topology.

Remark 2.17. Note that convergence in $C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ is equivalent to pointwise convergence (cf. [71, Chapter 0.1]).

Remark 2.18. Not all PCAFs belong to $C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$. Indeed, a PCAF may explode in finite time. Therefore, to study the convergence of PCAFs in the space $C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, certain restrictions on the underlying process or on the PCAFs under consideration are required. This issue will be discussed in more detail in Section 5.1 later.

We next introduce a space in which the convergence of STOMs will be discussed.

Definition 2.19. We define $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$ to be the set of Borel measures Σ on $S \times \mathbb{R}_{\geq 0}$ such that, for each $t \in \mathbb{R}_{\geq 0}$, $\Sigma|_{S \times [0, t]}$ is a finite Borel measure.

Recall from (1.1) that, given $(\xi, \varphi) \in D(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, representing a process and a PCAF respectively, the associated STOM is defined by

$$\Psi(\xi, \varphi)(\cdot) := \int_0^\infty \mathbf{1}_{(\xi_t, t)}(\cdot) d\varphi_t. \quad (2.9)$$

It follows immediately that $\Sigma = \Psi(\xi, \varphi)$ belongs to $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$. The function φ can be recovered from Σ via

$$\varphi(t) = \Sigma(S \times [0, t]) = \int_{S \times \mathbb{R}_{\geq 0}} \mathbf{1}_{S \times [0, t]}(x, s) \Sigma(dx ds).$$

From this observation, one sees that the vague topology cannot recover the convergence of PCAFs, since the test function $\mathbf{1}_{S \times [0, t]}$ is not compactly supported.

To resolve this issue, we introduce a metric that differs from the vague metric. For each $\Sigma_1, \Sigma_2 \in \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$, define

$$d_{S \times \mathbb{R}_{\geq 0}}^{\mathcal{M}_{(*, \text{fin})}}(\Sigma_1, \Sigma_2) := \int_0^\infty e^{-t} (1 \wedge d_{S \times \mathbb{R}_{\geq 0}}^P(\Sigma_1|_{S \times [0, t]}, \Sigma_2|_{S \times [0, t]})) dt. \quad (2.10)$$

Proposition 2.20. *The function $d_{S \times \mathbb{R}_{\geq 0}}^{\mathcal{M}_{(*, \text{fin})}}$ defines a complete, separable metric on $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$.*

Proof. This can be shown in exactly the same way as the proof that the vague metric (2.2) is a complete and separable metric, and so the proof is omitted. See [62, Theorem 3.19] for details. $\square$

For later use, we prove the following lemma, which states that the above-defined distance between elements is preserved under isometric embeddings of the underlying spaces. This property is particularly important for the metrization of Gromov–Hausdorff-type topologies, discussed in Section 3.

Lemma 2.21. *Let S_1 and S_2 be bcm spaces, and let $f: S_1 \rightarrow S_2$ be an isometric embedding. Then the following pushforward map is distance preserving with respect to the above-defined metrics:*

$$\mathcal{M}_{(*, \text{fin})}(S_1 \times \mathbb{R}_{\geq 0}) \rightarrow \mathcal{M}_{(*, \text{fin})}(S_2 \times \mathbb{R}_{\geq 0}), \quad \Sigma \mapsto \Sigma \circ (f \times \text{id}_{\mathbb{R}_{\geq 0}})^{-1}.$$

Proof. This property follows immediately from the corresponding result for the vague metric (see [62, Proposition 3.24]), and hence the details are omitted. $\square$

Henceforth, we equip $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$ with the topology induced by the metric $d_{S \times \mathbb{R}_{\geq 0}}^{\mathcal{M}_{(*, \text{fin})}}$. The following provides characterizations of convergence in this space. In particular, one can see that the topology is stronger than the vague topology.

Proposition 2.22. *Let $\Sigma, \Sigma_1, \Sigma_2, \dots$ be elements of $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$. The following statements are equivalent with each other.*

- (i) *The measures Σ_n converge to Σ in $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$.*
- (ii) *For all but countably many $t \in \mathbb{R}_{\geq 0}$, the finite Borel measures $\Sigma_n|_{S \times [0, t]}$ converge weakly to $\Sigma|_{S \times [0, t]}$.*
- (iii) *For any bounded and continuous function $f: S \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ such that $\text{supp}(f) \subseteq S \times [0, T]$ for some $T \in \mathbb{R}_{\geq 0}$, it holds that*

$$\lim_{n \rightarrow \infty} \int_{S \times \mathbb{R}_{\geq 0}} f(x, t) \Sigma_n(dx dt) = \int_{S \times \mathbb{R}_{\geq 0}} f(x, t) \Sigma(dx dt).$$

Proof. This can be shown in exactly the same way as the characterization of the vague topology given in [62, Theorem 3.20], so we will omit the proof. $\square$

Remark 2.23. Recall that the vague topology we adopt in this paper is defined by using continuous functions with compact support as test functions. A variant of this notion, in which the test functions are bounded continuous functions whose supports are bounded subsets, has been studied in detail in [43]. We refer to the resulting topology as the *vague topology with bounded support*. From Proposition 2.22, one sees that the topology of $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$ coincides with the vague topology with bounded support when the underlying metric d_S on S is replaced by a bounded, topologically equivalent one, for instance $d_S \wedge 1$. Hence, all the results of [43] concerning this topology are applicable to our setting.

Proposition 2.24. *Fix a countable family $\{\Pi_n\}_{n \geq 1}$ in $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$. The family is pre-compact if and only if the following two conditions are satisfied:*

- (i) $\limsup_{n \rightarrow \infty} \Pi_n(S \times [0, t]) < \infty$ for all $t > 0$,
- (ii) $\limsup_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \Pi_n((S \setminus S^{(R)}) \times [0, t]) = 0$ for all $t > 0$.

Proof. By Remark 2.23, the result is immediate from [43, Theorem 4.2]. $\square$

In the following proposition, we clarify the relationship between the convergence of PCAF-type functions and that of the associated STOM-type measures. In particular, it shows that convergence of STOMs in $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$ implies convergence of the corresponding PCAFs in $C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$.

Proposition 2.25. *Define a map*

$$\mathfrak{S}: D(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}) \ni (\xi, \varphi) \longmapsto (\xi, \Psi(\xi, \varphi)) \in D(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0}),$$

where $\Psi(\xi, \varphi)$ is defined in (2.9). Then the following statements hold.

- (i) *If both copies of $D(\mathbb{R}_{\geq 0}, S)$ in the domain and the codomain are equipped with the J_1 -Skorohod topology, then the map is a topological embedding, i.e., a homeomorphism onto its image.*
- (ii) *If both copies of $D(\mathbb{R}_{\geq 0}, S)$ are equipped with the L^0 topology, then the map is measurable and its inverse is continuous, whereas the map itself is not necessarily continuous.*

Proof. (i). It is clear that the map is injective. We first prove its continuity. Let $(\xi_n, \varphi_n)_{n \geq 1}$ be a sequence converging to (ξ, φ) in $D_{J_1}(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$. Fix a continuity point $T > 0$ of ξ . By the J_1 -Skorohod convergence of ξ_n to ξ , we can find a compact subset K of S such that all the trajectories of ξ_n , $n \geq 1$, up to time T are contained in K (cf. [44, Theorem A.5.4]). Moreover, there exist time-change functions $\lambda_n \in \Lambda_T^{J_1}$, $n \geq 1$, such that

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0, \quad \text{where} \quad \varepsilon_n := \sup_{0 \leq t \leq T} |\lambda_n(t) - t| \vee \sup_{0 \leq t \leq T} d_S(\xi_n(t), \xi \circ \lambda_n(t)).$$

Fix a bounded and continuous function $f: S \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that $\text{supp}(f) \subseteq S \times [0, T]$ for some $T > 0$. By the uniform continuity of f on $K \times [0, T]$, if we set, for each $\varepsilon > 0$,

$$w_f(\varepsilon) := \sup\{|f(x, s) - f(y, t)| \mid (x, s), (y, t) \in K \times [0, T] \text{ such that } d_S(x, y) \vee |s - t| \leq \varepsilon\},$$

then $w_f(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Write $\Sigma_n := \Psi(\xi_n, \varphi_n)$, $n \geq 1$, and $\Sigma := \Psi(\xi, \varphi)$. We deduce that

$$\begin{aligned} & \left| \int_{S \times \mathbb{R}_{\geq 0}} f(x, t) \Sigma_n^{(*, T)}(dx dt) - \int_{S \times \mathbb{R}_{\geq 0}} f(x, t) \Sigma^{(*, T)}(dx dt) \right| \\ &= \left| \int_0^T f(\xi_n(t), t) \varphi_n(dt) - \int_0^T f(\xi(t), t) \varphi(dt) \right| \\ &\leq \int_0^T |f(\xi_n(t), t) - f(\xi \circ \lambda_n(t), t)| \varphi_n(dt) + \left| \int_0^T f(\xi \circ \lambda_n(t), t) \varphi_n(dt) - \int_0^T f(\xi(t), t) \varphi(dt) \right| \\ &\leq w_f(\varepsilon_n) \varphi_n(T) + \left| \int_0^T f(\xi \circ \lambda_n(t), t) \varphi_n(dt) - \int_0^T f(\xi(t), t) \varphi(dt) \right|. \end{aligned}$$

Since $\varphi_n(T)$ is bounded uniformly in n by the convergence of φ_n to φ , the first term in the last inequality converges to 0 as $n \rightarrow \infty$. Thus, it suffices to show that the second term also vanishes. Define the right-continuous inverse of φ_n by

$$\varphi_n^{-1}(t) := \inf\{s \geq 0 \mid \varphi_n(s) > t\}.$$

Similarly, define φ^{-1} to be the right-continuous inverse of φ . By a change-of-variables formula (cf. [22, Lemma A.3.7]),

$$\begin{aligned} & \left| \int_0^T f(\xi \circ \lambda_n(t), t) \varphi_n(dt) - \int_0^T f(\xi(t), t) \varphi(dt) \right| \\ &= \left| \int_0^{\varphi_n(T)} f(\xi \circ \lambda_n \circ \varphi_n^{-1}(t), \varphi_n^{-1}(t)) dt - \int_0^{\varphi(T)} f(\xi \circ \varphi^{-1}(t), \varphi^{-1}(t)) dt \right|. \end{aligned} \quad (2.11)$$

By [79, Theorem 12.5.1 and Corollary 13.6.5], we have $\varphi_n^{-1}(t) \rightarrow \varphi^{-1}(t)$ for Lebesgue-almost every $t > 0$. For such t , it also holds that $\lambda_n \circ \varphi_n^{-1}(t) \rightarrow \varphi^{-1}(t)$. Let N_1 denote the negligible set of $t \geq 0$ for which the convergence $\lambda_n \circ \varphi_n^{-1}(t) \rightarrow \varphi^{-1}(t)$ fails. Let N_2 denote the set of $t \geq 0$ such that $\varphi^{-1}(t)$ is a discontinuity point of ξ . Since ξ is continuous except at countably many points and φ^{-1} is strictly increasing, the set N_2 is countable. Hence $\text{Leb}(N_1 \cup N_2) = 0$. For any $t \in \mathbb{R}_{\geq 0} \setminus (N_1 \cup N_2)$, we have $\lambda_n \circ \varphi_n^{-1}(t) \rightarrow \varphi^{-1}(t)$ and $\varphi^{-1}(t)$ is a continuity point of ξ , which implies $\xi(\lambda_n \circ \varphi_n^{-1}(t)) \rightarrow \xi(\varphi^{-1}(t))$. Therefore, by the dominated convergence theorem, the value of (2.11) converges to 0 as $n \rightarrow \infty$. This establishes the desired continuity.

It remains to prove that the inverse map is continuous. Let $(\xi_n, \varphi_n)_{n \geq 1}$ and (ξ, φ) be elements of $D_{J_1}(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$. Write $\Sigma_n := \Psi(\xi_n, \varphi_n)$, $n \geq 1$, and $\Sigma := \Psi(\xi, \varphi)$. Assume that $(\xi_n, \Sigma_n) \rightarrow (\xi, \Sigma)$. It suffices to show that $\varphi_n \rightarrow \varphi$ in the compact-convergence topology. By the convergence of Σ_n to Σ and Proposition 2.22, we deduce that $d\varphi_n \rightarrow d\varphi$ vaguely. Since φ is continuous, $d\varphi$ has no atoms. In particular, for any $t \geq 0$, $[0, t]$ is a continuity set of $d\varphi$. It then follows that $\varphi_n(t) \rightarrow \varphi(t)$ for each $t \geq 0$. This completes the proof.

(ii). By Proposition 2.12 and (i), the map is measurable. The continuity of the inverse map follows by the same argument as before. To see that the map fails to be continuous in general, assume that S consists of two distinct elements v_1 and v_2 , and take a function $\varphi \in C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ such that $d\varphi([0, 1]) = 1$, $\text{supp}(d\varphi) \subseteq [0, 1]$, and $\text{Leb}(\text{supp}(d\varphi)) = 0$. (For example, the Cantor function suffices.) From the last condition, for each $n \geq 1$, we can find a Borel set U_n which is a disjoint union of finitely many half-open intervals of the form $[a, b)$ and satisfies $\text{supp}(d\varphi) \subseteq U_n$ and $\text{Leb}(U_n) < 2^{-n}$. Define $\eta_n \in D(\mathbb{R}_{\geq 0}, S)$ by setting $\eta_n(t) = v_1$ if $t \in U_n$ and $\eta_n(t) = v_2$ otherwise. Then η_n converges to the constant function v_2 in the L^0 topology. On the other hand, $\Psi(\eta_n, \varphi)(\{v_1\} \times [0, 1]) = d\varphi([0, 1]) = 1$ for all $n \geq 1$, whereas for the limit $\eta \equiv v_2$ we have $\Psi(\eta, \varphi)(\{v_1\} \times [0, 1]) = 0$. Hence $\Psi(\eta_n, \varphi)$ does not converge to $\Psi(\eta, \varphi)$. $\square$

Remark 2.26. We record some remarks concerning Proposition 2.25.

- (a) The continuity of the inverse of $\mathfrak{S}$ implies that the convergence of STOM-type measures in $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$ entails the convergence of the associated PCAF-type functions.
- (b) By Proposition 2.25(i), when one works in the J_1 -Skorohod topology (or other relatively fine topologies), the joint convergence of processes and PCAFs is equivalent to that of processes and STOMs. However, in the main results of this paper, we mainly work under the L^0 topology in order to allow greater generality of assumptions. Consequently, the convergence of PCAFs does not necessarily imply that of STOMs, which makes the technical arguments considerably more involved.

3 Topology on the space of metric spaces equipped with additional structures

The Gromov–Hausdorff metric, introduced by Gromov [36], is a modification of the Hausdorff metric and measures the distance between compact metric spaces. It induces a Polish topology on the set of (equivalence classes of) compact metric spaces, called the *Gromov–Hausdorff topology*, and provides a natural setting for discussing convergence of metric spaces. In many situations, metric spaces are equipped with additional structures such as measures or stochastic processes, and it is then natural to consider convergence not only of the underlying metric spaces but also of the additional structures. For this purpose, several extensions of the Gromov–Hausdorff topology have been studied (cf. [1, 10, 46]), and general frameworks have recently been established in [47, 62]. Here, following [62], we introduce the topological framework needed for our main results. (The differences between [47] and [62] are discussed in [62, Section 1].)

3.1 General framework

In this subsection we explain how to formalize the addition of structures to metric spaces and how to define a generalized Gromov–Hausdorff metric. We summarize only the part of the framework needed for the discussion in this paper. Although the use of category-theoretic language would clarify the framework, we avoid this approach in order to keep the presentation self-contained and accessible. For further details see [62, Section 6].

We first define a rule that assigns an additional structure to each metric space. Recall that if a map $f: S_1 \rightarrow S_2$ between metric spaces is distance-preserving, then f is called an *isometric embedding*. An isometric embedding that is surjective (and hence bijective) is called an *isometry*.

Definition 3.1 (Structure). We call τ a *structure* (on bcm spaces) if it satisfies the following conditions.

- (i) For every bcm space S , there exists a metrizable topological space $\tau(S)$.
- (ii) For every isometric embedding $f: S_1 \rightarrow S_2$ between bcm spaces, there exists a topological embedding $\tau_f: \tau(S_1) \rightarrow \tau(S_2)$, i.e., τ_f is a homeomorphism onto its image.
- (iii) Let S_1, S_2, S_3 be bcm spaces. For any isometric embeddings $f: S_1 \rightarrow S_2$ and $g: S_2 \rightarrow S_3$, one has $\tau_{g \circ f} = \tau_g \circ \tau_f$.
- (iv) For any bcm space S , one has $\tau_{\text{id}_S} = \text{id}_{\tau(S)}$.

We say that τ is *separable* if each $\tau(S)$ is separable.

Readers who wish to keep a specific structure in mind may refer to Section 3.3 for a list of example structures.

Throughout this subsection, we fix a structure τ . Given $\mathcal{S}_i = (S_i, d_{S_i}, \rho_{S_i}, a_{S_i})$, $i = 1, 2$, where $(S_i, d_{S_i}, \rho_{S_i})$ is a rooted bcm space and $a_{S_i} \in \tau(S_i)$, we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *rooted- τ -isometric* if there exists a root-preserving isometry $f: S_1 \rightarrow S_2$ such that $\tau_f(a_{S_1}) = a_{S_2}$. Recall that f is

root-preserving if $f(\rho_{S_1}) = \rho_{S_2}$. We then define $\mathfrak{M}_\bullet(\tau)$ as the collection of rooted- τ -isometric equivalence classes.

Remark 3.2. From a rigorous set-theoretic point of view, one cannot directly take rooted- τ -isometric equivalence classes, since the collection of all tuples (S, d_S, ρ_S, a_S) is not a legitimate set. Nevertheless, it is possible to regard $\mathfrak{M}_\bullet(\tau)$ as a set. Indeed, one can construct a legitimate set $\mathcal{M}(\tau)$ such that every (S, d_S, ρ_S, a_S) is rooted- τ -isometric to a unique element of $\mathcal{M}(\tau)$. For details, see [62, Proposition 6.2]. In this paper, we tacitly fix such a representative system and thereafter treat $\mathfrak{M}_\bullet(\tau)$ as a set, so that we can proceed without repeatedly referring to these set-theoretic formalities.

To define a metric on $\mathfrak{M}_\bullet(\tau)$, we introduce the notion of a metrization of a structure.

Definition 3.3. We say that τ admits a *metrization* if and only if, for every rooted bcm space (S, ρ_S) , there exists a metric d_{S, ρ_S}^τ on $\tau(S)$ inducing the given topology such that the following condition holds:

- (v) Let (S_1, ρ_{S_1}) and (S_2, ρ_{S_2}) be rooted bcm spaces. For every root-preserving isometric embedding $f: S_1 \rightarrow S_2$, the map $\tau_f: \tau(S_1) \rightarrow \tau(S_2)$ is distance-preserving with respect to $d_{S_1, \rho_{S_1}}^\tau$ and $d_{S_2, \rho_{S_2}}^\tau$.

If each d_{S, ρ_S}^τ is complete, we say that τ admits a *complete metrization*.

Henceforth we assume that τ admits a metrization. We now define a distance between elements of $\mathfrak{M}_\bullet(\tau)$.

Definition 3.4. For $\mathcal{S}_1 = (S_1, d_{S_1}, \rho_{S_1}, a_{S_1})$ and $\mathcal{S}_2 = (S_2, d_{S_2}, \rho_{S_2}, a_{S_2})$ in $\mathfrak{M}_\bullet(\tau)$, set

$$d_{\mathfrak{M}_\bullet}^\tau(\mathcal{S}_1, \mathcal{S}_2) := \inf_{f, g, M} \left\{ d_{M, \rho_M}^F(f(S_1), g(S_2)) \vee d_{M, \rho_M}^\tau(\tau_f(a_{S_1}), \tau_g(a_{S_2})) \right\},$$

where the infimum is taken over all rooted bcm spaces (M, ρ_M) and root-preserving isometric embeddings $f: S_1 \rightarrow M, g: S_2 \rightarrow M$.

To ensure that the above defined function is a metric, we consider the following continuity condition on τ regarding embedding maps.

Definition 3.5. We say that τ is *embedding-continuous* if and only if the following condition holds.

- (EC) Fix bcm spaces S_1 and S_2 . Let $f, f_1, f_2, \dots$ be continuous maps from S_1 to S_2 . If $f_n \rightarrow f$ in the compact-convergence topology, then $\tau_{f_n}(a) \rightarrow \tau_f(a)$ in $\tau(S_2)$ for all $a \in \tau(S_1)$.

Lemma 3.6 ([62, Theorem 6.19]). *If τ is embedding-continuous, then $d_{\mathfrak{M}_\bullet}^\tau$ is a metric on $\mathfrak{M}_\bullet(\tau)$.*

Henceforth we assume that τ is embedding-continuous and endow $\mathfrak{M}_\bullet(\tau)$ with the topology induced by $d_{\mathfrak{M}_\bullet}^\tau$. Below we provide a characterization of convergence in $\mathfrak{M}_\bullet(\tau)$ that is useful in applications. In particular, it shows that the resulting topology on $\mathfrak{M}_\bullet(\tau)$ is independent of the choice of a metrization of τ .

Theorem 3.7 (Convergence). *Let $\mathcal{S}_n = (S_n, d_{S_n}, \rho_{S_n}, a_{S_n})$, $n \in \mathbb{N} \cup \{\infty\}$, be elements of $\mathfrak{M}_\bullet(\tau)$. The following conditions are equivalent:*

- (i) $\mathcal{S}_n \rightarrow \mathcal{S}_\infty$ in $\mathfrak{M}_\bullet(\tau)$;
- (ii) *there exist a rooted bcm space (M, ρ_M) and root-preserving isometric embeddings $f_n: S_n \rightarrow M$, $n \in \mathbb{N} \cup \{\infty\}$, such that $f_n(S_n) \rightarrow f_\infty(S_\infty)$ in the Fell topology as closed subsets of M , and $\tau_{f_n}(a_{S_n}) \rightarrow \tau_{f_\infty}(a_{S_\infty})$ in $\tau(M)$.*

Remark 3.8. In Theorem 3.7(ii), the roots ρ_{S_n} of S_n are mapped to a common root ρ_M of M . By relaxing this requirement, one can introduce another notion of convergence in $\mathfrak{M}_\bullet(\tau)$. Namely, we say that S_n converges to S_∞ if and only if

there exist a bcm space M and isometric embeddings $f_n: S_n \rightarrow M$ such that $f_n(S_n) \rightarrow f_\infty(S_\infty)$ in the Fell topology, $f_n(\rho_{S_n}) \rightarrow f_\infty(\rho_{S_\infty})$ in M , and $\tau_{f_n}(a_{S_n}) \rightarrow \tau_{f_\infty}(a_{S_\infty})$ in $\tau(M)$.

In general, this convergence is weaker than the convergence induced by $d_{\mathfrak{M}_\bullet}^\tau$ defined above. However, for most structures of interest the two notions of convergence coincide. See [62, Section 6.3] for details.

The following observation is immediate from the definition of $d_{\mathfrak{M}_\bullet}^\tau$ and will be useful for the approximation arguments in our main results.

Lemma 3.9. *For $\mathcal{S} = (S, d_S, \rho_S, a_S)$ and $\mathcal{S}' = (S, d_S, \rho_S, b_S)$ in $\mathfrak{M}_\bullet(\tau)$, it holds that*

$$d_{\mathfrak{M}_\bullet}^\tau(\mathcal{S}, \mathcal{S}') \leq d_{S, \rho_S}^\tau(a_S, b_S).$$

To discuss Polishness, we introduce another continuity condition on τ . For notational convenience, we adopt the convention that if a bcm space S is isometrically embedded into another bcm space M via an isometric embedding f , then we regard $\tau(S)$ as a subspace of $\tau(M)$ through the topological embedding $\tau_f: \tau(S) \rightarrow \tau(M)$.

Assumption 3.10. *Let S_n , $n \in \mathbb{N} \cup \{\infty\}$, be bcm spaces that are isometrically embedded into a common bcm space M such that $S_n \rightarrow S_\infty$ in the Fell topology as closed subsets of M .*

- (i) *If a sequence $a_n \in \tau(S_n)$ converges to some $a \in \tau(M)$, then $a \in \tau(S_\infty)$.*
- (ii) *For every $a \in \tau(S_\infty)$, there exist a subsequence $(n_k)_{k \geq 1}$ and elements $a_k \in \tau(S_{n_k})$ such that $a_k \rightarrow a$ in $\tau(M)$.*

Definition 3.11 (Continuity). We say that τ is *upper* (resp. *lower*) *semicontinuous* if and only if it satisfies Assumption 3.10(i) (resp. (ii)). We say that τ is *continuous* if and only if it is embedding-continuous and both upper and lower semicontinuous.

Theorem 3.12 ([62, Theorems 6.40 and 6.41]). *Assume that τ is separable and continuous, and admits a complete metrization. Then the induced metric $d_{\mathfrak{M}_\bullet}^\tau$ is a complete and separable metric on $\mathfrak{M}_\bullet(\tau)$.*

By the above theorem, $\mathfrak{M}_\bullet(\tau)$ is Polish for many structures τ . On the other hand, some important structures do not satisfy semicontinuity, and hence the theorem does not apply. Even in such cases, however, the Polishness of $\mathfrak{M}_\bullet(\tau)$ can still be established via Alexandrov's theorem (cf. [75, Theorem 2.2.1]), which asserts that any topological space that is topologically embedded into a Polish space as a G_δ -subset is itself Polish. To this end, we introduce several additional notions on structure, which are required to formulate the notion of a *Polish structure* (Definition 3.14), but they will not be used in the subsequent arguments. The essential result needed for later purposes is Theorem 3.15.

Definition 3.13 ([62, Definition 5.12]). Recall that we have a fixed structure τ . Let $\tilde{\tau}$ be another structure. A *topological embedding* $\eta: \tau \Rightarrow \tilde{\tau}$ is a family $\{\eta_S: \tau(S) \rightarrow \tilde{\tau}(S)\}_S$ indexed by bcm spaces S , satisfying the following conditions:

- (TE1) each $\eta_S: \tau(S) \rightarrow \tilde{\tau}(S)$ is a topological embedding;
- (TE2) for any bcm spaces S_1 and S_2 and isometric embedding $f: S_1 \rightarrow S_2$, $\tilde{\tau}_f \circ \eta_{S_1} = \eta_{S_2} \circ \tau_f$.

As the terminology suggests, if there exists a topological embedding $\eta: \tau \Rightarrow \tilde{\tau}$, then the following map is itself a topological embedding:

$$\mathfrak{M}_\bullet(\tau) \ni (S, d_S, \rho_S, a_S) \mapsto (S, d_S, \rho_S, \eta_S(a_S)) \in \mathfrak{M}_\bullet(\tilde{\tau}),$$

see [62, Lemma 6.44].

We are now ready to introduce the notion of Polishness for τ .

Definition 3.14 ([62, Definition 6.45]). We say that τ is *Polish* if there exist another structure $\tilde{\tau}$, a topological embedding $\eta: \tau \Rightarrow \tilde{\tau}$, and, for each rooted bcm space (S, ρ_S) , a sequence $(\tilde{\tau}_k(S, \rho_S))_{k \geq 1}$ of open subsets of $\tilde{\tau}(S)$ satisfying the following conditions:

- (P1) The structure $\tilde{\tau}$ is continuous and separable, and admits a complete metrization.
- (P2) Let (S_1, ρ_{S_1}) and (S_2, ρ_{S_2}) be rooted bcm spaces. For every root-preserving isometric embedding $f: S_1 \rightarrow S_2$, one has $\tilde{\tau}_f^{-1}(\tilde{\tau}_k(S_2, \rho_{S_2})) = \tilde{\tau}_k(S_1, \rho_{S_1})$ for all $k \geq 1$.
- (P3) For each rooted bcm space (S, ρ_S) , $\eta_S(\tau(S)) = \bigcap_{k \geq 1} \tilde{\tau}_k(S, \rho_S)$.

Condition (P1) ensures that $\mathfrak{M}_\bullet(\tilde{\tau})$ is Polish, while Conditions (P2) and (P3) guarantee that $\mathfrak{M}_\bullet(\tau)$ is topologically embedded into $\mathfrak{M}_\bullet(\tilde{\tau})$ as a G_δ -subset. Hence, by Alexandrov's theorem, we obtain the following.

Theorem 3.15 ([62, Theorem 6.46]). *If τ is Polish, then the topology on $\mathfrak{M}_\bullet(\tau)$ is Polish. (NB. The metric $d_{\mathfrak{M}_\bullet}^\tau$ is not necessarily complete.)*

So far, we have considered bcm spaces. When one considers only compact underlying spaces, it is more natural to use the Hausdorff topology rather than the Fell topology to describe the convergence of the underlying spaces. In what follows, we briefly introduce a corresponding modification of the above framework. See [62, Appendix B] for details.

Recall that we have a fixed embedding-continuous structure τ that admits a metrization. We define $\mathfrak{K}_\bullet(\tau)$ as the collection of $(S, d_S, \rho_S, a_S) \in \mathfrak{M}_\bullet(\tau)$ such that S is compact. We define a metric on $\mathfrak{K}_\bullet(\tau)$ as follows: for $\mathcal{S}_1 = (S_1, d_{S_1}, \rho_{S_1}, a_{S_1})$ and $\mathcal{S}_2 = (S_2, d_{S_2}, \rho_{S_2}, a_{S_2})$ in $\mathfrak{K}_\bullet(\tau)$, set

$$d_{\mathfrak{K}_\bullet}^\tau(\mathcal{S}_1, \mathcal{S}_2) := \inf_{f, g, M} \left\{ d_M^H(f(S_1), g(S_2)) \vee d_{M, \rho_M}^\tau(\tau_f(a_{S_1}), \tau_g(a_{S_2})) \right\},$$

where the infimum is taken over all rooted compact metric spaces (M, ρ_M) and all root-preserving isometric embeddings $f: S_1 \rightarrow M$ and $g: S_2 \rightarrow M$.

In the above definition, the Hausdorff metric d_M^H is employed instead of the Fell metric. Accordingly, the characterization of convergence in $\mathfrak{K}_\bullet(\tau)$ is given in the same manner as in statement (ii) of Theorem 3.7, with the Fell topology replaced by the Hausdorff topology. It should also be noted that, if τ is Polish, then the resulting topology on $\mathfrak{K}_\bullet(\tau)$ is again Polish.

Although the main convergence results of this paper are stated for $\mathfrak{M}_\bullet(\tau)$, there are many situations where the compact setting $\mathfrak{K}_\bullet(\tau)$ is more natural or convenient for discussion. However, convergence in $\mathfrak{M}_\bullet(\tau)$ can easily be converted into convergence in $\mathfrak{K}_\bullet(\tau)$, thanks to the following proposition.

Proposition 3.16. *Let $\mathcal{S}_n = (S_n, d_{S_n}, \rho_{S_n}, a_{S_n})$, $n \in \mathbb{N} \cup \{\infty\}$, be elements of $\mathfrak{K}_\bullet(\tau)$. Then $\mathcal{S}_n \rightarrow \mathcal{S}_\infty$ in $\mathfrak{K}_\bullet(\tau)$ if and only if $\mathcal{S}_n \rightarrow \mathcal{S}_\infty$ in $\mathfrak{M}_\bullet(\tau)$ and*

$$\sup_{n \geq 1} \sup_{x \in S_n} d_{S_n}(\rho_{S_n}, x) < \infty. \quad (3.1)$$

Proof. Suppose $\mathcal{S}_n \rightarrow \mathcal{S}_\infty$ in $\mathfrak{M}_\bullet(\tau)$. Then we can embed all the spaces S_n , $n \in \mathbb{N} \cup \{\infty\}$, isometrically into a common rooted bcm space (M, ρ_M) so that the statement (ii) of Theorem 3.7 holds. In addition, if (3.1) is satisfied, then, by the precompactness criterion for the Hausdorff topology (cf. [76, Lemma 3.5]), we deduce that $S_n \rightarrow S$ in the Hausdorff topology as subsets of M . This implies that $\mathcal{S}_n \rightarrow \mathcal{S}_\infty$ in $\mathfrak{K}_\bullet(\tau)$. The converse direction is immediate since the Hausdorff topology is finer than the Fell topology. This completes the proof. $\square$

3.2 Product and composition

In this subsection, we introduce two operations on structures, namely product and composition, which allow us to treat a wide variety of additional structures. For further details, see [62, Section 7].

We first define the product of structures.

Definition 3.17 ([62, Definition 5.12]). Fix $N \in \mathbb{N}$ and structures $(\tau^{(n)})_{n=1}^N$. The product structure $\tau = \prod_{n=1}^N \tau^{(n)}$ is defined as follows:

- (i) For each bcm space S , set $\tau(S) := \prod_{n=1}^N \tau^{(n)}(S)$, equipped with the product topology.
- (ii) For each isometric embedding $f: S_1 \rightarrow S_2$ between bcm spaces, define $\tau_f := \prod_{n=1}^N \tau_f^{(n)}$, that is,

$$\tau_f((a_n)_{n=1}^N) := (\tau_f^{(n)}(a_n))_{n=1}^N.$$

When $\tau^{(1)} = \dots = \tau^{(n)} =: \sigma$, we write $\tau = \sigma^{\otimes n}$. If each $\tau^{(n)}$ admits a metrization, then we define a metrization of τ by equipping $\tau(S)$, for each rooted bcm space S , with the associated max product metric.

The following result shows that the product structure inherits the Polishness of its component structures.

Theorem 3.18 ([62, Theorem 7.5]). Fix $N \in \mathbb{N}$ and structures $(\tau^{(n)})_{n=1}^N$. If each τ_n is Polish, then so is $\prod_{n=1}^N \tau^{(n)}$.

It is not always possible to compose two structures on bcm spaces. In Definition 3.19 below we introduce a variant of structures that allows for such compositions. Consider a map Ψ satisfying the following conditions.

- (ST1) For every bcm space S , there exists a bcm space $\Psi(S)$.
- (ST2) For every isometric embedding $f: S_1 \rightarrow S_2$ between bcm spaces, there exists an isometric embedding $\Psi_f: \Psi(S_1) \rightarrow \Psi(S_2)$.
- (ST3) Let S_1, S_2, S_3 be bcm spaces. For any isometric embeddings $f: S_1 \rightarrow S_2$ and $g: S_2 \rightarrow S_3$, one has $\Psi_{g \circ f} = \Psi_g \circ \Psi_f$.
- (ST4) For any bcm space S , one has $\Psi_{\text{id}_S} = \text{id}_{\Psi(S)}$.

By forgetting the metric on $\Psi(S)$ for each S , we may regard Ψ as a structure on bcm spaces in the sense of Definition 3.1. Hence the notion of continuity introduced in Definition 3.11 applies to Ψ as well.

Definition 3.19 ([62, Definition 7.6]). Let Ψ satisfy (ST1)–(ST4). We call Ψ a *space transformation* if it is continuous in the sense of Definition 3.11, and if it satisfies the following additional condition:

- (ST5) There exists a collection $\mathbf{r} = \{\mathbf{r}_S: S \rightarrow \Psi(S)\}_S$ of topological embeddings indexed by bcm spaces S such that, for any isometric embedding $f: S_1 \rightarrow S_2$ between bcm spaces, $\Psi_f \circ \mathbf{r}_{S_1} = \mathbf{r}_{S_2} \circ f$.

We refer to $\mathbf{r}$ as a *rooting system* of Ψ .

The terminology “rooting system” reflects the role of $\mathbf{r}$ in assigning a canonical root to each transformed space. Specifically, when S is a rooted space with root ρ_S , we define the root of $\Psi(S)$ by $\mathbf{r}_S(\rho_S)$.

The following lemma is an immediate consequence of the lower semicontinuity of Ψ (see [62, Remark 6.36]), and will be used later.

Lemma 3.20. *Let Ψ be a space transformation. Let $S, S_1, S_2, \dots$ be bcm spaces that are isometrically embedded into a common bcm space M in such a way that $S_n \rightarrow S$ in the Fell topology as closed subsets of M . Then $\Psi(S_n) \rightarrow \Psi(S)$ in the Fell topology as closed subsets of $\Psi(M)$.*

We now define the composition of a structure with a space transformation.

Definition 3.21. Let τ be a structure and Ψ a space transformation with rooting system $\mathbf{r}$. The composition $\tau \circ \Psi = \tau(\Psi)$ is defined as follows:

- (i) For each bcm space S , set $(\tau \circ \Psi)(S) := \tau(\Psi(S))$.
- (ii) For each isometric embedding $f: S_1 \rightarrow S_2$ between bcm spaces, define $(\tau \circ \Psi)_f := \tau_{\Psi_f}$.

If τ admits a metrization, then for each rooted bcm space (S, ρ_S) , we equip $\tau(\Psi(S))$ with the metric $d_{\Psi(S), \mathbf{r}_S(\rho_S)}^\tau$.

Similarly to the case of product structures, composition also preserves Polishness.

Theorem 3.22 ([62, Theorem 7.14]). *Let τ be a structure and Ψ be a space transformation. If τ is Polish, then so is $\tau \circ \Psi$.*

Example 3.23. Here, we provide space transformations, used in this paper. Fix $k \in \mathbb{N}$ and a rooted bcm space (Ξ, ρ_Ξ) . We define a space transformation $\Psi = \Psi_{\text{id}^k \times \Xi}$ as follows.

- For each bcm space S , define $\Psi(S) := S^k \times \Xi$ equipped with the max product metric.
- For each isometric embedding $f: S_1 \rightarrow S_2$ between bcm spaces, define

$$\Psi_f := \underbrace{f \times \dots \times f}_{k \text{ times}} \times \text{id}_\Xi.$$

Its rooting system $\mathbf{r}$ is given by $\mathbf{r}_X(x) = (x, \dots, x, \rho_\Xi) \in X^k \times \Xi$. Similarly, we define space transformations $\Psi_{\Xi \times \text{id}^k}$ and Ψ_{id^k} such that $\Psi_{\Xi \times \text{id}^k}(S) := \Xi \times S^k$ and $\Psi_{\text{id}^k}(S) = S^k$ for each bcm space S . When $k = 1$, we simply write $\Psi_{\text{id} \times \Xi} = \Psi_{\text{id}^1 \times \Xi}$, $\Psi_{\Xi \times \text{id}} = \Psi_{\Xi \times \text{id}^1}$, and $\Psi_{\text{id}} = \Psi_{\text{id}^1}$.

3.3 Structures used in the present paper

Here we list the structures that will be used in the present paper. We note that all the structures, except for $\tau_{\mathcal{M}(*, \text{fin})}$ defined in (S4) below, have already been introduced in [62].

(S1) **Fixed structure.** Fix a Polish space Ξ . We define a structure $\tau = \tau_\Xi$ as follows.

- For each bcm space S , define $\tau(S) := \Xi$.
- For each isometric embedding $f: S_1 \rightarrow S_2$ between bcm spaces, define $\tau_f := \text{id}_\Xi$.

Let d_Ξ be a metric on Ξ inducing the given topology. We define a metrization of τ by equipping $\tau(S) = \Xi$ with the metric d_Ξ .

(S2) **Point.** We define a structure $\tau = \tau_{\text{id}}$ as follows.

- For each bcm space S , define $\tau(S) := S$.
- For each isometric embedding $f: S_1 \rightarrow S_2$ between bcm spaces, define $\tau_f := f$.

We define a metrization of τ by equipping $\tau(S) = S$ with the associated metric d_S .

(S3) **Measure.** We define a structure $\tau = \tau_{\mathcal{M}}$ as follows.

- For each bcm space S , set $\tau(S) := \mathcal{M}(S)$ equipped with the vague topology.

- For each isometric embedding $f: S_1 \rightarrow S_2$ between **bcm** spaces, set $\tau_f(\mu) := \mu \circ f^{-1}$ for each $\mu \in \mathcal{M}(S_1)$, i.e., $\tau_f(\mu)$ is the pushforward of μ by f .

We define a metrization of τ by equipping, for each rooted **bcm** space (S, ρ_S) , $\tau(S) = \mathcal{M}(S)$ with the metric d_{S, ρ_S}^V , as recalled from (2.2).

(S4) **STOM structure.** We define a structure $\tau = \tau_{\mathcal{M}_{(*, \text{fin})}}$ as follows.

- For each **bcm** space S , define $\tau(S) := \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$.
- For each isometric embedding $f: S_1 \rightarrow S_2$ between **bcm** spaces, set $\tau_f(\Pi) := \Pi \circ (f \times \text{id}_{\mathbb{R}_{\geq 0}})^{-1}$ for each $\Pi \in \mathcal{M}_{(*, \text{fin})}(S_1 \times \mathbb{R}_{\geq 0})$, i.e., $\tau_f(\Pi)$ is the pushforward of Π by $f \times \text{id}_{\mathbb{R}_{\geq 0}}$.

Checking conditions in Definition 3.1 is straightforward, and thus $\tau_{\mathcal{M}_{(*, \text{fin})}}$ is indeed a structure. We define a metrization of τ by equipping $\tau(S) = \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$ with the metric $d_{S \times \mathbb{R}_{\geq 0}}^{\mathcal{M}_{(*, \text{fin})}}$, as recalled from (2.10). We note that, by Lemma 2.21, this indeed defines a metrization in the sense of Definition 3.3.

(S5) **Càdlàg functions.** We define a structure $\tau = \tau_{J_1}$ as follows.

- For each **bcm** space S , set $\tau(S) := D_{J_1}(\mathbb{R}_{\geq 0}, S)$ equipped with the J_1 -Skorohod topology.
- For each isometric embedding $f: S_1 \rightarrow S_2$ between **bcm** spaces, set $\tau_f(\xi) := f \circ \xi$ for each $\xi \in D_{J_1}(\mathbb{R}_{\geq 0}, S_1)$.

We define a metrization of τ by equipping $\tau(S) = D_{J_1}(\mathbb{R}_{\geq 0}, S)$ with the Skorohod metric $d_S^{J_1}$, as recalled from (2.6).

(S6) **Measurable functions.** We define a structure $\tau = \tau_{L^0}$ as follows.

- For each **bcm** space S , define $\tau(S) := L^0(\mathbb{R}_{\geq 0}, S)$.
- For each isometric embedding $f: S_1 \rightarrow S_2$ between **bcm** spaces, define $\tau_f(\xi) := f \circ \xi$ for each $\xi \in L^0(\mathbb{R}_{\geq 0}, S_1)$.

We define a metrization of τ by equipping $\tau(S) = L^0(\mathbb{R}_{\geq 0}, S)$ with the metric $d_S^{L^0}$, as recalled from (2.7).

(S7) **Continuous functions with space-dependent domains.** Let Ψ be space transformation and σ be a Polish structure. We define a structure $\tau = \tau_{\widehat{C}}(\Psi, \sigma)$ as follows.

- For each **bcm** space S , define $\tau(S) := \widehat{C}(\Psi(S), \sigma(S))$.
- For each isometric embedding $f: S_1 \rightarrow S_2$ between **bcm** spaces, define $\tau_f(F) := \sigma_f \circ F \circ \Psi_f^{-1}$ with $\text{dom}(\tau_f(F)) := \Psi_f(\text{dom}(F))$ for $F \in \tau(S_1)$.

We define a metrization of τ by equipping, for each rooted **bcm** space (S, ρ_S) , $\tau(S) = \widehat{C}(\Psi(S), \sigma(S))$ with the metric $d_{(\Psi(S), \tau(\rho_S)), \Xi}^{\widehat{C}}$, as recalled from (2.5).

(S8) **Law of structure.** Let σ be a Polish structure. We define a structure $\tau = \tau_{\mathcal{P}}(\sigma)$ as follows.

- For each **bcm** space S , define $\tau(S) := \mathcal{P}(\sigma(S))$, i.e., the set of probability measures on $\sigma(S)$ equipped with the weak topology.
- For each isometric embedding $f: S_1 \rightarrow S_2$ between **bcm** spaces, define $\tau_f(P) := P \circ \sigma_f^{-1}$, i.e., the pushforward of P by σ_f , for each $P \in \mathcal{P}(\sigma(S_1))$.

We define a metrization of τ by equipping, for each rooted **bcm** space (S, ρ_S) , $\tau(S) = \mathcal{P}(\sigma(S))$ with the Prohorov metric induced by the metric d_{S, ρ_S}^τ on $\sigma(S)$.

Proposition 3.24. *All the structures introduced above are Polish.*

Proof. It suffices to show that $\tau_{\mathcal{M}_{(*, \text{fin})}}$ is Polish, since the Polishness of the other structures was established in [62, Section 8]. The continuity of the embedding can be readily verified by the continuous mapping theorem. Moreover, the semicontinuity property can be proved by following the argument used for $\tau_{\mathcal{M}}$ in [62, Theorem 8.9]. Combining these facts with Proposition 2.20, we deduce that $\tau_{\mathcal{M}_{(*, \text{fin})}}$ is separable and continuous, and the associated metrization is complete. Hence $\tau_{\mathcal{M}_{(*, \text{fin})}}$ is Polish, as desired. $\square$

4 Dual processes, PCAFs, and smooth measures

In this section we introduce the class of processes studied in this paper and summarize basic results on their positive continuous additive functionals (PCAFs) in relation to smooth measures. Throughout this section, we fix a locally compact separable metrizable topological space S . We write $S_\Delta = S \cup \{\Delta\}$ for the one-point compactification of S . (NB. If S is compact, then we add Δ to S as an isolated point.) Any $[-\infty, \infty]$ -valued function f defined on S is regarded as a function on S_Δ by setting $f(\Delta) := 0$.

4.1 Dual processes

Here we introduce standard processes admitting dual processes and heat kernels, together with related notation used throughout this paper. For further background, see [18], for example.

Let

$$X = (\Omega^X, \mathcal{M}^X, (X_t)_{t \in [0, \infty)}, (P_x^X)_{x \in S_\Delta}, (\theta_t^X)_{t \in [0, \infty)})$$

be a standard process on S (see [18, Definition 9.2 in Chapter I] or [22, Definition A.1.23(i)]). Here $(\Omega^X, \mathcal{M}^X)$ denotes the underlying measurable space, and θ_t^X the shift operator, that is, the map $\theta_t^X: \Omega^X \rightarrow \Omega^X$ satisfying $X_s \circ \theta_t^X = X_{s+t}$ for all $s \in [0, \infty]$. Note that $X_\infty(\omega) = \Delta$ for every $\omega \in \Omega^X$. We write $\mathcal{F}_*^X = (\mathcal{F}_t^X)_{t \in [0, \infty]}$ for the minimal augmented admissible filtration (see [22, p. 397]), and set

$$\zeta^X := \inf\{t \in [0, \infty] \mid X_t = \Delta\}$$

for the *lifetime* of X . We say that X is *conservative* if

$$P^x(\zeta^X = \infty) = 1, \quad \forall x \in S. \quad (4.1)$$

We often suppress the subscript or superscript X when it is clear from the context.

Throughout this section, we assume the existence of a dual process and a heat kernel, as specified below. We will often refer to this assumption as the *duality and absolute continuity condition* (DAC condition, for short).

Assumption 4.1 (The DAC condition). *There exist a standard process $\check{X}$, a σ -finite Borel measure m on S , and a Borel measurable function $p = p_X: (0, \infty) \times S \times S \rightarrow [0, \infty]$ satisfying the following.*

(HK1) *For all $t > 0$ and $x \in S$, $P^x(X_t \in dy) = p(t, x, y) m(dy)$.*

(HK2) *For all $t > 0$ and $y \in S$, $P^y(\check{X}_t \in dx) = \check{p}(t, y, x) m(dx)$, where $\check{p}(t, y, x) := p(t, x, y)$.*

(HK3) *For any $s, t > 0$ and $x, y \in S$,*

$$p(t + s, x, y) = \int_S p(t, x, z) p(s, z, y) m(dz). \quad (4.2)$$

We recall from [18, Corollary 1.12 and Remark 1.13, Chapter VI] that m is a 0-excessive reference measure of X (in the sense of [18, Definitions 1.1, Chapter V and 1.10, Chapter VI]). Hence, under Assumption 4.1, the setting of [72] applies, and we may invoke the results therein.

Remark 4.2. The process $\check{X}$ is called a *dual process* of X , as it satisfies the following: for any non-negative measurable functions f and g on S , and any $t \geq 0$,

$$\int_S E_x[f(X_t)] g(x) m(dx) = \int_S f(x) E_x[g(\check{X}_t)] m(dx).$$

It is known that, if the above identity holds and the semigroups of X and $\check{X}$ are absolutely continuous with respect to a common σ -finite Borel measure, then the heat kernel exists (see [81, Theorem 1]).

We refer to p as the *heat kernel* of X with respect to m . For each $\alpha \geq 0$, we then define the α -potential (or resolvent) density $r^\alpha = r_p^\alpha: S \times S \rightarrow [0, \infty]$ by

$$r^\alpha(x, y) := \int_0^\infty e^{-\alpha t} p(t, x, y) dt.$$

Given a Borel measure ν on S , we define, the α -potential of ν by setting

$$\mathcal{R}^\alpha \nu(x) = \mathcal{R}_p^\alpha \nu(x) := \int_S r^\alpha(x, y) \nu(dy), \quad x \in S.$$

For a Borel subset $A \in \mathcal{B}(S_\Delta)$, we denote by $\sigma_A = \sigma_A^X$ and $\check{\sigma}_A = \check{\sigma}_A^X$ the first hitting and exit times of X to A , respectively, i.e.,

$$\begin{aligned} \sigma_A &= \sigma_A^X := \inf\{t > 0 \mid X_t \in A\}, \\ \check{\sigma}_A &= \check{\sigma}_A^X := \inf\{t > 0 \mid X_t \notin A\}. \end{aligned} \tag{4.3}$$

Below, we recall some classes of exceptional sets.

Definition 4.3 ([18, Definition 3.1 in Chapter II]). Let A be a Borel subset of S .

- (i) The set A is called *polar* (for X) if and only if $P^x(\sigma_A < \infty) = 0$ for all $x \in S$.
- (ii) The set A is called *thin* (for X) if and only if $P^x(\sigma_A = 0) = 0$ for all $x \in S$.
- (iii) The set A is called *semipolar* (for X) if and only if it is contained in a countable union of thin sets.

Remark 4.4. The above definition of semipolarity is slightly different from that in [18, Definition 3.1, Chapter II], where a set is called semipolar if it is contained in a countable union of *nearly Borel* thin sets. However, in the present setting, every nearly Borel thin set is contained in a Borel thin set (see the proof of [18, Corollary 1.15, Chapter V]). Consequently, the two definitions of semipolarity for Borel subsets coincide.

When T is a separable metric space and $Y: (\Omega, \mathcal{F}_\infty) \rightarrow (T, \mathcal{T})$ is measurable, we define the map $\mathcal{L}_Y: S \rightarrow \mathcal{P}(T)$ by

$$\mathcal{L}_Y(x)(\cdot) := P^x(Y \in \cdot). \tag{4.4}$$

We refer to this as the *law map* of Y . For example, $\mathcal{L}_X(x)$ denotes the law of X starting from x .

4.2 Smooth measures

In this subsection, we first recall the definition of smooth measures, and then introduce several classes of smooth measures that play an important role in our study. We also present some auxiliary results on smooth measures that will be used later. Our exposition mainly follows Revuz's framework [72]. For a more potential-theoretic perspective, see for example [15, 22, 32]. We work within the framework introduced in the previous subsection. In particular, the DAC condition (Assumption 4.1) is in force.

Definition 4.5 (Smooth measure, [72, Theorem VI.1]). A Borel measure μ on S is said to be a *smooth measure* of X if it satisfies the following conditions:

- (i) μ charges no semipolar sets, i.e., $\mu(E) = 0$ for any semipolar Borel subset E of S ;
- (ii) there exists an increasing sequence $(E_n)_{n \geq 1}$ of Borel subsets of S , called a *nest* of μ , such that $E = \bigcup_{n \geq 1} E_n$, $\mu(E_n) < \infty$ for each n ,

$$\|\mathcal{R}^1(\mu|_{E_n})\|_\infty = \sup_{x \in S} \int_{E_n} r^1(x, y) \mu(dy) < \infty, \quad \forall n \geq 1,$$

and $P^x(\lim_{n \rightarrow \infty} \check{\sigma}_{E_n} \geq \zeta) = 1$ for all $x \in S$.

In many examples, the following condition, known as *Hunt's hypothesis*, holds.

(H) Every semipolar set is polar.

For instance, if X is a Hunt process associated with a symmetric regular Dirichlet form, then condition (H) holds (by the DAC condition and [32, Theorems 4.1.2 and 4.1.3]). Under (H), condition (i) in Definition 4.5 is implied by condition (ii). This follows from the next result. The proof is given in Appendix A.1.

Proposition 4.6. *Let μ be a Borel measure on S . If $\|\mathcal{R}^\alpha \mu\|_\infty < \infty$ for some $\alpha > 0$, then μ charges no polar sets.*

Remark 4.7. The above is well known under the assumption that all α -excessive functions are lower semicontinuous (see the comments following Proposition 4.3 in Chapter VI of [18]).

Below, we introduce two important classes of smooth measures.

Definition 4.8. A finite Borel measure μ on S is said to be *in the Kato class (of X)* if and only if the following conditions are satisfied.

- (i) The Radon measure μ charges no semipolar sets.
- (ii) It holds that $\lim_{\alpha \rightarrow \infty} \|\mathcal{R}^\alpha \mu\|_\infty = 0$.

A Borel measure μ on S is said to be *in the local Kato class (of X)* if, for all compact subset K of S , the restriction $\mu|_K$ is in the Kato class. (In particular, μ is a Radon measure.)

Clearly, the local Kato class is wider than the Kato class. It is easy to verify that a Radon measure in the local Kato class is a smooth measure (cf. [58, Proposition 2.6]). These classes impose conditions on the decay of α -potentials as $\alpha \rightarrow \infty$. This corresponds to imposing a constraint on the short-time behavior of the heat kernel with respect to the measure in question. More precisely, the following result holds.

Proposition 4.9. *Let μ be a Borel measure on S . For any open subset D of S and $t > 0$,*

$$\begin{aligned} \frac{1}{e} \sup_{x \in S} \int_0^t \int_D p(s, x, y) \mu(dy) ds &\leq \|\mathcal{R}^{1/t}(\mu|_D)\|_\infty \\ &\leq \frac{1}{1 - e^{-1}} \sup_{x \in \text{cl}(D)} \int_0^t \int_D p(s, x, y) \mu(dy) ds, \end{aligned} \quad (4.5)$$

where $\text{cl}(D)$ denotes the closure of D in S . As a consequence, μ belongs to the local Kato class if and only if it charges no semipolar sets and, for all compact subsets K of S ,

$$\limsup_{t \rightarrow 0} \sup_{x \in K} \int_0^t \int_K p(s, x, y) \mu(dy) ds = 0.$$

To prove the above result, we use the following lemma, which is an immediate consequence of the Dynkin–Hunt formula in its heat-kernel version.

Lemma 4.10. *Let D be an open subset of S . It then holds that, for each $x \in S$, $y \in D$, and $t > 0$,*

$$p(t, x, y) = E^x[p(t - \sigma_D, X_{\sigma_D}, y) \cdot 1_{\{\sigma_D < t\}}].$$

Proof. We apply [34, Corollary 3.14], taking the hitting time σ_D as the terminal time in that corollary. Since D is open, every point of D is regular for σ_D , i.e., $P^x(\sigma_D = 0) = 1$ for all $x \in D$. Therefore, by [34, Corollary 3.14 and Remark 3.15], we obtain the desired result. $\square$

Proof of Proposition 4.9. Following the argument in [58, Equation (2.10)], one can verify that

$$\|\mathcal{R}^{1/t}(\mu|_D)\|_\infty \leq \frac{1}{1 - e^{-1}} \sup_{x \in S} \int_0^t \int_D p(s, x, y) \mu(dy) ds. \quad (4.6)$$

It follows from Lemma 4.10 that

$$\begin{aligned} \int_0^t \int_D p(s, x, y) \mu(dy) ds &= E_x \left[\int_0^t \int_D p(s - \sigma_D, X_{\sigma_D}, y) \cdot 1_{\{\sigma_D < s\}} \mu(dy) ds \right] \\ &= E_x \left[\int_0^{t - \sigma_D} \int_D p(s, X_{\sigma_D}, y) \mu(dy) ds \right] \\ &\leq \sup_{z \in \text{cl}(D)} \int_0^t \int_D p(s, z, y) \mu(dy) ds. \end{aligned}$$

This, combined with (4.6), yields that the second inequality in (4.5) holds. The other inequality follows from the calculation

$$\int_0^t \int_D p(s, x, y) \mu(dy) ds \leq e \int_0^t \int_D e^{-s/t} p(s, x, y) \mu(dy) ds \leq e \|\mathcal{R}^{1/t}(\mu|_D)\|_\infty.$$

$\square$

In the discrete space setting, any Radon measure belongs to the local Kato class, as shown below.

Proposition 4.11. *Suppose that S is finite or countably infinite, endowed with the discrete topology. Then any Radon measure μ on S is in the local Kato class of X .*

Proof. By the right-continuity of X , it follows that $P^x(\sigma_{\{x\}} = 0) = 1$ for any $x \in S$. Hence the empty set is the only semipolar set, and thus μ charges no semipolar set.

Observe that, for each $x, y \in S$ and $t > 0$,

$$p(t, x, y) = \frac{P^x(X_t = y)}{m(\{y\})} \leq \frac{1}{m(\{y\})}.$$

(Note that $m(\{y\}) > 0$ since m is a reference measure.) This yields

$$\sup_{x \in S} r^\alpha(x, y) \leq \frac{1}{\alpha m(\{y\})}, \quad \forall y \in S.$$

Therefore, for any compact subset K of S ,

$$\|\mathcal{R}^\alpha(\mu|_K)\|_\infty \leq \frac{1}{\alpha} \sum_{y \in K} \frac{\mu(\{y\})}{m(\{y\})}.$$

Because K is finite, the right-hand side converges to 0 as $\alpha \rightarrow \infty$, which establishes the desired result. $\square$

4.3 PCAFs, STOMs, and Revuz correspondence

In this subsection, we introduce PCAFs and associated STOMs, and then recall the Revuz correspondence [72]. Some basic results on STOMs are also presented. For a comprehensive theory of PCAFs, see [18]. For a potential-theoretic study of them in relation to symmetric Dirichlet forms, see [22, 32]. We proceed within the same framework as in the previous subsection.

Since several versions of the definitions of PCAFs exist, we clarify the definitions adopted in this paper, below.

Definition 4.12 (PCAF, cf. [32, p. 222 and 235]). Let $A = (A_t)_{t \geq 0}$ be an $\mathcal{F}_*$ -adapted non-negative stochastic process. It is called a *positive continuous additive functional (PCAF)* of X if there exists a set $\Lambda \in \mathcal{F}_\infty$, called a *defining set* of A , satisfying the following.

- (i) It holds that $P^x(\Lambda) = 1$ for all $x \in S_\Delta$.
- (ii) For every $t \in [0, \infty)$, $\theta_t(\Lambda) \subseteq \Lambda$.
- (iii) For every $\omega \in \Lambda$, $A_0(\omega) = 0$, the function $[0, \infty) \ni t \mapsto A_t(\omega) \in [0, \infty]$ is continuous, $A_t(\omega) < \infty$ for all $t < \zeta(\omega)$, $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for all $t \geq \zeta(\omega)$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$ for all $s, t \in [0, \infty)$.

We say that two PCAFs $A = (A_t)_{t \geq 0}$ and $B = (B_t)_{t \geq 0}$ are *equivalent* if there exists a common defining set $\Lambda \in \mathcal{F}_\infty$ such that $P^x(\Lambda) = 1$ for all $x \in S_\Delta$ and $A_t(\omega) = B_t(\omega)$ for all $t \geq 0$ and $\omega \in \Lambda$.

Remark 4.13. By [42, Remark 2.5], for any PCAF A with a defining set Λ , there exists an equivalent PCAF A' with defining set Λ' such that $A_t(\omega) = 0$ for all $t \geq 0$ and $\omega \notin \Lambda'$, and $\{\zeta = 0\} \subseteq \Lambda'$. Thus, in this paper, whenever we consider a PCAF A with a defining set Λ , we shall always assume that A and Λ satisfy this property.

To define STOMs, we introduce a measurable space consisting of Borel measures. Let $\mathcal{M}_{\text{Bor}}(S \times \mathbb{R}_{\geq 0})$ be the set of Borel measures on $S \times \mathbb{R}_{\geq 0}$. We equip $\mathcal{M}_{\text{Bor}}(S \times \mathbb{R}_{\geq 0})$ with the σ -algebra generated by the family of evaluation maps $\pi_E: \mathcal{M}_{\text{Bor}}(S \times \mathbb{R}_{\geq 0}) \rightarrow \mathbb{R}_{\geq 0}$, $E \in \mathcal{B}(S \times \mathbb{R}_{\geq 0})$, defined by $\pi_E(\Pi) := \Pi(E)$.

Definition 4.14 (STOM). For a PCAF $A = (A_t)_{t \geq 0}$, we define a measurable map $\Pi: (\Omega, \mathcal{F}_*) \rightarrow \mathcal{M}_{\text{Bor}}(S \times \mathbb{R}_{\geq 0})$ by

$$\Pi(\omega)(\cdot) := \int_0^\infty \mathbf{1}_{(X_t(\omega), t)}(\cdot) dA_t(\omega).$$

We refer to Π as the *space-time occupation measure* associated with A .

Remark 4.15. When X is conservative, i.e., $P^x(\zeta = \infty) = 1$ for all $x \in S$, we may assume that Π is a measurable map from Ω to $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$.

We now recall a fundamental result, known as the *Revuz correspondence*, which provides a one-to-one correspondence between PCAFs and smooth measures.

Theorem 4.16 ([72, Proposition V.1 and Theorem VI.1]). *There exists a one-to-one correspondence between the set of smooth measures and the set of equivalence classes of PCAFs. A smooth measure μ and a PCAF $A = (A_t)_{t \geq 0}$ correspond to each other if and only if for every $\alpha > 0$ and every non-negative Borel measurable function f on S ,*

$$E^x \left[\int_0^\infty e^{-\alpha t} f(X_t) dA_t \right] = \int_S r^\alpha(x, y) f(y) \mu(dy), \quad \forall x \in S.$$

Whenever we say that a PCAF A is associated with a smooth measure μ , we mean that they are associated via the Revuz correspondence. Through this correspondence, the corresponding notions for smooth measures also apply to PCAFs, and vice versa. For example, if μ belongs to the (local) Kato class, then we also say that A belongs to the (local) Kato class, and the STOM Π associated with A is then referred to as the STOM associated with μ .

Remark 4.17. Revuz's definition of PCAFs [72, Section I.3] is somewhat weak, and in particular does not guarantee the existence of a defining set. However, it is not difficult to modify his PCAF so that it satisfies the conditions of our definition. Indeed, by [18, Theorem 2.1 in Chapter V], his PCAF can be assumed to be a so-called perfect PCAF. The definition of perfect PCAFs consists of conditions (i) and (iii) of Definition 4.12 (see [18, Definition 1.3 in Chapter IV] for the precise formulation). Moreover, one can easily verify that the defining set of a perfect PCAF, as constructed in the proof of [18, Theorem 3.16 in Chapter IV], also satisfies condition (ii) of Definition 4.12. Thus, a perfect PCAF may be regarded as a PCAF in our sense.

Example 4.18. For any bounded Borel measurable function $f: S \rightarrow \mathbb{R}_{\geq 0}$, the stochastic process $A_t = \int_0^t f(X_s) ds$ is a PCAF and its corresponding smooth measure is $f(x) m(dx)$ (cf. [22, Theorem A.3.5(iii)]). When the Dirac measure δ_x at x is smooth, then the corresponding PCAF is a local time of X at x in the sense of [18, Definition 3.12 in Chapter V].

Below, we present some results (or applications) of [42, 65]. We note that, although the scope of these is restricted to Hunt processes associated with symmetric regular Dirichlet forms, the arguments still apply in the present setting.

The following estimate of the difference between PCAFs in terms of their potentials is crucial in the proof of our main results.

Lemma 4.19 ([65, Theorem 1.6]). *Let μ and ν be finite smooth measures such that $\|\mathcal{R}^1\mu\|_\infty < \infty$ and $\|\mathcal{R}^1\nu\|_\infty < \infty$. Let A and B be the associated PCAFs, respectively. It then holds that, for any $\alpha, T > 0$,*

$$\sup_{x \in S} E_x \left[\sup_{0 \leq t \leq T} |A_t - B_t|^2 \right] \leq 18(\|\mathcal{R}^\alpha\mu\|_\infty + \|\mathcal{R}^\alpha\nu\|_\infty) \|\mathcal{R}^\alpha\mu - \mathcal{R}^\alpha\nu\|_\infty + 4e^{2T}(1 - e^{-\alpha T})(\|\mathcal{R}^1\mu\|_\infty^2 + \|\mathcal{R}^1\nu\|_\infty^2).$$

The following is an application of a generalized Kac's moment formula for PCAFs established in [42].

Proposition 4.20. *Let μ be a smooth measure and Π be the associated STOM. Fix an arbitrary Borel measurable function $f: S \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ and $x \in S_\Delta$. For notational convenience, write*

$$\Pi(f) := \int_0^\infty \int_S f(y, t) \Pi(dy dt) = \int_0^\infty f(X_t, t) dA_t, \quad (4.7)$$

and $I_{\nearrow}^k = \{(t_1, \dots, t_k) \mid 0 < t_1 < \dots < t_k < \infty\}$ for each $k \in \mathbb{N}$. Then it holds that, for any $k \in \mathbb{N}$ and $x_0 \in S$,

$$E^{x_0} [\Pi(f)^k] = k! \int_{I_{\nearrow}^k} \int_{S^k} \prod_{i=1}^k (p(t_i - t_{i-1}, x_{i-1}, x_i) f(x_i, t_i)) \mu^{\otimes k}(dx) dt_1 \cdots dt_k,$$

where we set $t_0 := 0$. In particular

$$E^{x_0} [e^{\Pi(f)}] = 1 + \sum_{k=1}^\infty \int_{I_{\nearrow}^k} \int_{S^k} \prod_{i=1}^k (p(t_i - t_{i-1}, x_{i-1}, x_i) f(x_i, t_i)) \mu^{\otimes k}(dx) dt_1 \cdots dt_k.$$

Proof. This is a straightforward application of [42, Theorem 2.7]. Indeed, since we have

$$E^{x_0} [\Pi(f)^k] = k! E^{x_0} \left[\int_{I_{\nearrow}^k} \prod_{i=1}^k f(X_{t_i}, t_i) dA_{t_1} \cdots dA_{t_k} \right],$$

we can follow the same line as in the proof of [42, Theorem 2.9] to obtain (4.7). We thus omit the details. $\square$

The above proposition yields the following corollary on the support of Π (cf. [43, Corollary 2.5]). This can be also obtained from the fact that a PCAF increases only when the process hits the support of the associated smooth measure (cf. [18, Theorem 3.8 in Chapter V]).

Corollary 4.21. *Let μ be a smooth measure and Π be the associated STOM. For each $x \in S_\Delta$, the STOM Π is supported on $\text{supp}(\mu) \times \mathbb{R}_{\geq 0}$, P^x -a.s. That is,*

$$\text{supp}(\Pi) \subseteq \text{supp}(\mu) \times \mathbb{R}_{\geq 0}, \quad P^x\text{-a.s.}$$

Equivalently, for all $x \in S_\Delta$,

$$\int_0^t \mathbf{1}_{\text{supp}(\mu)}(X_s) dA_s = A_t, \quad \forall t \geq 0, \quad P^x\text{-a.s.}$$

When the state space S is discrete, the STOM associated with a Radon measure admits an explicit description, as follows. Recall from Proposition 4.11 that, in this case, every Radon measure belongs to the local Kato class, so the associated STOM is well defined.

Proposition 4.22. *Suppose that S is finite or countable, endowed with the discrete topology. Fix a Radon measure μ on S . Then the STOM Π associated with μ can be represented as follows:*

$$\Pi(dx dt) = \frac{1}{m(\{x\})} \mathbf{1}_{\{X_t=x\}} \mu(dx) dt.$$

(NB. Since m is a reference measure and $\{x\}$ is open, the denominator $m(\{x\})$ is strictly positive.)

Proof. If we set $f(x) := \mu(\{x\})/m(\{x\})$ for each $x \in S$, then $\mu(dx) = f(x) m(dx)$. Thus, the PCAF associated with μ is given by

$$A_t := \int_0^t \frac{\mu(\{X_s\})}{m(\{X_s\})} ds = \sum_{x \in S} \int_0^t \frac{\mu(\{x\})}{m(\{x\})} \mathbf{1}_{\{X_s=x\}} ds$$

(see Example 4.18). By the definition of STOMs, we obtain the desired result. $\square$

5 Approximation of PCAFs and STOMs

In Sections 5.1 and 5.2, we establish methods for approximating PCAFs and STOMs by those that are absolutely continuous. This construction plays a crucial role in the proofs of our main results. In the following two subsections, we apply these approximation results to show that the joint law of the process and its PCAFs/STOMs is continuous in the starting point, provided that the law of the process itself is continuous.

We work in the same setting as in Section 4.2. Throughout this section, we fix a boundedly compact metric d_S on S that induces the given topology (see [80, Theorem 1] for the existence of such a metric), and we fix an element $\rho \in S$ as the root of S . In particular, (S, d_S, ρ) is a rooted bcm space.

5.1 The approximation scheme for PCAFs

We first present an approximation result for PCAFs in the Kato class, see Proposition 5.1 below, and then extend the method to PCAFs in the local Kato class, see Propositions 5.3 and 5.5.

We start by considering PCAFs in the Kato class. The approximation is constructed by using the heat kernel p as a mollifier, which is inspired by approximations for mutual intersection measures [57, Definition 1.3]. The validity of this approximation will be justified by an application of Lemma 4.19.

Proposition 5.1. *Let μ be a finite Borel measure in the Kato class and let $A = (A_t)_{t \geq 0}$ be the associated PCAF. For each $\delta > 0$, define*

$$A_t^{(\delta,*)} = \frac{1}{\delta} \int_0^t \int_S \int_\delta^{2\delta} p(u, X_s, z) du \mu(dz) ds, \quad t \geq 0. \quad (5.1)$$

Then there exist universal constants $C_1, C_2 > 0$ such that the following holds: for any $T \in [0, \infty)$, $\alpha \in (0, 1)$, and $\delta \in (0, 1)$,

$$\begin{aligned} & \sup_{x \in S} E^x \left[\sup_{0 \leq t \leq T} |A_t^{(\delta,*)} - A_t|^2 \right] \\ & \leq C_1 \|\mathcal{R}^\alpha \mu\|_\infty \left(\delta \alpha \|\mathcal{R}^\alpha \mu\|_\infty + \|\mathcal{R}^{1/\delta} \mu\|_\infty \right) + C_2 e^{2T} (1 - e^{-\alpha T}) \|\mathcal{R}^1 \mu\|_\infty^2. \end{aligned} \quad (5.2)$$

In particular, for all $T > 0$,

$$\lim_{\delta \rightarrow 0} \sup_{x \in S} E^x \left[\sup_{0 \leq t \leq T} |A_t^{(\delta,*)} - A_t|^2 \right] = 0. \quad (5.3)$$

Consequently, $A^{(\delta,)} \xrightarrow[\delta \rightarrow 0]{P} A$ in $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ under P^x for each $x \in S$.*

Proof. Define

$$f(y) := \frac{1}{\delta} \int_S \int_\delta^{2\delta} p(s, y, z) ds \mu(dz), \quad y \in S.$$

Since μ is in the Kato class, Proposition 4.9 yields that f is bounded. Hence, by Example 4.18, we deduce that $A^{(\delta,*)}$ is a PCAF and the associated smooth measure is

$$\mu_\delta(dy) := f(y) m(dy) = \left(\frac{1}{\delta} \int_S \int_\delta^{2\delta} p(s, y, z) ds \mu(dz) \right) m(dy).$$

Using the Chapman–Kolmogorov equation (4.2), we obtain that

$$\begin{aligned} \mathcal{R}^\alpha \mu_\delta(x) &= \int_S r^\alpha(x, y) \mu_\delta(dy) \\ &= \int_S \int_0^\infty e^{-\alpha t} p(t, x, y) dt \left(\frac{1}{\delta} \int_S \int_\delta^{2\delta} p(s, y, z) ds \mu(dz) \right) m(dy) \\ &= \frac{1}{\delta} \int_\delta^{2\delta} \int_S \int_0^\infty e^{-\alpha t} p(t + s, x, z) dt \mu(dz) ds \\ &= \frac{1}{\delta} \int_\delta^{2\delta} e^{\alpha s} \int_S \int_s^\infty e^{-\alpha t} p(t, x, z) dt \mu(dz) ds. \end{aligned}$$

By definition,

$$\int_S \int_s^\infty e^{-\alpha t} p(t, x, z) dt \mu(dz) = \mathcal{R}^\alpha \mu(x) - \int_S \int_0^s e^{-\alpha t} p(t, x, z) dt \mu(dz).$$

Hence,

$$\mathcal{R}^\alpha \mu_\delta(x) = \frac{e^{2\delta\alpha} - e^{\delta\alpha}}{\delta\alpha} \mathcal{R}^\alpha \mu(x) - \frac{1}{\delta} \int_\delta^{2\delta} e^{\alpha s} \int_S \int_0^s e^{-\alpha t} p(t, x, z) dt \mu(dz) ds. \quad (5.4)$$

It follows that

$$\begin{aligned} & \|\mathcal{R}^\alpha \mu_\delta - \mathcal{R}^\alpha \mu\|_\infty \\ & \leq \left| \frac{e^{2\delta\alpha} - e^{\delta\alpha}}{\delta\alpha} - 1 \right| \|\mathcal{R}^\alpha \mu\|_\infty + \frac{e^{2\delta\alpha} - e^{\delta\alpha}}{\delta\alpha} \sup_{x \in S} \int_S \int_0^{2\delta} p(t, x, z) dt \mu(dz). \end{aligned}$$

Using the Taylor expansion of e^t , one can verify that there exists constants $c_1, c_2 > 0$ such that

$$\left| \frac{e^{2t} - e^t}{t} - 1 \right| \leq c_1 t, \quad \left| \frac{e^{2t} - e^t}{t} \right| \leq c_2, \quad \forall t \in [0, 1].$$

Moreover, following the proof of Proposition 4.9, we deduce that

$$\sup_{x \in S} \int_S \int_0^{2\delta} p(t, x, z) dt \mu(dz) \leq e^2 \|\mathcal{R}^{1/\delta} \mu\|_\infty.$$

Therefore, we obtain that

$$\|\mathcal{R}^\alpha \mu_\delta - \mathcal{R}^\alpha \mu\|_\infty \leq c_1 \delta \alpha \|\mathcal{R}^\alpha \mu\|_\infty + c_2 e^2 \|\mathcal{R}^{1/\delta} \mu\|_\infty.$$

From (5.4), we also obtain that

$$\|\mathcal{R}^\alpha \mu_\delta\|_\infty \leq c_2 \|\mathcal{R}^\alpha \mu\|_\infty, \quad \forall \alpha > 0.$$

Therefore, applying Lemma 4.19, we deduce that

$$\begin{aligned} \sup_{x \in S} E^x \left[\sup_{0 \leq t \leq T} |A_t^{(\delta,*)} - A_t|^2 \right] &\leq 18(c_2 \|\mathcal{R}^\alpha \mu\|_\infty + \|\mathcal{R}^\alpha \mu\|_\infty) \left(c_1 \delta \alpha \|\mathcal{R}^\alpha \mu\|_\infty + c_2 e^2 \|\mathcal{R}^{1/\delta} \mu\|_\infty \right) \\ &\quad + 4e^{2T} (1 - e^{-\alpha T}) (c_2^2 \|\mathcal{R}^1 \mu\|_\infty^2 + \|\mathcal{R}^1 \mu\|_\infty^2) \\ &\leq 18(c_2 + 1) \|\mathcal{R}^\alpha \mu\|_\infty \left(c_1 \delta \alpha \|\mathcal{R}^\alpha \mu\|_\infty + c_2 e^2 \|\mathcal{R}^{1/\delta} \mu\|_\infty \right) \\ &\quad + 4e^{2T} (1 - e^{-\alpha T}) (c_2^2 + 1) \|\mathcal{R}^1 \mu\|_\infty^2. \end{aligned}$$

Thus, (5.2) holds. Since μ is in the Kato class, it follows from (5.2)

$$\limsup_{\delta \rightarrow 0} \sup_{x \in S} E^x \left[\sup_{0 \leq t \leq T} |A_t^{(\delta,*)} - A_t|^2 \right] \leq C_2 e^{2T} (1 - e^{-\alpha T}) \|\mathcal{R}^1 \mu\|_\infty^2.$$

Letting $\alpha \rightarrow 0$ in the above inequality, we obtain (5.3). $\square$

We next consider PCAFs in the local Kato class and develop suitable approximation schemes. Here, we note that to continue discussing the approximation of PCAFs in $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, we need to impose additional assumptions. Indeed, PCAFs in the local Kato class may explode in finite time, which makes it impossible to discuss convergence in the space $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$. This observation leads to the following points.

- (a) To discuss convergence in the space $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, additional assumptions on the process X or on the PCAFs are required.
- (b) A different framework is needed to handle convergence of general PCAFs.

One possible approach in the direction of (a) is to assume that the process X is conservative (recall this property from (4.1)). Under this assumption, the definition of PCAFs immediately implies that each PCAF belongs to $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ almost surely. In this paper, we continue to discuss convergence of general PCAFs in $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ under this conservativeness assumption.

Nevertheless, it is worth noting that one can also develop a framework for studying convergence of PCAFs without imposing any additional assumptions, in accordance with (b). The key idea is to handle STOMs within a suitably chosen topology. See Remark 5.13 in the next subsection for further details.

We now return to our main line of argument. To establish an approximation method for PCAFs in the local Kato class, we introduce, in Definitions 5.2 and 5.4 below, two auxiliary maps

that will serve as building blocks for the construction of the approximations. Although these maps are formulated in rather general forms, independent of the process or PCAFs themselves, this level of generality will be useful for the main results discussed in Section 6.

The first map is used to truncate PCAFs so that associated smooth measures are restricted onto a compact ball.

Definition 5.2. For each $R > 1$ we define a map

$$\mathfrak{P}^{(*,R)}: D_{L^0}(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}) \rightarrow C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$$

by

$$(\eta, \varphi) \mapsto \left(t \mapsto \int_0^t \int_{R-1}^R \mathbf{1}_{S(r)}(\eta_s) dr d\varphi_s \right).$$

Proposition 5.3. Assume that X is conservative. Let μ be a smooth measure, and let $A = (A_t)_{t \geq 0}$ denote the associated PCAF. Fix $R > 1$. Set $A^{(*,R)} := \mathfrak{P}^{(*,R)}(X, A)$. Then $A^{(*,R)}$ is a PCAF, and its associated smooth measure is $\tilde{\mu}^{(R)}$, which is defined in (2.3). Moreover, we have

$$A_t^{(*,R)} = A_t \quad \text{for all } t \in [0, T] \text{ on the event } \{\check{\sigma}_{S(R-1)} > T\}. \quad (5.5)$$

In particular, for all $x \in S$ and $T > 0$,

$$\lim_{R \rightarrow \infty} E^x \left[\left(\sup_{0 \leq t \leq T} |A_t^{(*,R)} - A_t| \right) \wedge 1 \right] = 0. \quad (5.6)$$

Consequently, $A^{(*,R)} \xrightarrow[R \rightarrow \infty]{P} A$ in $C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ under P^x , for each $x \in S$.

Proof. The first assertion follows from [22, Theorem A.3.5]. The assertion (5.5) is immediate from the definition of $A^{(*,R)}$. Using the assertion, we deduce that, for any $x \in S$ and $T > 0$,

$$E^x \left[\left(\sup_{0 \leq t \leq T} |A_t^{(*,R)} - A_t| \right) \wedge 1 \right] \leq P^x(\check{\sigma}_{S(R-1)} \leq T).$$

The right-hand side converges to 0 as $R \rightarrow \infty$ by the conservativeness of X , which shows (5.6). The last assertion is elementary. $\square$

After the above truncation, we can approximate the truncated PCAF $A^{(*,R)}$ by absolutely continuous PCAFs, constructed from the following map. This map is indeed modeled after the approximation in (5.1).

Definition 5.4. For each $\delta > 0$ and $R > 1$, we define a map

$$\mathfrak{P}^{(\delta,R)}: C(\mathbb{R}_{>0} \times S \times S, \mathbb{R}_{\geq 0}) \times \mathcal{M}(S) \times D_{L^0}(\mathbb{R}_{\geq 0}, S) \rightarrow C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$$

by

$$(q, \nu, \eta) \mapsto \left(t \mapsto \frac{1}{\delta} \int_0^t \int_S \int_{\delta}^{2\delta} q(u, \eta_s, x) du \tilde{\nu}^{(R)}(dx) ds \right),$$

where the smooth truncation $\tilde{\nu}^{(R)}$ is recalled from (2.3).

The following is an immediate consequence of Proposition 5.1.

Proposition 5.5. Let μ be a Radon measure in the local Kato class and let $A = (A_t)_{t \geq 0}$ be the associated PCAF. For each $\delta > 0$ and $R > 1$, set

$$A^{(\delta,R)} := \mathfrak{P}^{(\delta,R)}(p, \mu, X), \quad A^{(*,R)} := \mathfrak{P}^{(*,R)}(X, A).$$

Let $C_1, C_2 > 0$ be the universal constants appearing in the assertion of Proposition 5.1. Then, for any $T \in [0, \infty)$, $\alpha \in (0, 1)$, and $\delta \in (0, 1)$,

$$\begin{aligned} \sup_{x \in S} E^x \left[\sup_{0 \leq t \leq T} |A_t^{(\delta, R)} - A_t^{(*, R)}|^2 \right] &\leq C_1 \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty \left(\delta \alpha \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty + \|\mathcal{R}^{1/\delta} \mu^{(R)}\|_\infty \right) \\ &\quad + C_2 e^{2T} (1 - e^{-\alpha T}) \|\mathcal{R}^1 \mu^{(R)}\|_\infty^2. \end{aligned}$$

In particular, for each $R > 1$,

$$\lim_{\delta \rightarrow 0} \sup_{x \in S} E^x \left[\sup_{0 \leq t \leq T} |A_t^{(\delta, R)} - A_t^{(*, R)}|^2 \right] = 0.$$

Proof. Recall from Proposition 5.3 that $A^{(*, R)}$ is a PCAF associated with the Kato class measure $\tilde{\mu}^{(R)}$. Thus, applying Proposition 5.1 to the PCAF $A^{(*, R)}$, we deduce that

$$\begin{aligned} \sup_{x \in S} E^x \left[\sup_{0 \leq t \leq T} |A_t^{(\delta, *)} - A_t|^2 \right] \\ \leq C_1 \|\mathcal{R}^\alpha \tilde{\mu}^{(R)}\|_\infty \left(\delta \alpha \|\mathcal{R}^\alpha \tilde{\mu}^{(R)}\|_\infty + \|\mathcal{R}^{1/\delta} \tilde{\mu}^{(R)}\|_\infty \right) + C_2 e^{2T} (1 - e^{-\alpha T}) \|\mathcal{R}^1 \tilde{\mu}^{(R)}\|_\infty^2 \end{aligned}$$

for all $T > 0$, $\alpha \in (0, 1)$, and $\delta \in (0, 1)$. By the definition of $\tilde{\mu}^{(R)}$ (see (2.3)), we have

$$\|\mathcal{R}^\alpha \tilde{\mu}^{(R)}\|_\infty \leq \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty,$$

which yields the desired result. $\square$

Combining Propositions 5.3 and 5.5, we see that PCAFs in the local Kato class can be approximated in two stages as follows.

Corollary 5.6. *Assume that X is conservative. Let μ be a smooth measure in the local Kato class, and let $A = (A_t)_{t \geq 0}$ denote the associated PCAF. For each $\delta > 0$ and $R > 1$, define*

$$A^{(\delta, R)} := \mathfrak{P}^{(\delta, R)}(p, \mu, X), \quad A^{(*, R)} := \mathfrak{P}^{(*, R)}(X, A).$$

Then, for each $x \in S$, the following convergences hold in $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ under P^x :

$$\begin{aligned} A^{(\delta, R)} &\xrightarrow[\delta \rightarrow 0]{P} A^{(*, R)}, \quad \forall R > 1, \\ A^{(*, R)} &\xrightarrow[R \rightarrow \infty]{P} A. \end{aligned}$$

5.2 The approximation scheme for STOMs

In this subsection, we extend the approximation method for PCAFs introduced in the previous section to STOMs. The approximation for STOMs in the Kato class is presented in Proposition 5.7 below, and for STOMs in the local Kato class is discussed in Propositions 5.10 and 5.11.

We first establish an approximation for STOMs in the Kato class, which corresponds to Proposition 5.1.

Proposition 5.7. *Let μ be a finite Borel measure in the Kato class and let Π be the associated STOM. For each $\delta > 0$, define*

$$\Pi^{(\delta, *)}(dz dt) = \frac{1}{\delta} \int_\delta^{2\delta} p(u, X_t, z) du \mu(dz) dt. \quad (5.7)$$

Then there exist universal constants $C_3, C_4 > 0$ such that the following holds: for any $T \in [0, \infty)$, $E \in \mathcal{B}(S)$, $\alpha \in (0, 1)$, and $\delta \in (0, 1)$,

$$\begin{aligned} & \sup_{x \in S} E^x \left[\sup_{0 \leq s < t \leq T} |\Pi^{(\delta, R)}(E \times [s, t)) - \Pi^{(*, R)}(E \times [s, t))|^2 \right] \\ & \leq C_3 \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty \left(\delta \alpha \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty + \|\mathcal{R}^{1/\delta} \mu^{(R)}\|_\infty \right) + C_4 e^{2T} (1 - e^{-\alpha T}) \|\mathcal{R}^1 \mu^{(R)}\|_\infty^2. \end{aligned} \quad (5.8)$$

In particular, for each $R > 1$, $E \in \mathcal{B}(S)$, and $T > 0$

$$\lim_{\delta \rightarrow 0} \sup_{x \in S} E^x \left[\sup_{0 \leq s < t \leq T} |\Pi^{(\delta, R)}(E \times [s, t)) - \Pi^{(*, R)}(E \times [s, t))|^2 \right] = 0. \quad (5.9)$$

Consequently, $\Pi^{(\delta, *)} \xrightarrow[\delta \rightarrow 0]{P} \Pi$ in $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$ under P^x for each $x \in S$.

Proof. Using the simple estimate that $(a + b)^2 \leq 2(a^2 + b^2)$ for any $a, b \geq 0$, we have

$$\begin{aligned} & \sup_{x \in S} E^x \left[\sup_{0 \leq s < t \leq T} |\Pi^{(\delta, *)}(E \times [s, t)) - \Pi(E \times [s, t))|^2 \right] \\ & \leq 4 \sup_{x \in S} E^x \left[\sup_{0 \leq t \leq T} |\Pi^{(\delta, *)}(E \times [0, t)) - \Pi(E \times [0, t))|^2 \right]. \end{aligned}$$

By the definition of the STOM Π , the process $B_t := \Pi(E \times [0, t))$, $t \geq 0$, is a PCAF associated with the smooth measure $\mu|_E$. If we let $B^{(\delta, *)} = (B_t^{(\delta, *)})_{t \geq 0}$ be the approximation of B defined as (5.1), then

$$B_t^{(\delta, *)} = \frac{1}{\delta} \int_0^t \int_S \int_\delta^{2\delta} p(u, X_s, z) du \mu|_E(dz) ds = \Pi^{(\delta, *)}(E \times [0, t)), \quad t \geq 0.$$

Hence, (5.8) follows from Proposition 5.5. In the inequality, letting $\delta \rightarrow 0$ and then $\alpha \rightarrow 0$ yields (5.9). The last assertion is now immediate from [43, Corollary 4.9(iii)] (see also the proof of Theorem 5.30 below). $\square$

We next consider STOMs in the local Kato class. Recall that, in Definitions 5.2 and 5.4, we introduced two maps $\mathfrak{P}^{(*, R)}$ and $\mathfrak{P}^{(\delta, R)}$ to approximate PCAFs. Below, we introduce STOM versions of these maps.

Definition 5.8. For each $R > 1$ we define a map

$$\mathfrak{S}^{(*, R)}: \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0}) \rightarrow \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$$

by

$$\Sigma \mapsto \left(\int_{R-1}^R \mathbf{1}_{S(r)}(x) dr \right) \Sigma(dx dt).$$

Definition 5.9. For each $\delta > 0$ and $R > 1$, we define a map

$$\mathfrak{S}^{(\delta, R)}: C(\mathbb{R}_{>0} \times S \times S, \mathbb{R}_{\geq 0}) \times \mathcal{M}(S) \times D_{L^0}(\mathbb{R}_{\geq 0}, S) \rightarrow \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$$

by

$$(q, \nu, \eta) \mapsto \frac{1}{\delta} \int_\delta^{2\delta} q(u, \eta_t, z) du \tilde{\nu}^{(R)}(dz) dt.$$

Below, we present STOM versions of Propositions 5.3 and 5.5.

Proposition 5.10. *Assume that X is conservative. Let μ be a smooth measure, and let Π denote the associated STOM. For each $R > 1$, set $\Pi^{(*,R)} := \mathfrak{S}^{(*,R)}(\Pi)$. Then $\Pi^{(*,R)}$ is a STOM associated with the smooth measure $\tilde{\mu}^{(R)}$. Moreover, it holds that, for any $T > 0$,*

$$\Pi^{(*,R)}|_{S \times [0,T]} = \Pi|_{S \times [0,T]} \quad \text{on the event} \quad \{\check{\sigma}_{S^{(R-1)}} > T\}. \quad (5.10)$$

In particular,

$$\lim_{R \rightarrow \infty} E^x \left[d_{S \times \mathbb{R}_{\geq 0}}^{\mathcal{M}_{(*,\text{fin})}}(\Pi^{(*,R)}, \Pi) \right] = 0, \quad \forall x \in S, \quad (5.11)$$

where the metric $d_{S \times \mathbb{R}_{\geq 0}}^{\mathcal{M}_{(*,\text{fin})}}$ is recalled from (2.10). Consequently, $\Pi^{(*,R)} \xrightarrow[R \rightarrow \infty]{\text{P}} \Pi$ in $\mathcal{M}_{(*,\text{fin})}(S \times \mathbb{R}_{\geq 0})$ under P^x , for each $x \in S$.

Proof. Write $A = (A_t)_{t \geq 0}$ for the PCAF associated with μ . By Proposition 5.3, $A^{(*,R)} := \mathfrak{P}^{(*,R)}(X, A)$ is the PCAF associated with $\tilde{\mu}^{(R)}$, and hence the associated STOM $\tilde{\Pi}^{(*,R)}$ is given by

$$\tilde{\Pi}^{(*,R)}(E \times [0, t]) = \int_0^t \mathbf{1}_E(\eta_s) \int_{R-1}^R \mathbf{1}_{S^{(r)}}(\eta_s) dr dA_s, \quad E \in \mathcal{B}(S), \quad t \geq 0.$$

By definition, we have, for any $E \in \mathcal{B}(S)$ and $t \geq 0$,

$$\Pi^{(*,R)}(E \times [0, t]) = \int_0^t \mathbf{1}_E(\eta_s) \int_{R-1}^R \mathbf{1}_{S^{(r)}}(\eta_s) dr dA_s.$$

Hence, $\tilde{\Pi}^{(*,R)} = \Pi^{(*,R)}$, which proves the first assertion. The second assertion (5.10) is straightforward from the definition of $\Pi^{(*,R)}$. Then, by the definition of $d_{S \times \mathbb{R}_{\geq 0}}^{\mathcal{M}_{(*,\text{fin})}}$, we have that, on the event $\{\check{\sigma}_{S^{(R-1)}} > T\}$,

$$\begin{aligned} d_{S \times \mathbb{R}_{\geq 0}}^{\mathcal{M}_{(*,\text{fin})}}(\Pi^{(*,R)}, \Pi) &= \int_0^\infty e^{-t} (1 \wedge d_{S \times \mathbb{R}_{\geq 0}}^P(\Pi^{(*,R)}|_{S \times [0,t]}, \Pi|_{S \times [0,t]})) dt \\ &\leq e^{-T} \end{aligned} \quad (5.12)$$

Since the metric does not exceed 1, it follows that, for any $x \in S$ and $T > 0$,

$$E^x \left[d_{S \times \mathbb{R}_{\geq 0}}^{\mathcal{M}_{(*,\text{fin})}}(\Pi^{(*,R)}, \Pi) \right] \leq e^{-T} + P^x(\check{\sigma}_{S^{(R-1)}} > T).$$

Letting $R \rightarrow \infty$ and then $T \rightarrow \infty$ in the above inequality, and using the conservativeness of X , we deduce (5.11). The last assertion is elementary (cf. [44, Lemma 5.2]). $\square$

Proposition 5.11. *Let μ be a Radon measure in the local Kato class and let Π be the associated STOM. Fix $R > 1$. For each $\delta > 0$, set*

$$\Pi^{(\delta,R)} := \mathfrak{S}^{(\delta,R)}(p, \mu, X), \quad \Pi^{(*,R)} := \mathfrak{S}^{(*,R)}(\Pi).$$

Then there exist universal constants $C_3, C_4 > 0$ such that the following holds: for any $T \in [0, \infty)$, $E \in \mathcal{B}(S)$, $\alpha \in (0, 1)$, and $\delta \in (0, 1)$,

$$\begin{aligned} &\sup_{x \in S} E^x \left[\sup_{0 \leq s < t \leq T} |\Pi^{(\delta,R)}(E \times [s, t]) - \Pi^{(*,R)}(E \times [s, t])|^2 \right] \\ &\leq C_3 \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty \left(\delta \alpha \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty + \|\mathcal{R}^{1/\delta} \mu^{(R)}\|_\infty \right) + C_4 e^{2T} (1 - e^{-\alpha T}) \|\mathcal{R}^1 \mu^{(R)}\|_\infty^2. \end{aligned} \quad (5.13)$$

In particular,

$$\lim_{\delta \rightarrow 0} \sup_{x \in S} E^x \left[\sup_{0 \leq s < t \leq T} |\Pi^{(\delta,R)}(E \times [s, t]) - \Pi^{(*,R)}(E \times [s, t])|^2 \right] = 0.$$

Consequently, $\Pi^{(\delta,R)} \xrightarrow[\delta \rightarrow 0]{\text{P}} \Pi^{(*,R)}$ in $\mathcal{M}_{(*,\text{fin})}(S \times \mathbb{R}_{\geq 0})$ under P^x , for each $x \in S$.

Proof. By Proposition 5.10, $\Pi^{(*,R)}$ is a STOM associated with $\tilde{\mu}^{(R)}$. On the other hand, by definition, we have

$$\Pi^{(\delta,R)}(dz dt) = \frac{1}{\delta} \int_{\delta}^{2\delta} q(u, \eta_t, z) du \tilde{\nu}^{(R)}(dz) dt.$$

Hence, we can apply Proposition 5.7 to the Kato class measure $\tilde{\mu}^{(R)}$. Noting that $\tilde{\mu}^{(R)} \leq \mu^{(R)}$, we obtain (5.13). The remaining assertions are verified in the same manner as the corresponding assertions of Proposition 5.7, and therefore we omit the details. $\square$

Combining the above two propositions, similarly to the PCAF case, we see that STOMs in the local Kato class can be approximated in two stages as follows.

Corollary 5.12. *Assume that X is conservative. Let μ be a smooth measure in the local Kato class, and let Π denote the associated PCAF. For each $\delta > 0$ and $R > 1$, define*

$$\Pi^{(\delta,R)} := \mathfrak{S}^{(\delta,R)}(p, \mu, X), \quad \Pi^{(*,R)} := \mathfrak{S}^{(*,R)}(\Pi).$$

Then, for each $x \in S$, the following convergences hold in $\mathcal{M}_{(,\text{fin})}(S \times \mathbb{R}_{\geq 0})$ under P^x :*

$$\begin{aligned} \Pi^{(\delta,R)} &\xrightarrow[\delta \rightarrow 0]{P} \Pi^{(*,R)}, \quad \forall R > 1, \\ \Pi^{(*,R)} &\xrightarrow[R \rightarrow \infty]{P} \Pi. \end{aligned}$$

Remark 5.13. Here, we provide several reasons why STOMs offer a more flexible framework than PCAFs.

- (a) In the previous discussion, to approximate PCAFs in the local Kato class within the space $C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, we imposed the conservativeness of X . In fact, by working with STOMs instead, one can study such approximations *without* assuming conservativeness. Let us explain this in a bit more detail. Suppose that Π is a STOM in the local Kato class. Since the associated PCAF may explode in finite time, Π may fail to belong to $\mathcal{M}_{(*,\text{fin})}(S \times \mathbb{R}_{\geq 0})$. However, by Proposition 4.20, we know that Π is a Radon measure on $S \times \mathbb{R}_{\geq 0}$ with probability one. Therefore, although the topology becomes coarser, we can still discuss its convergence within $\mathcal{M}(S \times \mathbb{R}_{\geq 0})$ endowed with the vague topology. See Proposition 5.14 below for an application of this idea.
- (b) Convergence of STOMs can also be discussed for general smooth measures. The idea is to introduce a variant of the vague topology that is adapted to the nest of the smooth measure. Let μ be a general smooth measure and let Π denote the corresponding STOM. Fix an nest $\mathfrak{E} = (E_N)_{N \geq 1}$ of μ (recall it from Definition 4.5). Define $\mathcal{M}_{\mathfrak{E}}(S \times \mathbb{R}_{\geq 0})$ to be the collection of measures on $S \times \mathbb{R}_{\geq 0}$ that are finite on each $E_N \times [0, N]$. Then, by an argument similar to the above using Proposition 4.20, we see that Π belongs to $\mathcal{M}_{\mathfrak{E}}(S \times \mathbb{R}_{\geq 0})$ with probability one. By endowing $\mathcal{M}_{\mathfrak{E}}(S \times \mathbb{R}_{\geq 0})$ with a suitable modification of the vague topology adapted to the underlying nest, we can discuss the convergence of STOMs. Such nest-dependent modifications of the vague topology were recently introduced in [68] for the purpose of studying the convergence of general smooth measures (not necessarily Radon measures).

As pointed out in Remark 5.13(a), we provide a result which shows the flexibility of the STOM framework. Namely, we present an approximation result for STOMs in local Kato class in the vague topology, without assuming conservativeness.

Proposition 5.14. *Let μ be a smooth measure, and let Π denote the associated STOM. For each $R > 1$, set $\Pi^{(*,R)} := \mathfrak{S}^{(*,R)}(\Pi)$. For any $x \in S$, it holds P^x -a.s. that*

$$d_{S \times \mathbb{R}_{\geq 0}}^V(\Pi^{(*,R)}, \Pi) \leq e^{-R+1}, \quad \forall R > 1, \quad (5.14)$$

where $d_{S \times \mathbb{R}_{\geq 0}}^V$ denotes the vague metric as recalled from (2.2). Consequently, $\Pi^{(*,R)} \xrightarrow[R \rightarrow \infty]{\text{a.s.}} \Pi$ vaguely under P^x , for each $x \in S$.

Proof. By definition, if $r < R - 1$, then we have, for any $E \in \mathcal{B}(S)$ and $t \geq 0$,

$$\begin{aligned} \Pi^{(*,R)}|_{S^{(r)} \times [0,r]}(E \times [0,t]) &= \int_0^{t \wedge r} \mathbf{1}_{E \cap S^{(r)}}(\eta_s) \int_{R-1}^R \mathbf{1}_{S^{(u)}}(\eta_s) du dA_s \\ &= \int_0^{t \wedge r} \mathbf{1}_{E \cap S^{(r)}}(\eta_s) dA_s \\ &= \Pi|_{S^{(r)} \times [0,r]}(E \times [0,t]). \end{aligned}$$

This implies $\Pi^{(*,R)}|_{S^{(r)} \times [0,r]} = \Pi|_{S^{(r)} \times [0,r]}$ for all $r < R - 1$. It follows that

$$d_{S \times \mathbb{R}_{\geq 0}}^V(\Pi^{(*,R)}, \Pi) = \int_0^\infty e^{-r} \{1 \wedge d_{S \times \mathbb{R}_{\geq 0}}^P(\Pi^{(*,R)}|_{S^{(r)} \times [0,r]}, \Pi|_{S^{(r)} \times [0,r]})\} dr \leq e^{-R+1},$$

which yields (5.14). This completes the proof. $\square$

5.3 Continuity of joint laws with PCAFs

In a wide class of stochastic processes, such as Feller processes, the law of the process depends continuously on the starting point. Here, we employ the approximation introduced in the previous subsection to show that the joint law with a PCAF also enjoys this continuity; see Theorems 5.21 and 5.25 below.

To formulate the results in a suitable topological framework, we consider the following conditions for the underlying process X and its heat kernel. We recall the notation $\mathcal{L}$ from (4.4).

Assumption 5.15 (The weak L^0 -continuity condition).

- (i) The process X is conservative, and the law map $\mathcal{L}_X: S \rightarrow \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S))$ is continuous.
- (ii) The heat kernel $p(t, x, y)$ is $[0, \infty)$ -valued and is continuous in $(t, x, y) \in \mathbb{R}_{>0} \times S \times S$.

Assumption 5.16 (The spatial tightness condition). For all $r > 0$ and $T > 0$,

$$\lim_{R \rightarrow \infty} \sup_{x \in S^{(r)}} P^x(\check{\sigma}_{S^{(R)}} < T) = 0.$$

If the law map $\mathcal{L}_X$ is weakly continuous with respect to a topology that captures the path behavior more precisely than the L^0 topology, such as the J_1 -Skorohod topology, then the spatial tightness condition is automatically satisfied. For instance, the following condition suffices.

Assumption 5.17 (The weak J_1 -continuity condition).

- (i) The process X is conservative, and the map $\mathcal{L}_X: S \rightarrow \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, S))$ is continuous.
- (ii) The heat kernel $p(t, x, y)$ takes finite values and is continuous in $(t, x, y) \in \mathbb{R}_{>0} \times S \times S$.

Lemma 5.18. Condition (i) of Assumption 5.17 implies Assumption 5.16.

Proof. Condition (i) of Assumption 5.17 implies that, for each $r > 0$, the family $\{\mathcal{L}_X(x)\}_{x \in S^{(r)}}$ is tight as a collection of probability measures on $D_{J_1}(\mathbb{R}_{\geq 0}, S)$. Recall that J_1 -tightness ensures that, for every $T > 0$, the trajectories $\{X_t \mid t \leq T\}$ starting from points in $S^{(r)}$ remain, with high probability, within a common compact subset of S (see [44, Theorem 23.8]). Hence, Assumption 5.16 follows. This completes the proof. $\square$

Remark 5.19. Condition (i) of Assumption 5.17 holds for a wide class of processes. For instance, any conservative Feller process enjoys this property. (This follows from the Aldous tightness criterion for the J_1 -Skorohod topology; see [44, Theorem 23.11].)

Remark 5.20. The joint continuity of the heat kernel is more delicate than the weak continuity of the law map. There is no simple sufficient condition for this in general, but sufficient conditions are known in several important frameworks.

- (a) When X is a Lévy process, the integrability of its characteristic function is sufficient (see [73, Proposition 2.5(xii)]).
- (b) When X is associated with a regular strongly local symmetric Dirichlet form, the parabolic Harnack inequality is sufficient [12].
- (c) Any process associated with a resistance form (see Section 8 below) admits a unique jointly continuous heat kernel.

In the theorem below, we show that, under the weak L^0 -continuity condition (Assumption 5.15), the continuity of $\mathcal{L}_X$ is inherited by the joint law $\mathcal{L}_{(X,A)}$ for any PCAF A in the Kato class.

Theorem 5.21. *Suppose that Assumption 5.15 is satisfied. Let μ be a finite Borel measure in the Kato class and let $A = (A_t)_{t \geq 0}$ be the associated PCAF. For each $\delta > 0$, let $A^{(\delta,*)}$ denote the PCAF defined in (5.1). Then the map*

$$\mathcal{L}_{(X,A^{(\delta,*)})} : S \rightarrow \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}))$$

is continuous for each $\delta > 0$. Likewise, $\mathcal{L}_{(X,A)}$ is continuous. Moreover,

$$\mathcal{L}_{(X,A^{(\delta,*)})} \xrightarrow[\delta \rightarrow 0]{} \mathcal{L}_{(X,A)} \quad \text{in} \quad C(S, \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}))).$$

Proof. The proof proceeds as follows. Using the joint continuity of the heat kernel, we first verify the continuity of $\mathcal{L}_{(X,A^{(\delta,*)})}$. Then, by Proposition 5.5, we deduce that $\mathcal{L}_{(X,A^{(\delta,*)})}$ converges to $\mathcal{L}_{(X,A)}$ as $\delta \rightarrow 0$ locally uniformly, which yields the desired conclusion. The details are omitted here, since the argument follows the same line as in the case of PCAFs in the local Kato class, which will be treated next. $\square$

We next consider PCAFs in the local Kato class. We first state basic properties of the approximation map $\mathfrak{P}^{(\delta,R)}$ introduced in Definition 5.4. The proof is deferred to Appendix A.2.

Lemma 5.22. *Fix $\delta > 0$ and $R > 1$.*

- (i) *The map $\mathfrak{P}^{(\delta,R)}$ is Borel measurable.*
- (ii) *Fix an arbitrary sequence $(q_n, \nu_n, \eta_n)_{n \geq 1}$ converging to (q, ν, η) in the domain of $\mathfrak{P}^{(\delta,R)}$. Assume that*

$$\sup_{n \geq 1} \sup_{y \in S} \int_{\delta}^{2\delta} \int_S q_n(u, y, z) \nu_n^{(R)}(dz) du < \infty.$$

Then $\mathfrak{P}^{(\delta,R)}(q_n, \nu_n, \eta_n)$ converges to $\mathfrak{P}^{(\delta,R)}(q, \nu, \eta)$ in $C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$.

The above lemma immediately implies the continuity of the approximation $\mathcal{L}_{(X,A^{(\delta,R)})}$, as stated below.

Lemma 5.23. *Suppose that Assumption 5.15 is satisfied. Let μ be a Radon measure in the local Kato class and let $A = (A_t)_{t \geq 0}$ be the associated PCAF. Fix $\delta > 0$ and $R > 1$, and set*

$$A^{(\delta,R)} := \mathfrak{P}^{(\delta,R)}(p, \mu, X).$$

Then the map

$$\mathcal{L}_{(X,A^{(\delta,R)})} : S \rightarrow \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}))$$

is continuous.

Proof. This is an immediate consequence of the assumption, Lemma 5.22(ii), and the continuous mapping theorem. $\square$

Using the approximation of $\mathcal{L}_{(X,A^{(*,R)})}$ by $\mathcal{L}_{(X,A^{(\delta,R)})}$ established in Proposition 5.5, we show that $\mathcal{L}_{(X,A^{(*,R)})}$ is also continuous.

Proposition 5.24. *Suppose that Assumption 5.15 is satisfied. Let μ be a Radon measure in the local Kato class, and let $A = (A_t)_{t \geq 0}$ be the associated PCAF. Fix $R > 1$. For each $\delta > 0$, define*

$$A^{(\delta,R)} := \mathfrak{P}^{(\delta,R)}(p, \mu, X), \quad A^{(*,R)} := \mathfrak{P}^{(*,R)}(X, A).$$

Then the following statements hold.

- (i) *The map $\mathcal{L}_{(X,A^{(*,R)})}$ is continuous.*
- (ii) *Let $C_1, C_2 > 0$ be the universal constants appearing in Proposition 5.1. Then, for any $N \in \mathbb{N}$, $\alpha \in (0, 1)$, and $\delta \in (0, 1)$, we have*

$$\begin{aligned} & d_{S, \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}))}^{\widehat{C}} \left(\mathcal{L}_{(X,A^{(\delta,R)})}, \mathcal{L}_{(X,A^{(*,R)})} \right) \\ & \leq 2^{-N+1} + 2^{2N} C_1 \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty \left(\delta \alpha \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty + \|\mathcal{R}^{1/\delta} \mu^{(R)}\|_\infty \right) \\ & \quad + C_2 (2e)^{2N} (1 - e^{-\alpha N}) \|\mathcal{R}^1 \mu^{(R)}\|_\infty^2. \end{aligned}$$

- (iii) *It holds that*

$$\mathcal{L}_{(X,A^{(\delta,R)})} \xrightarrow[\delta \rightarrow 0]{} \mathcal{L}_{(X,A^{(*,R)})} \quad \text{in } C(S, \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}))).$$

Proof. We prove the assertions simultaneously. For each $N \in \mathbb{N}$, define

$$E_N := \left\{ \sup_{0 \leq t \leq N} |A_t^{(\delta,R)} - A_t^{(*,R)}| \leq 2^{-N} \right\}.$$

By Lemma 5.5 and Markov's inequality, we have

$$\begin{aligned} \sup_{x \in S} P^x(E_N^c) & \leq 2^{2N} C_1 \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty \left(\delta \alpha \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty + \|\mathcal{R}^{1/\delta} \mu^{(R)}\|_\infty \right) \\ & \quad + C_2 (2e)^{2N} (1 - e^{-\alpha N}) \|\mathcal{R}^1 \mu^{(R)}\|_\infty^2. \end{aligned}$$

On the event E_N , we have

$$d_{C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})}(A^{(\delta,R)}, A^{(*,R)}) \leq 2^{-N+1},$$

where we recall the metric from (2.8). Therefore, by Lemma 2.3, we obtain

$$\begin{aligned} & \sup_{x \in S} d_{L^0(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})}^P \left(\mathcal{L}_{(X,A^{(\delta,R)})}(x), \mathcal{L}_{(X,A^{(*,R)})}(x) \right) \\ & \leq 2^{-N+1} + 2^{2N} C_1 \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty \left(\delta \alpha \|\mathcal{R}^\alpha \mu^{(R)}\|_\infty + \|\mathcal{R}^{1/\delta} \mu^{(R)}\|_\infty \right) \\ & \quad + C_2 (2e)^{2N} (1 - e^{-\alpha N}) \|\mathcal{R}^1 \mu^{(R)}\|_\infty^2. \end{aligned} \tag{5.15}$$

Letting $\delta \rightarrow 0$, then $\alpha \rightarrow 0$, and finally $N \rightarrow \infty$ in the above inequality yields

$$\lim_{\delta \rightarrow 0} \sup_{x \in S} d_{L^0(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})}^P \left(\mathcal{L}_{(X,A^{(\delta,R)})}(x), \mathcal{L}_{(X,A^{(*,R)})}(x) \right) = 0.$$

This, combined with Lemma 5.23, yields (i) and (iii). Finally, applying Lemma 2.8 to (5.15) gives (ii). $\square$

Finally, using Proposition 5.3 and the spatial tightness condition, we verify that the continuity of $\mathcal{L}_{(X,A^{(*,R)})}$ established above is inherited by $\mathcal{L}_{(X,A)}$, as stated below.

Theorem 5.25. *Suppose that Assumptions 5.15 and 5.16 are satisfied. Let μ be a Radon measure in the local Kato class and let $A = (A_t)_{t \geq 0}$ be the associated PCAF. For each $R > 1$, set*

$$A^{(*,R)} := \mathfrak{P}^{(*,R)}(X, A).$$

Then the following statements hold.

- (i) *The map $\mathcal{L}_{(X,A)}$ is continuous.*
- (ii) *It holds that, for any $r > 0$, $N \in \mathbb{N}$, and $R > 1$,*

$$\begin{aligned} & d_{S, \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}))}^{\widehat{C}} \left(\mathcal{L}_{(X, A^{(*,R)})}, \mathcal{L}_{(X, A)} \right) \\ & \leq e^{-r} + 2^{-N} + \sup_{x \in S^{(r)}} P^x(\check{\sigma}_{S^{(R-1)}} \leq N). \end{aligned}$$

- (iii) *It holds that*

$$\mathcal{L}_{(X, A^{(*,R)})} \xrightarrow{R \rightarrow \infty} \mathcal{L}_{(X, A)} \quad \text{in } C(S, \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}))).$$

Proof. The proof follows the same line as that of the previous proposition. For each $N \in \mathbb{N}$ and $R > 1$, define

$$E_{N,R} := \{\check{\sigma}_{S^{(R-1)}} > N\}.$$

On this event, we have $A_t^{(*,R)} = A_t$ for all $t \leq N$, which implies that

$$d_{C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})}(A^{(*,R)}, A) \leq 2^{-N}.$$

Thus, by Lemma 2.3, we obtain, for each $r > 0$,

$$\begin{aligned} & \sup_{x \in S^{(r)}} d_{L^0(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})}^P \left(\mathcal{L}_{(X, A^{(*,R)})}(x), \mathcal{L}_{(X, A^{(R)})}(x) \right) \\ & \leq 2^{-N} + \sup_{x \in S^{(r)}} P^x(\check{\sigma}_{S^{(R-1)}} \leq N). \end{aligned} \tag{5.16}$$

Letting $R \rightarrow \infty$ and then $N \rightarrow \infty$ in the above inequality, the spatial tightness condition (Assumption 5.16) yields

$$\lim_{R \rightarrow \infty} \sup_{x \in S^{(r)}} d_{L^0(\mathbb{R}_{\geq 0}, S) \times C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})}^P \left(\mathcal{L}_{(X, A^{(*,R)})}(x), \mathcal{L}_{(X, A^{(R)})}(x) \right) = 0, \quad \forall r > 0.$$

This, together with Proposition 5.24(i), establishes (i) and (iii). Finally, applying Lemma 2.8 to (5.16) gives (ii). $\square$

5.4 Continuity of joint laws with STOMs

The aim of this subsection is to extend the results of the previous subsection to STOMs. The desired continuity of the joint laws of the process and STOMs in the Kato and local Kato classes is established in Theorems 5.26 and 5.30 below.

The following is the STOM counterpart of Theorem 5.21. Again, we omit the proof, as it is essentially the same as that for STOMs in the local Kato class discussed below.

Theorem 5.26. *Suppose that Assumption 5.15 is satisfied. Let μ be a finite Borel measure in the Kato class and let Π be the associated STOM. For each $\delta > 0$, let $\Pi^{(\delta,*)}$ denote the PCAF defined in (5.7). Then the map*

$$\mathcal{L}_{(X, \Pi^{(\delta,*)})} : S \rightarrow \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0}))$$

is continuous for each $\delta > 0$. Likewise, $\mathcal{L}_{(X, \Pi)}$ is continuous. Moreover,

$$\mathcal{L}_{(X, \Pi^{(\delta,*)})} \xrightarrow{\delta \rightarrow 0} \mathcal{L}_{(X, \Pi)} \quad \text{in } C(S, \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0}))).$$

Recall the approximation map $\mathfrak{S}^{(\delta, R)}$ from Definition 5.9. The following lemma corresponds to Lemma 5.22. Again, the proof is deferred to Appendix A.2.

Lemma 5.27. *Fix $\delta > 0$ and $R > 1$.*

- (i) *The map $\mathfrak{S}^{(\delta, R)}$ is Borel measurable.*
- (ii) *Fix an arbitrary sequence $(q_n, \nu_n, \eta_n)_{n \geq 1}$ converging to (q, ν, η) in the domain of $\mathfrak{S}^{(\delta, R)}$. Assume that*

$$\sup_{n \geq 1} \sup_{y \in S} \int_{\delta}^{2\delta} \int_S q_n(u, y, z) \nu_n^{(R)}(dz) du < \infty.$$

Then $\mathfrak{S}^{(\delta, R)}(q_n, \nu_n, \eta_n)$ converges to $\mathfrak{S}^{(\delta, R)}(q, \nu, \eta)$ in $\mathcal{M}_{(, \text{fin})}(S \times \mathbb{R}_{\geq 0})$.*

Lemma 5.28. *Suppose that Assumption 5.15 is satisfied. Let μ be a Radon measure in the local Kato class. Fix $\delta > 0$ and $R > 1$, and set*

$$\Pi^{(\delta, R)} := \mathfrak{S}^{(\delta, R)}(p, \mu, X).$$

Then the map

$$\mathcal{L}_{(X, \Pi^{(\delta, R)})} : S \rightarrow \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0}))$$

is continuous.

Proof. This is an immediate consequence of the assumption, Lemma 5.27(ii), and the continuous mapping theorem. $\square$

The following proposition corresponds to Proposition 5.24. However, unlike the case of PCAFs, we cannot directly estimate the distance between the truncated STOM $\Pi^{(*, R)}$ and its approximation $\Pi^{(\delta, R)}$ in $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$. Instead, we establish the validity of the approximation by means of a coupling argument.

Proposition 5.29. *Suppose that Assumption 5.15 is satisfied. Let μ be a Radon measure in the local Kato class, and let $A = (A_t)_{t \geq 0}$ be the associated PCAF. Fix $R > 1$. For each $\delta > 0$, define*

$$\Pi^{(\delta, R)} := \mathfrak{S}^{(\delta, R)}(p, \mu, X), \quad \Pi^{(*, R)} := \mathfrak{S}^{(*, R)}(\Pi).$$

Then the following statements hold.

- (i) *The map $\mathcal{L}_{(X, \Pi^{(*, R)})}$ is continuous.*
- (ii) *We have*

$$\mathcal{L}_{(X, \Pi^{(\delta, R)})} \xrightarrow{\delta \rightarrow 0} \mathcal{L}_{(X, \Pi^{(*, R)})} \quad \text{in } C(S, \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0}))).$$

Proof. We prove both assertions simultaneously. Fix a sequence $(\delta_n)_{n \geq 1}$ in $(0, 1)$ converging to 0, and a sequence $(x_n)_{n \geq 1}$ converging to x in S . By Proposition 2.7, it suffices to show that

$$P^{x_n}((X, \Pi^{(\delta_n, R)}) \in \cdot) \rightarrow P^x((X, \Pi^{(*, R)}) \in \cdot) \quad (5.17)$$

weakly as probability measures on $D_{L^0}(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$. By Assumption 5.15(i), we have $\mathcal{L}_X(x_n) \rightarrow \mathcal{L}_X(x)$. Hence, by the Skorohod representation theorem, there exist random elements $(Y_n, \Sigma_n, \Sigma_n^*)$, $n \geq 1$, and (Y, Σ) defined on a common probability space with probability measure P such that

$$\begin{aligned} P((Y_n, \Sigma_n^{\delta_n}, \Sigma_n^*) \in \cdot) &= P^{x_n}((X, \Pi^{(\delta_n, R)}, \Pi^{(*, R)}) \in \cdot), \quad \forall n \geq 1, \\ P((Y, \Sigma^*) \in \cdot) &= P^x((X, \Pi^{(*, R)}) \in \cdot), \\ Y_n &\rightarrow Y, \quad \text{almost surely.} \end{aligned} \quad (5.18)$$

The convergence in (5.17) follows once we show that $\Sigma_n^{\delta_n} \xrightarrow{P} \Sigma$ in $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$. Recall from Remark 2.23 that if S is equipped with the bounded metric $1 \wedge d_S$, then the topology on $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$ coincides with the vague topology with bounded support. Hence, in what follows, we replace the metric d_S on S by $1 \wedge d_S$ and prove the desired convergence by applying a characterization of this topology given in [43, Corollary 4.9(iii)].

Define a collection $\mathcal{I}$ of subsets of $S \times \mathbb{R}_{\geq 0}$ by

$$\mathcal{I} := \{E \times [s, t) \mid 0 \leq s < t < \infty, E \in \mathcal{B}(M) \text{ such that } \mu(\partial E) = 0\}.$$

It is straightforward to check that $\mathcal{I}$ is a dissecting semi-ring in the sense of [43], i.e., it satisfies the following properties.

(DSR1) Every open subset of $S \times \mathbb{R}_{\geq 0}$ is a countable union of elements of $\mathcal{I}$.

(DSR2) Every bounded Borel subset of $S \times \mathbb{R}_{\geq 0}$ is covered by finitely many elements of $\mathcal{I}$.

(DSR3) The collection $\mathcal{I}$ is closed under finite intersections.

(DSR4) The difference between any two sets in $\mathcal{I}$ is a finite disjoint union of sets in $\mathcal{I}$.

(The first two properties mean that $\mathcal{I}$ is dissecting, and the last two mean that it is a semi-ring.) By [43, Corollary 4.9(iii)], it is enough to show that, for each $H \in \mathcal{I}$, $\Sigma_n^{\delta_n}(H) \xrightarrow{P} \Sigma^*(H)$ as random variables. Fix such a set $H = E \times [s, t)$. By Proposition 5.11, we have

$$\limsup_{\delta \rightarrow 0} E^y \left[|\Pi^{(\delta, R)}(H) - \Pi^{(*, R)}(H)| \wedge 1 \right] = 0. \quad (5.19)$$

The triangle inequality then yields, for any $\delta \in (0, 1)$,

$$\begin{aligned} &E[|\Sigma_n^{\delta_n}(H) - \Sigma^*(H)| \wedge 1] \\ &\leq E[|\Sigma_n^{\delta_n}(H) - \Sigma_n^*(H)| \wedge 1] + E[|\Sigma_n^*(H) - \mathfrak{S}^{(\delta, R)}(p, \mu, Y_n)(H)| \wedge 1] \\ &\quad + E[|\mathfrak{S}^{(\delta, R)}(p, \mu, Y_n)(H) - \mathfrak{S}^{(\delta, R)}(p, \mu, Y)(H)| \wedge 1] \\ &\quad + E[|\mathfrak{S}^{(\delta, R)}(p, \mu, Y)(H) - \Sigma^*(H)| \wedge 1] \\ &= E^{x_n}[|\Pi_n^{(\delta_n, R)}(H) - \Pi_n^{(*, R)}(H)| \wedge 1] + E^{x_n}[|\Pi^{(*, R)}(H) - \Pi^{(\delta, R)}(H)| \wedge 1] \\ &\quad + E[|\mathfrak{S}^{(\delta, R)}(p, \mu, Y_n)(H) - \mathfrak{S}^{(\delta, R)}(p, \mu, Y)(H)| \wedge 1] + E^x[|\Pi^{(\delta, R)}(H) - \Pi(H)| \wedge 1]. \end{aligned}$$

Letting $n \rightarrow \infty$ in the last display, the first term converges to 0 by (5.19), and the third term converges to 0 by (5.18) and Lemma 5.27(ii). Hence,

$$\limsup_{n \rightarrow \infty} E[|\Sigma_n^{\delta_n}(H) - \Sigma^*(H)| \wedge 1] \leq 2 \sup_{y \in S} E^y \left[|\Pi^{(\delta, R)}(H) - \Pi^{(*, R)}(H)| \wedge 1 \right].$$

Letting $\delta \rightarrow 0$ and using (5.19) once more, we obtain $\Sigma_n^{\delta_n}(H) \rightarrow \Sigma^*(H)$ in probability, which completes the proof. $\square$

Finally, we can prove the continuity of joint laws, using the above result, Proposition 5.10, and the spatial tightness condition.

Theorem 5.30. *Suppose that Assumptions 5.15 and 5.16 hold. Let μ be a Radon measure in the local Kato class and let $A = (A_t)_{t \geq 0}$ be the associated PCAF. For each $R > 1$, set*

$$\Pi^{(*,R)} := \mathfrak{S}^{(*,R)}(\Pi).$$

Then the following statements hold.

- (i) *The map $\mathcal{L}_{(X,\Pi)}$ is continuous.*
- (ii) *It holds that, for any $r > 0$, $N \in \mathbb{N}$, and $R > 1$,*

$$\begin{aligned} & d_{S, \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin}}(S \times \mathbb{R}_{\geq 0}))}^{\hat{C}} \left(\mathcal{L}_{(X, \Pi^{(*,R)})}, \mathcal{L}_{(X, \Pi)} \right) \\ & \leq e^{-r} + e^{-N} + \sup_{x \in S^{(r)}} P^x(\check{\sigma}_{S^{(R-1)}} < N). \end{aligned}$$

- (iii) *It holds that*

$$\mathcal{L}_{(X, \Pi^{(*,R)})} \xrightarrow{R \rightarrow \infty} \mathcal{L}_{(X, \Pi)} \quad \text{in} \quad C(S, \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin}}(S \times \mathbb{R}_{\geq 0}))).$$

Proof. For each $N \in \mathbb{N}$ and $R > 1$, we define an event

$$E_{N,R} := \{\check{\sigma}_{S^{(R-1)}} > N\}.$$

On this event, by (5.12), we have

$$d_{S \times \mathbb{R}_{\geq 0}}^{\mathcal{M}_{(*, \text{fin})}}(\Pi^{(*,R)}, \Pi) \leq e^{-N}.$$

Lemma 2.3 then yields that, for each $x \in S$,

$$\begin{aligned} & d_{L^0(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin}}(S \times \mathbb{R}_{\geq 0}))}^P \left(\mathcal{L}_{(X, \Pi^{(*,R)})}(x), \mathcal{L}_{(X, \Pi)}(x) \right) \\ & \leq e^{-N} + P^x(\check{\sigma}_{S^{(R-1)}} > N). \end{aligned}$$

Now, we can follow the same argument as in the proof of Theorem 5.25 to complete the proof. $\square$

By using Proposition 5.14 and working with the vague topology instead of the topology on $\mathcal{M}_{(*, \text{fin}}(S \times \mathbb{R}_{\geq 0})$, we can derive the continuity of $\mathcal{L}_{(X, \Pi)}$ without assuming the spatial tightness condition, although the continuity is then obtained only with respect to the vague topology.

Theorem 5.31. *Suppose that Assumption 5.15 holds. Let μ be a Radon measure in the local Kato class and let $A = (A_t)_{t \geq 0}$ be the associated PCAF. For each $R > 1$, set*

$$\Pi^{(*,R)} := \mathfrak{S}^{(*,R)}(\Pi).$$

Then the following statements hold.

- (i) *The map*

$$\mathcal{L}_{(X, \Pi)} : S \rightarrow \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}(S \times \mathbb{R}_{\geq 0}))$$

is continuous. (NB. In the codomain, we have $\mathcal{M}(S \times \mathbb{R}_{\geq 0})$ equipped with the vague topology, not $\mathcal{M}_{(, \text{fin}}(S \times \mathbb{R}_{\geq 0}))$.)*

(ii) For any $R > 1$,

$$d_{S, \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}(S \times \mathbb{R}_{\geq 0}))}^{\widehat{C}} \left(\mathcal{L}_{(X, \Pi(*, R))}, \mathcal{L}_{(X, \Pi)} \right) \leq e^{-R+1}$$

(iii) It holds that

$$\mathcal{L}_{(X, \Pi(*, R))} \xrightarrow{R \rightarrow \infty} \mathcal{L}_{(X, \Pi)} \quad \text{in} \quad C(S, \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}(S \times \mathbb{R}_{\geq 0}))).$$

Proof. For any $x \in S$, we have from Proposition 5.14 that

$$d_{L^0(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}(S \times \mathbb{R}_{\geq 0})}^P \left(\mathcal{L}_{(X, \Pi(*, R))}(x), \mathcal{L}_{(X, \Pi)}(x) \right) \leq e^{-R+1}.$$

Thus, we deduce the results in the same way as in the proof of Theorem 5.30. $\square$

6 Convergence of PCAFs and STOMs

In this section, we present the main results of this paper, which establish the convergence of PCAFs and STOMs from the convergence of processes within the framework of Gromov–Hausdorff topologies introduced in Section 3. In Section 6.1, we provide results in the setting where the underlying spaces are deterministic. This is extended to random spaces in the following subsection, with applications to random graph models in mind.

Throughout this section, we fix a Polish structure τ and a space transformation Ψ .

6.1 Deterministic spaces

The main result of this subsection is Theorem 6.6. It asserts that if processes and their smooth measures converge in a Gromov–Hausdorff-type topology, and if the local Kato class condition is satisfied uniformly by these measures, then the associated STOMs converge jointly with the processes.

We first clarify the setting for the result. For each $n \in \mathbb{N}$, we suppose the following:

- $(S_n, d_{S_n}, \rho_n, \mu_n, a_n)$ is an element of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau)$, i.e., (S_n, d_{S_n}) is a bcm space, ρ_n is the root of S_n , μ_n is a Radon measure on $\Psi(S_n)$, and $a_n \in \tau(S_n)$;
- $X_n = ((X_n(t))_{t \geq 0}, (P_n^x)_{x \in \Psi(S_n)})$ is a standard process on $\Psi(S_n)$ satisfying the DAC and L^0 -weak continuity conditions (Assumptions 4.1 and 5.15), with heat kernel p_n ;
- the Radon measure μ_n is in the local Kato class of X_n .

We also suppose the following:

- (S, d_S, ρ, μ, a) is an element of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau)$;
- $X = ((X(t))_{t \geq 0}, (P^x)_{x \in \Psi(S)})$ is a standard process on $\Psi(S)$ satisfying the DAC and L^0 -weak continuity conditions (Assumptions 4.1 and 5.15); with heat kernel p ,
- the Radon measure μ charges no sets semipolar for X .

The presence of Ψ in our setting is motivated by applications to collisions of stochastic processes, which will be studied from Section 7.1 onward. The incorporation of τ is also important, as it broadens the applicability of our main results. See Remarks 6.8 and 6.9 at the end of this subsection for further details.

Recall from Assumption 5.15(i) that the law maps $\mathcal{L}_{X_n}$ and $\mathcal{L}_X$ are continuous with respect to the weak topology induced by the L^0 topology. Our fundamental assumption for the main

result is the convergence of the underlying spaces together with the law maps and smooth measures. To put this in the framework introduced in Section 3, we define a structure by

$$\tau_{L^0\text{-SP}} := \tau_{\widehat{C}}(\Psi_{\text{id}}, \tau_{\mathcal{P}}(\tau_{L^0})),$$

where “SP” stands for “stochastic process”. In particular, for each bcm space M ,

$$\tau_{L^0\text{-SP}}(\Psi)(M) = \widehat{C}(\Psi(M), \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(M)))),$$

where we recall from Definition 3.21 that $\tau_{L^0\text{-SP}}(\Psi)$ denotes the composition $\tau_{L^0\text{-SP}} \circ \Psi$. To discuss convergence of heat kernels, we define another structure by

$$\tau_{\text{HK}} := \tau_{\widehat{C}}(\Psi_{\mathbb{R}_{>0} \times \text{id}^2}, \tau_{\mathbb{R}_{\geq 0}}). \quad (6.1)$$

In particular, for each bcm space M ,

$$\tau_{\text{HK}}(\Psi)(M) = \widehat{C}(\mathbb{R}_{>0} \times \Psi(M) \times \Psi(M), \mathbb{R}_{\geq 0}).$$

Assumption 6.1.

(i) *It holds that*

$$(S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{X_n}) \rightarrow (S, d_S, \rho, \mu, a, p, \mathcal{L}_X)$$

$$\text{in } \mathfrak{M}_{\bullet}(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP}}(\Psi)).$$

(ii) *For all $r > 0$,*

$$\lim_{\alpha \rightarrow \infty} \limsup_{n \rightarrow \infty} \|\mathcal{R}_{p_n}^{\alpha} \mu_n^{(r)}\|_{\infty} = 0.$$

Remark 6.2. By Proposition 4.9, condition (ii) of Assumption 6.1 is equivalent to the following:

(iii) *For all $r > 0$,*

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{x \in \Psi(S_n^{(r)})} \int_0^{\delta} \int_{\Psi(S_n^{(r)})} p_n(t, x, y) \mu_n(dy) dt = 0.$$

As observed in Section 5.4, the following (uniform) spatial tightness condition is instrumental in obtaining a stronger mode of convergence for STOMs than that given by the vague topology.

Assumption 6.3. *Each process X_n , $n \geq 1$, as well as the limiting process X , satisfies the spatial tightness condition (Assumption 5.16). Moreover, for all $r > 0$ and $T > 0$, it holds that*

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{x \in S_n^{(r)}} P_n^x(\check{\sigma}_{S_n^{(R)}} < T) = 0. \quad (6.2)$$

Remark 6.4. By the same argument as in the proof of Lemma 5.18, if the convergence of law maps in Assumption 6.1(i) takes place under a topology finer than the L^0 topology, such as the J_1 -Skorohod topology, then (6.2) can be automatically satisfied. For instance, set

$$\tau_{\text{SP}} := \tau_{\widehat{C}}(\Psi_{\text{id}}, \tau_{\mathcal{P}}(\tau_{J_1})). \quad (6.3)$$

If the convergence in Assumption 6.1(i) holds with $\tau_{L^0\text{-SP}}$ replaced by τ_{SP} , then Assumption 6.3 is satisfied.

Under the above assumptions, it follows from Fatou’s lemma that the limiting measure μ also belongs to the local Kato class.

Lemma 6.5. *Under Assumption 6.1, the Radon measure μ is in the local Kato class of X .*

Proof. It is enough to show that, for all but countably many $r > 0$,

$$\lim_{\alpha \rightarrow \infty} \|\mathcal{R}_p^\alpha \mu^{(r)}\|_\infty = 0, \quad (6.4)$$

By Assumption 6.1(i) and Theorem 3.7, all the rooted metric spaces (S_n, ρ_n) and (S, ρ) are embedded isometrically into a common rooted bcm space (M, ρ_M) in such a way that $S_n \rightarrow S$ in the Fell topology as closed subsets of M , $\mu_n \rightarrow \mu$ vaguely as measures on $\Psi(M)$, and $p_n \rightarrow p$ in $\widehat{C}(\mathbb{R}_{>0} \times \Psi(M) \times \Psi(M), \mathbb{R}_{\geq 0})$. Fix $r > 0$ such that $\mu_n^{(r)} \rightarrow \mu^{(r)}$ weakly as measures on $\Psi(M)$, and fix $x \in S$. By Lemma 3.20, we can find elements $x_n \in \Psi(S_n)$ converging to x in $\Psi(M)$. It is then the case that, for each $t > 0$,

$$p_n(t, x_n, \cdot) \rightarrow p(t, x, \cdot) \quad \text{in } \widehat{C}(\Psi(M), \mathbb{R}_{\geq 0}).$$

We deduce from Lemma 2.9 that, for each $t > 0$,

$$\lim_{n \rightarrow \infty} \int_{\Psi(M)} p_n(t, x_n, y) \mu_n^{(r)}(dy) = \int_{\Psi(M)} p(t, x, y) \mu^{(r)}(dy).$$

Fatou's lemma then yields that, for each $\alpha > 0$,

$$\begin{aligned} \int_0^\infty e^{-\alpha t} \int_{\Psi(M)} p(t, x, y) \mu^{(r)}(dy) dt &\leq \liminf_{n \rightarrow \infty} \int_0^\infty e^{-\alpha t} \int_{\Psi(M)} p_n(t, x_n, y) \mu_n^{(r)}(dy) dt \\ &\leq \liminf_{n \rightarrow \infty} \|\mathcal{R}_{p_n}^\alpha \mu_n^{(r)}\|_\infty. \end{aligned}$$

Taking the supremum over $x \in \Psi(S)$, it follows that

$$\|\mathcal{R}_p^\alpha \mu^{(r)}\|_\infty \leq \liminf_{n \rightarrow \infty} \|\mathcal{R}_{p_n}^\alpha \mu_n^{(r)}\|_\infty. \quad (6.5)$$

This, combined with Assumption 6.1(ii), yields (6.4). $\square$

Let A_n and Π_n (resp. A and Π) denote the PCAF and the STOM of X_n (resp. X) associated with μ_n (resp. μ). The main result below asserts that, under the above assumptions, the joint laws of the processes together with their associated PCAFs/STOMs converge, locally uniformly in the starting point. For notational convenience, set

$$\begin{aligned} \tau_{L^0\text{-SP,PCAF}} &:= \tau_{\widehat{C}}\left(\Psi_{\text{id}}, \tau_{\mathcal{P}}\left(\tau_{L^0} \times \Psi_{C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})}\right)\right), \\ \tau_{L^0\text{-SP,STOM}} &:= \tau_{\widehat{C}}\left(\Psi_{\text{id}}, \tau_{\mathcal{P}}\left(\tau_{L^0} \times \tau_{\mathcal{M}(*, \text{fin})}\right)\right), \\ \tau_{L^0\text{-SP,v-STOM}} &:= \tau_{\widehat{C}}\left(\Psi_{\text{id}}, \tau_{\mathcal{P}}\left(\tau_{L^0} \times \tau_{\mathcal{M}}(\Psi_{\text{id}} \times \Psi_{\mathbb{R}_{\geq 0}})\right)\right), \end{aligned}$$

Here, the first structure is used for describing the convergence of PCAFs, the second for that of STOMs, and the third also for STOMs, but with respect to the vague topology (which is weaker than the topology introduced in Section 2.5). In particular, for each bcm space M ,

$$\begin{aligned} \tau_{L^0\text{-SP,PCAF}}(\Psi)(M) &= \widehat{C}(\Psi(M), \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(M)) \times C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}))), \\ \tau_{L^0\text{-SP,STOM}}(\Psi)(M) &= \widehat{C}(\Psi(M), \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(M)) \times \mathcal{M}(*, \text{fin})(\Psi(M) \times \mathbb{R}_{\geq 0}))), \\ \tau_{L^0\text{-SP,v-STOM}}(\Psi)(M) &= \widehat{C}(\Psi(M), \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(M)) \times \mathcal{M}(\Psi(M) \times \mathbb{R}_{\geq 0}))). \end{aligned}$$

Theorem 6.6.

(i) Under Assumptions 6.1 and 6.3, it holds that

$$(S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{(X_n, A_n)}) \rightarrow (S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, A)})$$

$$\text{in } \mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP,PCAF}}(\Psi)).$$

(ii) Under Assumptions 6.1 and 6.3, it holds that

$$(S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{(X_n, \Pi_n)}) \rightarrow (S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, \Pi)})$$

$$\text{in } \mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP, STOM}}(\Psi)).$$

(iii) Under Assumption 6.1, it holds that

$$(S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{(X_n, \Pi_n)}) \rightarrow (S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, \Pi)})$$

$$\text{in } \mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP, v-STOM}}(\Psi)).$$

Since convergence of STOMs implies convergence of PCAFs (see Remark 2.26), if we establish (ii), then (i) follows immediately from Proposition 2.25. Nevertheless, we provide separate proofs here, since several intermediate results obtained in their proofs will be used to extend the results to random spaces in the next subsection.

The idea of the proofs is common to all three assertions. First, we employ the continuity of the approximation maps introduced in Definitions 5.2 and 5.4 to lift the convergence of the laws of the processes to that of the joint laws of the processes and the approximated PCAFs. Then, using Assumptions 6.1(ii) and 6.3, we verify the uniformity of the approximation and finally deduce the desired convergence. We note that in (iii), the lack of Assumption 6.3 leads to convergence in a weaker topology than that in (ii). This situation has already appeared in Proposition 5.14 and Theorem 5.31.

Proof of Theorem 6.6(i). Before proceeding to the detailed argument, we introduce some notation. For each $\delta > 0$ and $R > 1$, we set

$$\begin{aligned} A^{(\delta, R)} &:= \mathfrak{P}^{(\delta, R)}(p, \mu, X), & A^{(*, R)} &:= \mathfrak{P}^{(*, R)}(X, A), \\ A_n^{(\delta, R)} &:= \mathfrak{P}^{(\delta, R)}(p_n, \mu_n, X_n), & A_n^{(*, R)} &:= \mathfrak{P}^{(*, R)}(X_n, A_n), \end{aligned}$$

where the approximation maps are recalled from Definitions 5.2 and 5.4. We define

$$\begin{aligned} \mathcal{X}_n &:= (S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{(X_n, A_n)}), \\ \mathcal{X}_n^{(*, R)} &:= (S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{(X_n, A_n^{(*, R)})}), \\ \mathcal{X}_n^{(\delta, R)} &:= (S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{(X_n, A_n^{(\delta, R)})}), \end{aligned}$$

and similarly define $\mathcal{X}$, $\mathcal{X}^{(*, R)}$, and $\mathcal{X}^{(\delta, R)}$ by removing the subscript n from the corresponding symbols above. For notational convenience, we write

$$\tau_1 := \tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP, PCAF}}(\Psi).$$

By Theorems 5.24 and 5.25, we have

$$\mathcal{X}^{(\delta, R)} \xrightarrow[\delta \rightarrow 0]{} \mathcal{X}^{(*, R)} \quad \text{in } \mathfrak{M}_\bullet(\tau_1), \tag{6.6}$$

$$\mathcal{X}^{(*, R)} \xrightarrow[R \rightarrow \infty]{} \mathcal{X} \quad \text{in } \mathfrak{M}_\bullet(\tau_1). \tag{6.7}$$

By Assumption 6.1(i) and Theorem 3.7, there exists a rooted bcmspace (M, ρ_M) into which all the metric spaces S_n and S can be isometrically embedded so that the following hold:

(A1) $S_n \rightarrow S$ in the Fell topology as closed subsets of M ,

(A2) $\rho_n = \rho = \rho_M$ as elements of M ,

(A3) $\mu_n \rightarrow \mu$ vaguely as measures on $\Psi(M)$,

(A4) $a_n \rightarrow a$ in $\tau(M)$,

(A5) $p_n \rightarrow p$ in $\widehat{C}(\mathbb{R}_{>0} \times \Psi(M) \times \Psi(M), \mathbb{R}_{\geq 0})$,

(A6) $\mathcal{L}_{X_n} \rightarrow \mathcal{L}_X$ in $\widehat{C}(\Psi(M), \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(M))))$.

By (A5) and Lemma 2.6, we may assume that the domains of p_n and p are extended continuously to $\mathbb{R}_{>0} \times \Psi(M) \times \Psi(M)$ so that

$$p_n \rightarrow p \quad \text{in} \quad C(\mathbb{R}_{>0} \times \Psi(M) \times \Psi(M), \mathbb{R}_{\geq 0}). \quad (6.8)$$

Fix points $x_n \in \Psi(S_n)$ converging to some $x \in \Psi(S)$ in $\Psi(M)$. Then, by (A6),

$$\mathcal{L}_{X_n}(x_n) = P_n^{x_n}(X_n \in \cdot) \xrightarrow[n \rightarrow \infty]{d} \mathcal{L}_X(x) = P^x(X \in \cdot).$$

Hence, by the Skorohod representation theorem, we may assume that X_n under $P_n^{x_n}$ and X under P^x are coupled so that $X_n \rightarrow X$ in $L^0(\mathbb{R}_{\geq 0}, \Psi(M))$ almost surely. By (A3), (6.8), and this coupling,

$$(p_n, \mu_n, X_n) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} (p, \mu, X).$$

Moreover, from (ii), it is clear that

$$\sup_{n \geq 1} \|\mathcal{R}_{p_n}^1 \mu_n^{(r)}\|_\infty < \infty, \quad (6.9)$$

which, combined with Proposition 4.9, implies

$$\sup_{n \geq 1} \sup_{x \in \Psi(S_n)} \int_\delta^{2\delta} \int_S p_n(t, x, y) \mu_n^{(R)}(dy) dt < \infty.$$

Hence, by Lemma 5.22(ii), we obtain

$$A_n^{(\delta, R)} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} A^{(\delta, R)}. \quad (6.10)$$

In particular,

$$\mathcal{L}_{(X_n, A_n^{(\delta, R)})}(x_n) \xrightarrow[n \rightarrow \infty]{d} \mathcal{L}_{(X, A^{(\delta, R)})}(x).$$

Since the choice of $(x_n)_{n \geq 1}$ was arbitrary, we deduce that, for each $\delta > 0$ and $R > 1$,

$$\mathcal{L}_{(X_n, A_n^{(\delta, R)})} \xrightarrow[n \rightarrow \infty]{} \mathcal{L}_{(X, A^{(\delta, R)})} \quad \text{in} \quad \widehat{C}(\Psi(M), \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(M)) \times C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}))).$$

Therefore, for each $\delta > 0$ and $R > 1$,

$$\mathcal{X}_n^{(\delta, R)} \xrightarrow[n \rightarrow \infty]{} \mathcal{X}^{(\delta, R)} \quad \text{in} \quad \mathfrak{M}_\bullet(\tau_1). \quad (6.11)$$

(Note that so far we have used only Assumption 6.1(i) and (6.9).)

By Lemma 3.9 and Proposition 5.24, for any $n \geq 1$, $N \in \mathbb{N}$, $\alpha \in (0, 1)$, and $\delta \in (0, 1)$,

$$\begin{aligned} d_{\mathfrak{M}_\bullet}^{\tau_1}(\mathcal{X}_n^{(\delta, R)}, \mathcal{X}_n^{(*, R)}) &\leq d_{S, \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(S_n)) \times C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}))}^{\widehat{C}}(\mathcal{L}_{(X_n, A_n^{(\delta, R)})}, \mathcal{L}_{(X_n, A_n^{(*, R)})}) \\ &\leq 2^{-N+1} + 2^{2N} C_1 \|\mathcal{R}^\alpha \mu_n^{(R)}\|_\infty \left(\delta \alpha \|\mathcal{R}^\alpha \mu_n^{(R)}\|_\infty + \|\mathcal{R}^{1/\delta} \mu_n^{(R)}\|_\infty \right) \\ &\quad + C_2 (2e)^{2N} (1 - e^{-\alpha N}) \|\mathcal{R}^1 \mu_n^{(R)}\|_\infty^2. \end{aligned} \quad (6.12)$$

By Assumption 6.1(ii), letting $n \rightarrow \infty$, $\delta \rightarrow 0$, then $\alpha \rightarrow 0$, and finally $N \rightarrow \infty$ in the above inequality yields

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} d_{\mathfrak{M}_\bullet}^{\tau_1}(\mathcal{X}_n^{(\delta, R)}, \mathcal{X}_n^{(*, R)}) = 0. \quad (6.13)$$

Combining this with (6.6) and (6.11) gives

$$\mathcal{X}_n^{(*,R)} \xrightarrow[n \rightarrow \infty]{} \mathcal{X}^{(*,R)} \quad \text{in } \mathfrak{M}_\bullet(\tau_1). \quad (6.14)$$

Similarly, by Lemma 3.9 and Theorem 5.25, for any $n \geq 1$, $N \in \mathbb{N}$, and $r > 0$,

$$\begin{aligned} d_{\mathfrak{M}_\bullet}^{\tau_1}(\mathcal{X}_n^{(*,R)}, \mathcal{X}_n) &\leq d_{S, \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(S_n)) \times C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}))}^{\widehat{C}}(\mathcal{L}_{(X_n, A_n^{(*,R)})}, \mathcal{L}_{(X_n, A_n)}) \\ &\leq e^{-r} + 2^{-N+1} + 2^N \sup_{x \in \Psi(S_n^{(r)})} P^x(\check{\sigma}_{\Psi(S_n^{(R-1)})} < N). \end{aligned} \quad (6.15)$$

By Assumption 6.3, letting $n \rightarrow \infty$, $R \rightarrow \infty$, then $N \rightarrow \infty$, and finally $r \rightarrow \infty$ in the above inequality gives

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} d_{\mathfrak{M}_\bullet}^{\tau_1}(\mathcal{X}_n^{(*,R)}, \mathcal{X}_n) = 0. \quad (6.16)$$

Combining this with (6.7) and (6.14), we obtain

$$\mathcal{X}_n \xrightarrow[n \rightarrow \infty]{} \mathcal{X} \quad \text{in } \mathfrak{M}_\bullet(\tau_1),$$

which completes the proof. $\square$

Proof of Theorem 6.6(ii). We use the same notation as in the proof of Theorem 6.6(i), and introduce some new notation. For each $\delta > 0$ and $R > 1$, we set

$$\begin{aligned} \Pi^{(\delta, R)} &:= \mathfrak{S}^{(\delta, R)}(p, \mu, X), & \Pi^{(*, R)} &:= \mathfrak{S}^{(*, R)}(X, A), \\ \Pi_n^{(\delta, R)} &:= \mathfrak{S}^{(\delta, R)}(p_n, \mu_n, X_n), & \Pi_n^{(*, R)} &:= \mathfrak{S}^{(*, R)}(X_n, A_n), \end{aligned}$$

where the approximation maps are recalled from Definitions 5.8 and 5.9. We define

$$\begin{aligned} \mathcal{Y}_n &:= (S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{(X_n, \Pi_n)}) , \\ \mathcal{Y}_n^{(*, R)} &:= (S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{(X_n, \Pi_n^{(*, R)})}) , \end{aligned}$$

and similarly define $\mathcal{Y}$ and $\mathcal{Y}^{(*, R)}$ by removing the subscript n from the corresponding symbols above. We write

$$\tau_2 := \tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP, STOM}}(\Psi).$$

By Theorem 5.30, we have

$$\mathcal{Y}^{(*, R)} \xrightarrow[R \rightarrow \infty]{} \mathcal{Y} \quad \text{in } \mathfrak{M}_\bullet(\tau_2). \quad (6.17)$$

Fix $R > 1$. We first show that

$$\mathcal{Y}_n^{(*, R)} \xrightarrow[n \rightarrow \infty]{} \mathcal{Y}^{(*, R)} \quad \text{in } \mathfrak{M}_\bullet(\tau_2) \quad (6.18)$$

by using only Assumption 6.1(i) and the following condition:

$$\lim_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{y \in \Psi(S_n)} \int_0^t \int_{\Psi(S_n)} p_n(s, y, z) \mu^{(R)}(dz) ds = 0, \quad (6.19)$$

which is equivalent to

$$\lim_{\alpha \rightarrow \infty} \limsup_{n \rightarrow \infty} \|\mathcal{R}_{p_n}^\alpha \mu_n^{(R)}\|_\infty = 0 \quad (6.20)$$

by Proposition 4.9. (Note that this condition is indeed satisfied in the present setting by Assumption 6.1(ii).) By Assumption 6.1(i), there exists a rooted bcm space (M, ρ_M) into which all

the metric spaces S_n and S can be isometrically embedded so that (A1)–(A6) hold. The desired convergence follows once we show that

$$\mathcal{L}_{(X_n, \Pi_n^{(*,R)})} \rightarrow \mathcal{L}_{(X, \Pi^{(*,R)})} \quad \text{in} \quad \widehat{C}(\Psi(M), \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(M)) \times \mathcal{M}_{(*, \text{fin}}(\Psi(M) \times \mathbb{R}_{\geq 0}))).$$

Fix elements $x_n \in \Psi(S_n)$ converging to $x \in \Psi(S)$ in $\Psi(M)$. The above convergence is obtained by showing that

$$\mathcal{L}_{(X_n, \Pi_n^{(*,R)})}(x_n) \rightarrow \mathcal{L}_{(X, \Pi^{(*,R)})}(x) \quad \text{in} \quad \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(M)) \times \mathcal{M}_{(*, \text{fin}}(\Psi(M) \times \mathbb{R}_{\geq 0})).$$

As before, by (A6) and the Skorohod representation theorem, we may assume that X_n under $P_n^{x_n}$ and X under P^x are coupled so that $X_n \rightarrow X$ in $L^0(\mathbb{R}_{\geq 0}, \Psi(M))$ almost surely. We denote by P the underlying probability measure for this coupling. We verify the above convergence by showing that

$$\Pi_n^{(*,R)} \xrightarrow[n \rightarrow \infty]{P} \Pi^{(*,R)} \quad \text{in} \quad \mathcal{M}_{(*, \text{fin}}(\Psi(M) \times \mathbb{R}_{\geq 0}) \text{ under } P. \quad (6.21)$$

Similarly to (6.10), one can verify that, for each $\delta > 0$,

$$\Pi_n^{(\delta, R)} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \Pi^{(\delta, R)} \quad \text{in} \quad \mathcal{M}_{(*, \text{fin}}(\Psi(M) \times \mathbb{R}_{\geq 0}). \quad (6.22)$$

Indeed, the only change is that one uses Lemma 5.27(ii) instead of Lemma 5.22(ii).

We are now able to follow the proof of Proposition 5.29. Define a dissecting semi-ring $\mathcal{I}$ by setting

$$\mathcal{I} := \{E \times [s, t) \mid 0 \leq s < t < \infty, E \in \mathcal{B}(\Psi(M)) \text{ such that } \mu(\partial_{\Psi(M)} E) = 0\}.$$

Fix a set $H = E \times [s, t) \in \mathcal{I}$. It follows from (5.19) that

$$\lim_{\delta \rightarrow 0} \sup_{y \in \Psi(S)} E^y \left[|\Pi^{(*,R)}(H) - \Pi^{(\delta, R)}(H)| \wedge 1 \right] = 0. \quad (6.23)$$

In the same manner as we established (5.19), using (5.13) and (6.20), we deduce that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{y \in \Psi(S_n)} E_n^y \left[|\Pi_n^{(*,R)}(H) - \Pi_n^{(\delta, R)}(H)| \wedge 1 \right] = 0. \quad (6.24)$$

The triangle inequality yields that

$$\begin{aligned} E \left[|\Pi_n^{(*,R)}(H) - \Pi^{(*,R)}(H)| \wedge 1 \right] &\leq E_n^{x_n} \left[|\Pi_n^{(*,R)}(H) - \Pi_n^{(\delta, R)}(H)| \wedge 1 \right] \\ &\quad + E \left[|\Pi_n^{(\delta, R)}(H) - \Pi^{(\delta, R)}(H)| \wedge 1 \right] \\ &\quad + E^x \left[|\Pi^{(\delta, R)}(H) - \Pi^{(*,R)}(H)| \wedge 1 \right]. \end{aligned}$$

Let $n \rightarrow \infty$ and then $\delta \rightarrow 0$ in the above inequality. By (6.23) and (6.24), the first and the third terms converge to 0. The second term also converges to 0 by (6.22). Hence, $\Pi_n^{(*,R)}(H) \xrightarrow{P} \Pi^{(*,R)}(H)$ under P , which implies (6.21). We thus obtain (6.18).

By Lemma 3.9 and Theorem 5.30, for any $n \geq 1$, $N \in \mathbb{N}$, $r > 0$, and $R > 1$,

$$\begin{aligned} d_{\mathfrak{M}_\bullet}^{T_2}(\mathcal{Y}_n^{(*,R)}, \mathcal{Y}_n) &\leq d_{\Psi(S_n), \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(S_n)) \times \mathcal{M}_{(*, \text{fin}}(\Psi(S_n) \times \mathbb{R}_{\geq 0}))}^{\widehat{C}} \left(\mathcal{L}_{(X_n, \Pi_n^{(*,R)})}, \mathcal{L}_{(X_n, \Pi_n)} \right) \\ &\leq e^{-r} + e^{-N} + \sup_{x \in \Psi(S_n^{(r)})} P_n^x \left(\check{\sigma}_{\Psi(S_n^{(R-1)})} < N \right). \end{aligned}$$

Thus, the same argument that led to (6.16) yields that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} d_{\mathfrak{M}_\bullet}^{T_2}(\mathcal{Y}_n^{(*,R)}, \mathcal{Y}_n) = 0. \quad (6.25)$$

Combining this with (6.17) and (6.18), we obtain

$$\mathcal{Y}_n \xrightarrow[n \rightarrow \infty]{} \mathcal{Y} \quad \text{in} \quad \mathfrak{M}_\bullet(\tau_2),$$

which completes the proof. $\square$

Proof of Theorem 6.6(iii). We use the same notation as before, and additionally introduce

$$\tau_3 := \tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP}, \text{v-STOM}}(\Psi).$$

By Theorem 5.31, we obtain

$$\mathcal{Y}^{(*,R)} \xrightarrow[R \rightarrow \infty]{} \mathcal{Y} \quad \text{in} \quad \mathfrak{M}_\bullet(\tau_3). \quad (6.26)$$

Note that in the proof above, (6.18) is derived solely from Assumption 6.1. Since the topology on $\mathfrak{M}_\bullet(\tau_3)$ is weaker than that on $\mathfrak{M}_\bullet(\tau_2)$, it follows that

$$\mathcal{Y}_n^{(*,R)} \xrightarrow[n \rightarrow \infty]{} \mathcal{Y}^{(*,R)} \quad \text{in} \quad \mathfrak{M}_\bullet(\tau_3), \quad \forall R > 1. \quad (6.27)$$

By Lemma 3.9 and Theorem 5.31, for any $n \geq 1$ and $R > 1$ we have

$$\begin{aligned} d_{\mathfrak{M}_\bullet}^{\tau_3}(\mathcal{Y}_n^{(*,R)}, \mathcal{Y}_n) &\leq \widehat{d}_{\Psi(S_n), \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(S_n)) \times \mathcal{M}(\Psi(S_n) \times \mathbb{R}_{\geq 0}))}^{\widehat{C}}(\mathcal{L}_{(X_n, \Pi_n^{(*,R)})}, \mathcal{L}_{(X_n, \Pi_n)}) \\ &\leq e^{-R+1}. \end{aligned} \quad (6.28)$$

This immediately yields

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} d_{\mathfrak{M}_\bullet}^{\tau_3}(\mathcal{Y}_n^{(*,R)}, \mathcal{Y}_n) = 0. \quad (6.29)$$

Therefore, the desired convergence follows from (6.26), (6.27), and (6.29). $\square$

Remark 6.7. When the convergence of processes holds in a stronger Polish topology, such as the J_1 -Skorohod topology, the joint convergence established in Theorem 6.6 holds accordingly in the corresponding stronger topology. For example, as discussed in Remark 6.4, if the convergence in Assumption 6.1(i) holds with $\tau_{L^0\text{-SP}}$ replaced by τ_{SP} , then the convergences in Theorem 6.6(i) and (ii), holds with $\tau_{L^0\text{-SP}, \text{PCAF}}(\Psi)$ and $\tau_{L^0\text{-SP}, \text{STOM}}$ replaced by the following structures, respectively:

$$\begin{aligned} \tau_{\text{SP}, \text{PCAF}} &:= \widehat{C}(\Psi_{\text{id}}, \tau_{\mathcal{P}}(\tau_{J_1} \times \Psi_{C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})})), \\ \tau_{\text{SP}, \text{STOM}} &:= \widehat{C}(\Psi_{\text{id}}, \tau_{\mathcal{P}}(\tau_{J_1} \times \tau_{\mathcal{M}_{(*, \text{fin})}})). \end{aligned}$$

This replacement is readily verified via a standard tightness argument, see Appendix C. As a consequence of this replacement argument, Theorem 1.3 follows directly from Theorem 6.6(ii).

Remark 6.8. We note that the convergence of the additional structures a_n assumed in Assumption 6.1 is preserved in Theorem 6.6. Put differently, the convergence of PCAFs and STOMs is established while retaining the other structures of the underlying spaces. In this sense, our formulation shares a similar philosophy with that of stable convergence [5]. For a concrete application of this formulation, see Remark 6.9 below.

Remark 6.9. In some examples, there exists a canonical space M into which both S_n and S can be embedded. For instance, when S_n are the standard graph approximations converging to the two-dimensional Sierpiński gasket S , the Euclidean space $\mathbb{R}^2$ serves as such a space M . In this case, we set $\tau := \tau_{\widehat{C}}(\Psi_{\text{id}}, \tau_M)$, where we recall the fixed structure τ_M from (S1), and let $a_n \in \tau(S_n)$ denote the canonical embedding $a_n: S_n \rightarrow M$. Then, by Theorems 3.7 and 6.6, together with the continuous mapping theorem, we obtain convergence of the processes and their STOMs within the fixed space M .

6.2 Random spaces

The main result of this subsection is Theorem 6.14 below, which is a version of Theorem 6.6 for random spaces.

We first clarify the setting for the result. For each $n \in \mathbb{N}$, we let $(\Omega_n, \mathcal{G}_n, \mathbf{P}_n)$ be a complete probability space, and assume that, for $\mathbf{P}_n$ -a.s. $\omega \in \Omega_n$,

- $(S_n^\omega, d_{S_n}^\omega, \rho_n^\omega, \mu_n^\omega, a_n^\omega) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau)$;
- X_n^ω is a standard process on $\Psi(S_n^\omega)$ satisfying the DAC and L^0 -weak continuity conditions (Assumptions 4.1 and 5.15), with heat kernel p_n^ω ;
- the Radon measure μ_n^ω belongs to the local Kato class of X_n^ω .

Similarly, we let $(\Omega, \mathcal{G}, \mathbf{P})$ be a complete probability space, and assume that, for $\mathbf{P}$ -a.s. $\omega \in \Omega$,

- $(S^\omega, d_S^\omega, \rho^\omega, \mu^\omega, a^\omega) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau)$;
- X^ω is a standard process on $\Psi(S^\omega)$ satisfying the DAC and L^0 -weak continuity conditions (Assumptions 4.1 and 5.15), with heat kernel p^ω ;
- the Radon measure μ^ω charges no sets semipolar for X^ω .

To discuss the laws of these objects, we assume that the maps

$$\begin{aligned} (\Omega_n, \mathcal{G}_n) \ni \omega &\mapsto (S_n^\omega, d_{S_n}^\omega, \rho_n^\omega, \mu_n^\omega, a_n^\omega, p_n^\omega, \mathcal{L}_{X_n^\omega}) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP}}(\Psi)), \\ (\Omega, \mathcal{G}) \ni \omega &\mapsto (S^\omega, d_S^\omega, \rho^\omega, \mu^\omega, a^\omega, p^\omega, \mathcal{L}_{X^\omega}) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP}}(\Psi)) \end{aligned} \quad (6.30)$$

are measurable, where the codomain is equipped with the Borel σ -algebra. (Strictly speaking, since the objects are defined only for $\mathbf{P}_n$ - and $\mathbf{P}$ -a.s. ω , we extend them arbitrarily on null sets. As each probability space is complete, this does not affect measurability of the maps.)

We now consider the following versions of Assumptions 6.1 and 6.3. As usual, we suppress the dependence on ω in the notation.

Assumption 6.10.

(i) It holds that

$$(S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{X_n}) \xrightarrow{d} (S, d_S, \rho, \mu, a, p, \mathcal{L}_X) \quad (6.31)$$

in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{\text{SP}}(\Psi))$.

(ii) For all $r > 0$,

$$\lim_{\alpha \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[\|\mathcal{R}_{p_n}^\alpha \mu_n^{(r)}\|_\infty \wedge 1 \right] = 0.$$

Remark 6.11. By Proposition 4.9, condition (ii) of Assumption 6.10 is equivalent to the following:

(iii) For all $r > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[\left(\sup_{x \in \Psi(S_n^{(r)})} \int_0^\delta \int_{\Psi(S_n^{(r)})} p_n(t, x, y) \mu_n(dy) dt \right) \wedge 1 \right] = 0.$$

Assumption 6.12. For almost every realization ω_n and ω of the random spaces, the processes $X_n^{\omega_n}$, $n \geq 1$, and X^ω satisfy the spatial tightness condition (Assumption 5.16). Moreover, for all $r > 0$ and $T > 0$, it holds that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[\sup_{x \in \Psi(S_n^{(r)})} P_n^x \left(\check{\sigma}_{\Psi(S_n^{(R)})} < T \right) \right] = 0.$$

Similarly to Lemma 6.5, we first verify that the limiting measure μ is in the local Kato class, almost surely.

Lemma 6.13. *Under Assumption 6.10, $\mathbf{P}$ -a.s., the Radon measure μ is in the local Kato class of X .*

Proof. By the Skorohod representation theorem, we may assume that the convergence in (6.31) takes place almost surely in some probability space with probability measure $\mathbb{P}$. Fix $r > 0$. By Assumption 6.10(ii) and Fatou's lemma,

$$0 = \lim_{\alpha \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left[\left\| \mathcal{R}_{p_n}^\alpha \mu_n^{(r)} \right\|_\infty \wedge 1 \right] \geq \mathbb{E} \left[\lim_{\alpha \rightarrow \infty} \liminf_{n \rightarrow \infty} \left(\left\| \mathcal{R}_{p_n}^\alpha \mu_n^{(r)} \right\|_\infty \wedge 1 \right) \right].$$

It follows that

$$\lim_{\alpha \rightarrow \infty} \liminf_{n \rightarrow \infty} \left\| \mathcal{R}_{p_n}^\alpha \mu_n^{(r)} \right\|_\infty = 0, \quad \mathbb{P}\text{-a.s.}$$

This, combined with (6.5), implies that, $\mathbf{P}$ -a.s.,

$$\lim_{\alpha \rightarrow \infty} \left\| \mathcal{R}_p^\alpha \mu^{(r)} \right\|_\infty = 0.$$

By the monotonicity with respect to r , the above assertion holds for all $r > 0$, $\mathbf{P}$ -a.s. This completes the proof. $\square$

For each $n \geq 1$, we let A_n and Π_n be the PCAF and STOM of X_n associated with μ_n , respectively. By Lemma 6.13, $\mathbf{P}$ -a.s., there exists a PCAF A and STOM Π of X associated with μ . We now state the main result.

Theorem 6.14.

(i) *Under Assumptions 6.10 and 6.12, it holds that*

$$(S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{(X_n, A_n)}) \xrightarrow{d} (S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, A)})$$

as random elements of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP,PCAF}}(\Psi))$.

(ii) *Under Assumptions 6.10 and 6.12, it holds that*

$$(S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{(X_n, \Pi_n)}) \xrightarrow{d} (S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, \Pi)})$$

as random elements of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP,STOM}}(\Psi))$.

(iii) *Under Assumption 6.10, it holds that*

$$(S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{(X_n, \Pi_n)}) \xrightarrow{d} (S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, \Pi)})$$

as random elements of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP,v-STOM}}(\Psi))$.

Before proving the above theorem, we justify the measurability of the random objects appearing in its statements.

Lemma 6.15. *All the tuples appearing in the assertions of Theorem 6.14 are random elements (that is, measurable).*

Proof. We prove only the measurability of

$$(S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, A)}),$$

as an element of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP,PCAF}}(\Psi))$, since the measurability of the remaining tuples can be shown in the same way. The idea is, again, to use the approximation results established in Section 5.

For each $\delta > 0$ and $R > 1$, we set

$$A^{(\delta, R)} := \mathfrak{P}^{(\delta, R)}(p, \mu, X), \quad A^{(*, R)} := \mathfrak{P}^{(*, R)}(X, A),$$

where the approximation maps are recalled from Definitions 5.2 and 5.4. By the measurability of the map in (6.30) and the measurability of $\mathfrak{P}^{(\delta, R)}$ established in Lemma 5.22, we deduce that

$$\left(S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, A^{(\delta, R)})} \right)$$

is a random element. By Theorems 5.24 and 5.25, we have

$$\begin{aligned} \left(S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, A^{(\delta, R)})} \right) &\xrightarrow[\delta \rightarrow 0]{\text{a.s.}} \left(S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, A^{(*, R)})} \right), \\ \left(S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, A^{(*, R)})} \right) &\xrightarrow[R \rightarrow \infty]{\text{a.s.}} \left(S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, A)} \right). \end{aligned}$$

Hence $(S, d_S, \rho, \mu, a, p, \mathcal{L}_{(X, A)})$ is a random element. $\square$

In the proof of Theorem 6.14, the following tightness of potentials plays a crucial role.

Lemma 6.16. *Under Assumption 6.10(ii), it holds that*

$$\lim_{\alpha \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}_n \left(\|\mathcal{R}_{p_n}^1 \mu_n^{(r)}\|_\infty > \alpha \right) = 0, \quad \forall r > 0.$$

Proof. By [58, Equation (2.7)] (with the parameters (p, α, β) therein set to $(1, 1, \alpha)$), we deduce that

$$\|\mathcal{R}_{p_n}^1 \mu_n^{(r)}\|_\infty \leq \alpha \|\mathcal{R}_{p_n}^\alpha \mu_n^{(r)}\|_\infty, \quad \forall \alpha > 1.$$

It follows that

$$\mathbf{P}_n \left(\|\mathcal{R}_{p_n}^1 \mu_n^{(r)}\|_\infty > \alpha/2 \right) \leq \mathbf{P}_n \left(\|\mathcal{R}_{p_n}^\alpha \mu_n^{(r)}\|_\infty > 1/2 \right) \leq 2\mathbf{E}_n \left[\|\mathcal{R}_{p_n}^\alpha \mu_n^{(r)}\|_\infty \wedge 1 \right],$$

where we use Markov's inequality to obtain the last inequality. Hence, Assumption 6.10(ii) yields the desired result. $\square$

We are now ready to prove Theorem 6.14(i). Before presenting the proof, we briefly outline the main idea. The basic strategy is the same as in the deterministic case discussed in the previous subsection: we propagate the convergence of the processes to that of the joint laws with the approximated STOMs by using the continuity of the approximation maps (Lemma 5.27(ii)). However, the continuity of these maps requires the uniform boundedness of the 1-potentials of the smooth measures. This difficulty is overcome by Lemma 6.16. Specifically, by applying the Skorohod representation theorem, we show that for any subsequence, one can further extract a subsubsequence along which both the convergence of the processes and the uniform boundedness of the 1-potentials hold almost surely on a suitably constructed probability space. This allows us to apply the continuity of the approximation maps appropriately, so that the approximation argument proceeds in the same manner as in the deterministic case.

Proof of Theorem 6.14(i). We use the notation introduced in the proof of Theorem 6.6. We first note that Lemma 6.16 implies that $\{\|\mathcal{R}_{p_n}^1 \mu_n^{(r)}\|_\infty\}_{n \geq 1}$ is tight for each $r > 0$. From this and Assumption 6.10(i), it follows that the family of random elements

$$\left(S_n, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{X_n}, (\|\mathcal{R}_{p_n}^1 \mu_n^{(r)}\|_\infty)_{r \in \mathbb{N}} \right), \quad n \geq 1,$$

is tight in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP}}(\Psi)) \times \mathbb{R}_{\geq 0}^\mathbb{N}$. Fix an arbitrary subsequence $(n_k)_{k \geq 1}$ of $\mathbb{N}$. By taking a further subsequence and relabeling for simplicity, we can assume that

$$\left(S_{n_k}, \rho_{n_k}, \mu_{n_k}, a_{n_k}, p_{n_k}, \mathcal{L}_{X_{n_k}}, (\|\mathcal{R}_{p_{n_k}}^1 \mu_{n_k}^{(r)}\|_\infty)_{r \in \mathbb{N}} \right) \xrightarrow[n \rightarrow \infty]{\text{d}} (S, \rho, \mu, a, p, \mathcal{L}_X, (\xi_r)_{r \in \mathbb{N}}),$$

where $(\xi_r)_{r \in \mathbb{N}}$ is a certain collection of random variables. By the Skorohod representation theorem, we may assume that the above convergence holds almost surely in some probability space with probability measure $\mathbb{P}$. Now, we fix a realization such that the above convergence holds. It is then the case that

$$\sup_{k \geq 1} \|\mathcal{R}_{p_{n_k}}^1 \mu_{n_k}^{(r)}\|_\infty < \infty, \quad \forall r > 0.$$

Thus, similarly to (6.11), we deduce that, for each $\delta > 0$ and $R > 1$.

$$\mathcal{X}_{n_k}^{(\delta, R)} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mathcal{X}^{(\delta, R)} \quad \text{in } \mathfrak{M}_\bullet(\tau_1).$$

In particular, the above convergence holds in distribution. Since the sequence $(n_k)_{k \geq 1}$ was arbitrary, it follows that

$$\mathcal{X}_n^{(\delta, R)} \xrightarrow[n \rightarrow \infty]{\text{d}} \mathcal{X}^{(\delta, R)} \quad \text{in } \mathfrak{M}_\bullet(\tau_1). \quad (6.32)$$

In a similar manner as we checked (6.13), using (6.12) and Assumption 6.10(ii), we obtain that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[d_{\mathfrak{M}_\bullet}^{\tau_1} \left(\mathcal{X}_n^{(\delta, R)}, \mathcal{X}_n^{(*, R)} \right) \wedge 1 \right] = 0.$$

This, combined with (6.32), yields that

$$\mathcal{X}_n^{(*, R)} \xrightarrow[n \rightarrow \infty]{\text{d}} \mathcal{X}^{(*, R)} \quad \text{in } \mathfrak{M}_\bullet(\tau_1). \quad (6.33)$$

Moreover, similarly to (6.16), using (6.15) and Assumption 6.3, we obtain that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[d_{\mathfrak{M}_\bullet}^{\tau_1} \left(\mathcal{X}_n^{(*, R)}, \mathcal{X}_n \right) \wedge 1 \right] = 0.$$

This, combined with (6.7) and (6.33), yields the desired result. $\square$

We next prove Theorem 6.14(ii), which is the STOM counterpart of the above result. We begin by taking a coupling under which the joint laws of the processes and the PCAFs converge and the corresponding 1-potentials are uniformly bounded, as guaranteed by the previous result. Under this coupling, the distributional convergence of the PCAFs implies the convergence of their moments. Consequently, the uniform local Kato-type condition required in the deterministic case (Assumption 6.1(ii)) holds almost surely. This allows us to apply exactly the same argument as in the deterministic setting, which yields the desired convergence of the STOMs.

Proof of Theorem 6.14(ii). We use the same notation as in the proof of Theorem 6.6. Note that Theorem 6.14(i) has already been established, so that all the results obtained in its proof are available here as well. Fix $R > 1$. By the same argument as at the beginning of the previous proof, we deduce from Lemma 6.16 and (6.33) that the family of random elements

$$\left(S_n, \rho_n, \mu_n, a_n, p_n, \mathcal{L}_{(X_n, A_n^{(R+1)})}, (\|\mathcal{R}_{p_n}^1 \mu_n^{(r)}\|_\infty)_{r \in \mathbb{N}} \right), \quad n \geq 1,$$

is tight in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}(\Psi) \times \tau \times \tau_{\text{HK}}(\Psi) \times \tau_{L^0\text{-SP,PCAF}}(\Psi)) \times \mathbb{R}_{\geq 0}^{\mathbb{N}}$. Fix an arbitrary subsequence $(n_k)_{k \geq 1}$ of $\mathbb{N}$. By taking a further subsequence and relabeling it for simplicity, we may assume that

$$\begin{aligned} & \left(S_{n_k}, \rho_{n_k}, \mu_{n_k}, a_{n_k}, p_{n_k}, \mathcal{L}_{(X_{n_k}, A_{n_k}^{(R+1)})}, (\|\mathcal{R}_{p_{n_k}}^1 \mu_{n_k}^{(r)}\|_\infty)_{r \in \mathbb{N}} \right) \\ & \xrightarrow[k \rightarrow \infty]{\text{d}} \left(S, \rho, \mu, a, p, \mathcal{L}_{(X, A^{(R+1)})}, (\xi_r)_{r \in \mathbb{N}} \right), \end{aligned}$$

where $(\xi_r)_{r \in \mathbb{N}}$ is a certain collection of random variables. By the Skorohod representation theorem, we may further assume that the above convergence holds almost surely on some probability

space with probability measure $\mathbb{P}$. We hence fix a realization such that this convergence holds. By the convergence of the 1-potentials, we have

$$\sup_{k \geq 1} \|\mathcal{R}_{p_{n_k}}^1 \mu_{n_k}^{(r)}\|_\infty < \infty. \quad (6.34)$$

We will show that, on the fixed realization,

$$\lim_{t \rightarrow 0} \limsup_{k \rightarrow \infty} \sup_{y \in \Psi(S_{n_k})} \int_0^t \int_{\Psi(S_{n_k})} p_{n_k}(s, y, z) \mu_{n_k}^{(R)}(dz) ds = 0, \quad (6.35)$$

which corresponds to (6.19) along the subsequence $(n_k)_{k \geq 1}$.

By Theorem 3.7, there exists a rooted bcm space (M, ρ_M) into which all the metric spaces S_n and S can be isometrically embedded so that (A1)–(A6) and the following condition hold:

$$(A7) \quad \mathcal{L}_{(X_{n_k}, A_{n_k}^{(*, R+1)})} \rightarrow \mathcal{L}_{(X, A^{(*, R+1)})} \text{ in } \widehat{C}(\Psi(M), \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, \Psi(M)) \times C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}))).$$

Fix $t \geq 0$, and take elements $x_n \in \Psi(S_n)$ converging to $x \in \Psi(S)$ in $\Psi(M)$. By (A7),

$$P_{n_k}^{x_{n_k}}(A_{n_k}^{(*, R+1)}(t) \in \cdot) \xrightarrow[k \rightarrow \infty]{d} P^x(A^{(*, R+1)}(t) \in \cdot). \quad (6.36)$$

Since $\tilde{\mu}_{n_k}^{(R+1)}$ is the smooth measure associated with $A_{n_k}^{(*, R+1)}$, as in the proof of [65, Theorem 1.6], one readily verifies that

$$E_{n_k}^{x_{n_k}}[A_{n_k}^{(*, R+1)}(t)^2] \leq 2e^{2t} \|\mathcal{R}_{p_{n_k}}^1 \tilde{\mu}_{n_k}^{(R+1)}\|_\infty.$$

The right-hand side is bounded uniformly in k by (6.34). Hence, the distributional convergence in (6.36) implies the convergence of the corresponding first moments. In particular, we obtain

$$\lim_{k \rightarrow \infty} E_{n_k}^{x_{n_k}}[A_{n_k}^{(*, R+1)}(t)] = E^x[A^{(*, R+1)}(t)].$$

Since the sequence $(x_{n_k})_{k \geq 1}$ was arbitrary, it follows that

$$E_{n_k}[A_{n_k}^{(*, R+1)}(t)] \xrightarrow[k \rightarrow \infty]{} E[A^{(*, R+1)}(t)] \text{ in } \widehat{C}(\Psi(M), \mathbb{R}_{\geq 0}).$$

In particular,

$$\lim_{k \rightarrow \infty} \sup_{y \in \Psi(S_{n_k}^{(R+1)})} E_{n_k}^y[A_{n_k}^{(*, R+1)}(t)] = \sup_{y \in \Psi(S^{R+1})} E^y[A^{(*, R+1)}(t)].$$

By Proposition 4.20, this is equivalent to

$$\begin{aligned} & \lim_{k \rightarrow \infty} \sup_{y \in \Psi(S_{n_k}^{(R+1)})} \int_0^t \int_{\Psi(S_{n_k})} p_{n_k}(s, y, z) \tilde{\mu}_{n_k}^{(R+1)}(dz) ds \\ &= \sup_{y \in \Psi(S^{R+1})} \int_0^t \int_{\Psi(S)} p(s, y, z) \tilde{\mu}^{(R+1)}(dz) ds. \end{aligned}$$

Since the limiting measure μ belongs to the local Kato class of X by Lemma 6.13, letting $t \rightarrow 0$ in the above equation yields

$$\lim_{t \rightarrow 0} \lim_{k \rightarrow \infty} \sup_{y \in \Psi(S_{n_k}^{(R+1)})} \int_0^t \int_{\Psi(S_{n_k})} p_{n_k}(s, y, z) \tilde{\mu}_{n_k}^{(R+1)}(dz) ds = 0.$$

From this, Proposition 4.9 and the inequality $\mu^{(R)} \leq \tilde{\mu}^{(R+1)}$ together yield (6.35).

Using (6.35), we can follow the argument that led to (6.18), and conclude that, on the fixed realization,

$$\mathcal{Y}_{n_k}^{(*,R)} \xrightarrow[k \rightarrow \infty]{} \mathcal{Y}^{(*,R)} \quad \text{in } \mathfrak{M}_\bullet(\tau_2).$$

In particular, the above convergence holds in distribution. Since the subsequence $(n_k)_{k \geq 1}$ was arbitrary, we obtain

$$\mathcal{Y}_n^{(*,R)} \xrightarrow[n \rightarrow \infty]{d} \mathcal{Y}^{(*,R)} \quad \text{in } \mathfrak{M}_\bullet(\tau_2). \quad (6.37)$$

Finally, in the same way as in (6.25), using (6.16) and Assumption 6.12, we obtain

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[d_{\mathfrak{M}_\bullet}^{\tau_2} \left(\mathcal{Y}_n^{(*,R)}, \mathcal{Y}_n \right) \wedge 1 \right] = 0.$$

Combining this with (6.17) and (6.37) completes the proof. $\square$

We complete the proof of Theorem 6.14 by establishing (iii).

Proof of Theorem 6.14(iii). This is established in the same way as that in the deterministic case. By (6.37), we have

$$\mathcal{Y}_n^{(*,R)} \xrightarrow[n \rightarrow \infty]{d} \mathcal{Y}^{(*,R)} \quad \text{in } \mathfrak{M}_\bullet(\tau_3),$$

which corresponds to (6.26). From (6.28), it follows that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[d_{\mathfrak{M}_\bullet}^{\tau_3} \left(\mathcal{Y}_n^{(*,R)}, \mathcal{Y}_n \right) \wedge 1 \right] = 0.$$

Now the result is immediate. $\square$

7 Collision measures and their convergence

In this section, we apply our main results on STOMs to the study of collisions between stochastic processes. In Section 7.1, we introduce random measures that capture the collision times and locations of two given stochastic processes. We refer to these random measures as *collision measures*. In Section 7.2, we then discuss convergence of collision measures of stochastic processes living on different spaces.

7.1 Collision measures

Throughout this subsection, we fix a locally compact separable metrizable topological space S . For two given processes X^1 and X^2 on S , we aim to study the locations and times of their collisions, that is, the set of pairs (x, t) such that $X_t^1 = X_t^2 = x$. In this subsection, we introduce random measures that capture such collisions. They are defined using the idea of STOMs; see Definition 7.3 below. Although we restrict attention to the case where X^1 and X^2 are independent, the same construction of STOMs can be extended to non-independent processes, as briefly discussed in Remark 7.10 below.

For $i \in \{1, 2\}$, we let

$$X^i = (\Omega_i, \mathcal{M}_i, (X_t^i)_{t \in [0, \infty]}, (P_i^x)_{x \in S_\Delta}, (\theta_t^i)_{t \in [0, \infty]})$$

be a standard process satisfying the DAC condition (Assumption 4.1). Let $(S \times S) \cup \{\Delta\}$ be the one-point compactification of $S \times S$. For each $\mathbf{x} = (x_1, x_2) \in S \times S$, we define $\hat{P}^{\mathbf{x}} := P_1^{x_1} \otimes P_2^{x_2}$, which is a probability measure on the product measurable space $(\hat{\Omega}, \hat{\mathcal{M}}) := (\Omega_1 \times \Omega_2, \mathcal{M}_1 \otimes \mathcal{M}_2)$. Also, we define $\hat{P}^\Delta := P_1^\Delta \otimes P_2^\Delta$. For each $\omega = (\omega_1, \omega_2) \in \hat{\Omega}$ and $t \in [0, \infty]$, we set

$$\hat{X}_t(\omega) := \begin{cases} (X_t^1(\omega_1), X_t^2(\omega_2)), & t < \zeta_1(\omega_1) \wedge \zeta_2(\omega_2), \\ \Delta, & t \geq \zeta_1(\omega_1) \wedge \zeta_2(\omega_2), \end{cases}$$

where ζ_i denotes the lifetime of X^i for each $i = 1, 2$. Finally, we define, for each $\omega = (\omega_1, \omega_2) \in \hat{\Omega}$ and $t \in [0, \infty]$,

$$\hat{\theta}_t(\omega) := (\theta_t^1(\omega_1), \theta_t^2(\omega_2)).$$

Then by Theorem D.1 and Proposition D.6,

$$\hat{X} := \left(\hat{\Omega}, \hat{\mathcal{M}}, (\hat{X}_t)_{t \in [0, \infty]}, (\hat{P})_{x \in (S \times S) \cup \{\Delta\}}, (\hat{\theta}_t)_{t \in [0, \infty]} \right)$$

is a standard process on $S \times S$ satisfying the DAC condition. Let m^i and p^i denote the reference measure and the heat kernel of X^i , respectively, for $i \in \{1, 2\}$. Then a reference measure of $\hat{X}$ is given by $m^1 \otimes m^2$, and the corresponding heat kernel is

$$\hat{p}(t, (x_1, x_2), (y_1, y_2)) = p^1(t, x_1, y_1) p^2(t, x_2, y_2), \quad (7.1)$$

with the convention $0 \cdot \infty := 0$. (See Proposition D.6 for these facts.)

Definition 7.1. We refer to the above-defined process $\hat{X}$ as the *product process* of X^1 and X^2 .

To study the collisions of X^1 and X^2 in S , we embed S into the diagonal line in $S \times S$ by the following diagonal map.

Definition 7.2 (The diagonal map). We define the *diagonal map* $\text{diag}_S : S \rightarrow S \times S$ by $\text{diag}(x) := (x, x)$. When the context is clear, we simply write $\text{diag} = \text{diag}_S$.

The collision of X^1 and X^2 can be rephrased as the event of $\hat{X}$ hitting the diagonal line. Specifically, for any $t \geq 0$ and a subset $B \subseteq S$, we have that

$$X_t^1 = X_t^2 \in B \quad \Leftrightarrow \quad \hat{X}_t \in \text{diag}(B).$$

With this background, we introduce collision measures below.

Fix a Radon measure on μ on S . We write $\mu^{\text{diag}} := \mu \circ \text{diag}^{-1}$, i.e., μ^{diag} is the pushforward measure of μ by diag . Assume that μ^{diag} is a smooth measure of $\hat{X}$. Let $(C_t)_{t \geq 0}$ and $\hat{\Pi}$ be the associated PCAF and STOMs of $\hat{X}$, respectively. By Corollary 4.21, $\hat{\Pi}$ is supported on $\text{diag}(S) \times \mathbb{R}_{\geq 0}$ and formally speaking, for each Borel subset $E \subseteq S$ and $t \geq 0$,

$$\hat{\Pi}(\text{diag}(E) \times [0, t]) = \text{“the total collision time of } X^1 \text{ and } X^2 \text{ in } E \text{ up to } t\text{”}.$$

We thus define a random measure Π on $S \times [0, \infty)$ by

$$\Pi := \hat{\Pi} \circ (\text{diag} \times \text{id}_{\mathbb{R}_{\geq 0}}).$$

(NB. Since $\text{diag} \times \text{id}_{\mathbb{R}_{\geq 0}}$ is injective, Π is indeed a measure on $S \times \mathbb{R}_{\geq 0}$.) By Corollary 4.21, we have

$$\Pi(\cdot) = \int_0^\infty \mathbf{1}_{(X_t^1, t)}(\cdot) dC_t = \int_0^\infty \mathbf{1}_{(X_t^2, t)}(\cdot) dC_t$$

as random measures on $S \times [0, \infty)$, $\hat{P}^x$ -a.s., for all $x \in S \times S$.

Definition 7.3 (Collision measure). We refer to the random measure Π as the *collision measure* of X^1 and X^2 associated with μ , and we call μ the *weighting measure*.

Remark 7.4. As mentioned in Section 1.3, Nguyen [59] obtained the scaling limit of the point process consisting of collision sites and collision times of two independent simple random walks on $\mathbb{Z}$. Although the limiting random measure in his construction is characterized only through moment identities, it can now be realized as a collision measure within our framework. Assume that X^1 is a standard one-dimensional Brownian motion on $\mathbb{R}$, and X^2 is an independent copy. Then, the collision measure of X^1 and X^2 associated with $2 \cdot \text{Leb}$ coincides with the limiting random measure obtained by Nguyen [59]. Indeed, using his characterization [59, Equation (1.2)] together with our moment formula for STOMs (Proposition 4.20), one readily verifies that the two random measures have the same distribution (cf. [43, Theorem 2.2]).

Remark 7.5. We also note that Nguyen [59] treated the general case of $k \geq 3$ independent simple random walks $S^1, \dots, S^k$ on $\mathbb{Z}$, and considered the point process consists of all $(t, x) \in \mathbb{N} \times \mathbb{Z}$ such that $S_t^i = S_t^j = x$ for some distinct i, j . The scaling limit of this point process obtained in his work can also be interpreted within our framework. Let $X^1, \dots, X^k$ be independent one-dimensional Brownian motions on $\mathbb{R}$. Then the limiting random measure should coincide with the sum of the pairwise collision measures $\sum_{1 \leq i < j \leq k} \Pi_{i,j}$ of X^i and X^j , associated with a suitable multiple of the Lebesgue measure. Indeed, as simultaneous collisions of three or more walks vanish in the scaling limit [59, Proof of Theorem 1.3], only binary collisions of Brownian motions contribute in the limit.

By Theorem 5.26, under the weak J_1 -continuity condition (Assumption 5.17) on X , we have the continuity of the joint law $\mathcal{L}_{(X, \hat{\Pi})}$ with respect to the starting point. It is not difficult to verify that this continuity is also inherited by the collision measure Π . However, since Π is not a pushforward of $\hat{\Pi}$, this does not follow directly from the usual continuous mapping theorem. The following lemma fills this minor gap. A more general version of the result can be found in Appendix B.

Lemma 7.6. *Let $(\Sigma_n)_{n \geq 1}$ be a sequence converging to Σ in $\mathcal{M}_{(*, \text{fin})}(S \times S \times \mathbb{R}_{\geq 0})$. If $\text{supp}(\Sigma_n) \subseteq \text{diag}(S) \times \mathbb{R}_{\geq 0}$, then $\Sigma_n \circ (\text{diag} \times \text{id}_{\mathbb{R}_{\geq 0}}) \rightarrow \Sigma \circ (\text{diag} \times \text{id}_{\mathbb{R}_{\geq 0}})$ in $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$.*

Proof. By Proposition B.1, we obtain that $\Sigma_n \circ (\text{diag} \times \text{id}_{\mathbb{R}_{\geq 0}}) \rightarrow \Sigma \circ (\text{diag} \times \text{id}_{\mathbb{R}_{\geq 0}})$ vaguely. Moreover, by Proposition 2.24, we deduce that the family $\{\Sigma_n \circ (\text{diag} \times \text{id}_{\mathbb{R}_{\geq 0}})\}_{n \geq 1}$ is precompact in $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$. Using the vague convergence, one can readily verify that the limit of any convergent subsequence is unique and coincides with $\Sigma \circ (\text{diag} \times \text{id}_{\mathbb{R}_{\geq 0}})$, which yields the desired result. $\square$

Using the above lemma, we obtain the following result, which is a counterpart of the continuity assertion in Theorem 5.26.

Proposition 7.7. *Assume that the following conditions hold.*

- (i) *The measure μ^{diag} on $S \times S$ belongs to the local Kato class of $\hat{X}$.*
- (ii) *For each $i \in \{1, 2\}$, the process X^i satisfies the weak J_1 -continuity condition (Assumption 5.17).*

Then the following map is continuous:

$$\mathcal{L}_{(\hat{X}, \Pi)}: S \times S \longrightarrow \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, S) \times D_{J_1}(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})).$$

Proof. By Theorem 5.30, the map

$$\mathcal{L}_{(\hat{X}, \hat{\Pi})}: S \times S \longrightarrow \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times D_{L^0}(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin})}(S \times S \times \mathbb{R}_{\geq 0}))$$

is continuous. By Lemma 7.6, we then deduce that the following map is also continuous:

$$\mathcal{L}_{(\hat{X}, \Pi)}: S \times S \longrightarrow \mathcal{P}(D_{L^0}(\mathbb{R}_{\geq 0}, S) \times D_{L^0}(\mathbb{R}_{\geq 0}, S) \times \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})).$$

It remains to upgrade the continuity to the J_1 -Skorohod topology. By condition (ii) and the independence of X^1 and X^2 , the map

$$\mathcal{L}_{\hat{X}}: S \times S \longrightarrow \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, S) \times D_{J_1}(\mathbb{R}_{\geq 0}, S))$$

is continuous. Combining this with a tightness argument, we can upgrade the L^0 -continuity established above to the J_1 -continuity. This completes the proof. (See Appendix C for details.) $\square$

The following is immediate from Propositions 4.11 and 4.22.

Proposition 7.8. *Suppose that S is finite or countable, endowed with the discrete topology. Then μ^{diag} belongs to the local Kato class of X , and the collision measure Π associated with μ can be represented as follows:*

$$\Pi(dx dt) = \frac{1}{m^1(\{x\})m^2(\{x\})} \mathbf{1}_{\{X_t^1=X_t^2=x\}} \mu(dx) dt.$$

Remark 7.9. As seen in Proposition 7.8, the weighting measure $\mu(dx)$ represents the contribution of collisions at each point x . In the setting of that proposition, if we take the pointwise product $m^1 \cdot m^2$ as the weighting measure μ , i.e., $\mu(\{x\}) := m^1(\{x\})m^2(\{x\})$, then the associated collision measure assigns equal weight to the contribution of collisions at each point. We refer to this collision measure as the *uniform collision measure*, i.e., it is defined as follows:

$$\Pi(dx dt) = \sum_{y \in S} \mathbf{1}_{\{X_t^1=X_t^2=y\}} \delta_y(dx) dt.$$

In particular, for each $x \in S$ and $t \geq 0$,

$$\Pi(\{x\} \times [0, t]) = \int_0^t \mathbf{1}_{\{X_s^1=X_s^2=x\}} ds,$$

which represents the total amount of time up to t during which X^1 and X^2 collide at x . We call the associated weighting measure $\mu = m^1 \cdot m^2$ the *canonical weighting measure*.

Remark 7.10. Although X^1 and X^2 are independent in the above framework, one can also study collisions of correlated processes. Indeed, if a standard process $X = (X^1, X^2)$ on $S \times S$ satisfies the DAC condition (Assumption 4.1), then a collision measure of X^1 and X^2 can be defined through a STOM of X . However, since research on collisions has primarily focused on independent stochastic processes, we restrict our attention in this paper to this case.

7.2 Convergence of collision measures

In this subsection, by applying Theorems 6.6 and 6.14, we prove convergence of collision measures of independent stochastic processes living on spaces that vary along a sequence. The main results of this subsection are found in Theorems 7.13 and 7.16 below. Throughout this subsection, we fix a Polish structure τ .

We first state a result for deterministic spaces, obtained as an application of Theorem 6.6. For each $n \in \mathbb{N}$, we suppose that

- $(S_n, d_{S_n}, \rho_n, \mu_n, a_n)$ is an element of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$;
- X_n^1 and X_n^2 are standard processes on S_n satisfying the DAC and weak J_1 -continuity conditions (Assumptions 4.1 and 5.17) with heat kernels p_n^1 and p_n^2 , respectively;
- the Radon measure μ_n^{diag} on $S_n \times S_n$ is in the local Kato class of the product process $\hat{X}_n$ of X_n^1 and X_n^2 .

We also suppose that

- (S, d_S, ρ, μ, a) is an element of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$;
- X^1 and X^2 are standard processes on S satisfying the DAC and weak J_1 -continuity conditions, with heat kernels p^1 and p^2 , respectively;
- the Radon measure μ^{diag} on $S \times S$ charges no sets semipolar for the product process $\hat{X}$ of X^1 and X^2 .

Remark 7.11. Recall that the convergence results in Theorems 6.6 and 6.14 were formulated under the convergence of processes with respect to the L^0 topology. Here we instead work with the stronger J_1 -Skorohod topology. This choice is not essential in itself, but it is made in view of the framework of resistance metric spaces considered later, where convergence in the J_1 -Skorohod topology arises naturally from the convergence of the underlying spaces.

Since we employ the J_1 -Skorohod topology instead of the L^0 topology, we accordingly use the structure τ_{SP} defined in (6.3):

$$\tau_{\text{SP}} := \tau_{\widehat{C}}(\Psi_{\text{id}}, \tau_{\mathcal{P}}(\tau_{J_1})). \quad (7.2)$$

We now consider the following version of Assumption 6.1 (see also Remark 6.2).

Assumption 7.12.

(i) *It holds that*

$$(S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n^1, p_n^2, \mathcal{L}_{X_n^1}, \mathcal{L}_{X_n^2}) \rightarrow (S, d_S, \rho, \mu, a, p^1, p^2, \mathcal{L}_{X^1}, \mathcal{L}_{X^2})$$

$$\text{in } \mathfrak{M}_{\bullet}(\tau_{\mathcal{M}} \times \tau \times \tau_{\text{HK}}^{\otimes 2} \times \tau_{\text{SP}}^{\otimes 2}).$$

(ii) *For all $r > 0$,*

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{x_1, x_2 \in S_n^{(r)}} \int_0^\delta \int_{S_n^{(r)}} p^1(t, x_1, y) p^2(t, x_2, y) \mu_n(dy) dt = 0.$$

Under the above assumptions, Lemma 6.5 implies that μ^{diag} belongs to the local Kato class of $\hat{X}$. Let Π_n (resp. Π) denote the collision measure of X_n^1 and X_n^2 (resp. X^1 and X^2) associated with μ_n (resp. μ). Below, by using Theorem 6.6, we establish the joint convergence of the processes and the collision measures. We use the following structure to formulate the result precisely:

$$\tau_{\text{CM}} := \tau_{\widehat{C}}\left(\Psi_{\text{id}^2}, \tau_{\mathcal{P}}(\tau_{J_1} \times \tau_{J_1} \times \tau_{\mathcal{M}_{(*, \text{fin})}})\right). \quad (7.3)$$

In particular, for each bcm space M ,

$$\tau_{\text{CM}}(M) = \widehat{C}(M \times M, \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, M) \times D_{J_1}(\mathbb{R}_{\geq 0}, M) \times \mathcal{M}_{(*, \text{fin})}(M \times \mathbb{R}_{\geq 0}))),$$

where the domain $M \times M$ corresponds to the starting points of the two processes, and the codomain represents the joint law of the processes together with their associated collision measure.

Theorem 7.13. *Under Assumption 7.12, we have*

$$(S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n^1, p_n^2, \mathcal{L}_{(\hat{X}_n, \Pi_n)}) \rightarrow (S, d_S, \rho, \mu, a, p^1, p^2, \mathcal{L}_{(\hat{X}, \Pi)})$$

$$\text{in } \mathfrak{M}_{\bullet}(\tau_{\mathcal{M}} \times \tau \times \tau_{\text{HK}}^{\otimes 2} \times \tau_{\text{CM}}).$$

Proof. Let $\hat{p}_n$ and $\hat{p}$ be the heat kernels of the product processes $\hat{X}_n$ and $\hat{X}$, respectively, defined in (7.1). By Lemma 2.15, Assumption 7.12(i), and the independence of X_n^1 and X_n^2 , we have

$$\begin{aligned} & (S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n^1, p_n^2, \mathcal{L}_{X_n^1}, \mathcal{L}_{X_n^2}, \mu_n^{\text{diag}}, \hat{p}_n, \mathcal{L}_{\hat{X}_n}) \\ & \rightarrow (S, d_S, \rho, \mu, a, p^1, p^2, \mathcal{L}_{X^1}, \mathcal{L}_{X^2}, \mu^{\text{diag}}, \hat{p}, \mathcal{L}_{\hat{X}}) \end{aligned}$$

in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau \times \tau_{\text{HK}}^{\otimes 2} \times \tau_{\text{SP}}^{\otimes 2} \times \tau_{\mathcal{M}}(\Psi_{\text{id}^2}) \times \tau_{\text{HK}}(\Psi_{\text{id}^2}) \times \tau_{L^0\text{-SP}}(\Psi_{\text{id}^2}))$. Since the product processes converge with respect to the product J_1 topology, it follows immediately, as noted in Remark 6.4, that the uniform spatial tightness condition (Assumption 6.3) is satisfied for the product processes. Thus, applying Theorem 6.6(ii) to the sequence of product processes with the spatial transformation $\Psi = \Psi_{\text{id}^2}$, we obtain

$$(S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n^1, p_n^2, \mathcal{L}_{X_n^1}, \mathcal{L}_{X_n^2}, \mathcal{L}_{(\hat{X}_n, \hat{\Pi}_n)}) \rightarrow (S, d_S, \rho, \mu, a, p^1, p^2, \mathcal{L}_{X^1}, \mathcal{L}_{X^2}, \mathcal{L}_{(\hat{X}, \hat{\Pi})})$$

in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau \times \tau_{\text{HK}}^{\otimes 2} \times \tau_{\text{SP}}^{\otimes 2} \times \tau_{L^0\text{-SP, STOM}}(\Psi_{\text{id}^2}))$, where $\hat{\Pi}_n$ (resp. $\hat{\Pi}$) denotes the STOM of $\hat{X}_n$ (resp. $\hat{X}$) associated with μ_n^{diag} (resp. μ^{diag}).

By Theorem 3.7, we may assume that all rooted bcm spaces (S_n, ρ_n) and (S, ρ) are isometrically embedded into a common bcm space (M, ρ_M) in such a way that

- (B1) $S_n \rightarrow S$ in the Fell topology as closed subsets of M ;
- (B2) $\rho_n = \rho = \rho_M$ when viewed as elements on M ;
- (B3) $\mu_n \rightarrow \mu$ vaguely as measures on M ;
- (B4) $a_n \rightarrow a$ in $\tau(M)$;
- (B5) $p_n^i \rightarrow p^i$ in $\hat{C}(\mathbb{R}_{\geq 0} \times M \times M, \mathbb{R}_{\geq 0})$ for each $i \in \{1, 2\}$;
- (B6) $\mathcal{L}_{X_n^i} \rightarrow \mathcal{L}_{X^i}$ in $\hat{C}(M, \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, M)))$ for each $i \in \{1, 2\}$;
- (B7) $\mathcal{L}_{(\hat{X}_n, \hat{\Pi}_n)} \rightarrow \mathcal{L}_{(\hat{X}, \hat{\Pi})}$ in $\hat{C}(M \times M, \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, M \times M) \times \mathcal{M}_{(*, \text{fin})}(M \times M \times \mathbb{R}_{\geq 0})))$.

Applying Lemma 7.6 to (B7), we deduce that

$$\mathcal{L}_{(\hat{X}_n, \Pi_n)} \rightarrow \mathcal{L}_{(\hat{X}, \Pi)} \quad \text{in} \quad \hat{C}(M \times M, \mathcal{P}(L^0(\mathbb{R}_{\geq 0}, M \times M) \times \mathcal{M}_{(*, \text{fin})}(M \times \mathbb{R}_{\geq 0}))).$$

It remains to upgrade the L^0 topology to the J_1 -Skorohod topology. As in the proof of Proposition 7.7, this follows directly from (B6) together with a tightness argument (see Appendix C for details). Consequently, we obtain

$$\mathcal{L}_{(\hat{X}_n, \Pi_n)} \rightarrow \mathcal{L}_{(\hat{X}, \Pi)} \quad \text{in} \quad \hat{C}(M \times M, \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, M) \times D_{J_1}(\mathbb{R}_{\geq 0}, M) \times \mathcal{M}_{(*, \text{fin})}(M \times \mathbb{R}_{\geq 0}))),$$

which completes the proof. $\square$

Remark 7.14. As the above proof illustrates, the adoption of the L^0 topology is particularly convenient in the study of collision measures. In the proof, we applied the convergence theorem for STOMs (Theorem 6.6) to the product processes $\hat{X}_n = (X_n^1, X_n^2)$. These processes converge only in the product of the J_1 topologies, that is, in $D_{J_1}(\mathbb{R}_{\geq 0}, M) \times D_{J_1}(\mathbb{R}_{\geq 0}, M)$, rather than in $D_{J_1}(\mathbb{R}_{\geq 0}, M \times M)$ itself. Therefore, if the main STOM convergence theorems were formulated under the J_1 -Skorohod topology, they could not be applied to this product setting in a straightforward manner. Of course, one could still modify the proofs to make them work under the J_1 topology, but this would require additional technical effort. The L^0 topology, on the other hand, is stable under product constructions and thus provides a cleaner and more flexible framework for treating collision measures.

We next provide a version of Theorem 7.13 for random spaces. For each $n \in \mathbb{N}$, we let $(\Omega_n, \mathcal{G}_n, \mathbf{P}_n)$ be a complete probability space, and assume that, for $\mathbf{P}_n$ -a.s. $\omega \in \Omega_n$,

- $(S_n^\omega, d_{S_n}^\omega, \rho_n^\omega, \mu_n^\omega, a_n^\omega) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$;
- $X_n^{1, \omega}$ and $X_n^{2, \omega}$ are standard processes on S_n^ω satisfying the DAC and weak J_1 -continuity conditions, with heat kernels $p_n^{1, \omega}$ and $p_n^{2, \omega}$, respectively;

- the Radon measure $(\mu_n^\omega)^{\text{diag}}$ on $S_n^\omega \times S_n^\omega$ belongs to the local Kato class of the product process $\hat{X}_n^\omega$ of $X_n^{1,\omega}$ and $X_n^{2,\omega}$.

Similarly, we let $(\Omega, \mathcal{G}, \mathbf{P})$ be a complete probability space, and assume that, for each $\omega \in \Omega$,

- $(S^\omega, d_S^\omega, \rho^\omega, \mu^\omega, a^\omega) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$;
- $X^{1,\omega}$ and $X^{2,\omega}$ are standard processes on S^ω satisfying the DAC and weak J_1 -continuity conditions, with heat kernels $p^{1,\omega}$ and $p^{2,\omega}$, respectively;
- the Radon measure $(\mu^\omega)^{\text{diag}}$ on $S^\omega \times S^\omega$ charges no sets semipolar for the product process $\hat{X}^\omega$ of $X^{1,\omega}$ and $X^{2,\omega}$.

To discuss the laws of these objects, we assume that the maps

$$\begin{aligned} (\Omega_n, \mathcal{G}_n) \ni \omega &\mapsto (S_n^\omega, d_{S_n}^\omega, \rho_n^\omega, \mu_n^\omega, a_n^\omega, p_n^{1,\omega}, p_n^{2,\omega}, \mathcal{L}_{X_n^{1,\omega}}, \mathcal{L}_{X_n^{2,\omega}}) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau \times \tau_{\text{HK}}^{\otimes 2} \times \tau_{\text{SP}}^{\otimes 2}), \\ (\Omega, \mathcal{G}) \ni \omega &\mapsto (S^\omega, d_S^\omega, \rho^\omega, \mu^\omega, a^\omega, p^{1,\omega}, p^{2,\omega}, \mathcal{L}_{X^{1,\omega}}, \mathcal{L}_{X^{2,\omega}}) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau \times \tau_{\text{HK}}^{\otimes 2} \times \tau_{\text{SP}}^{\otimes 2}) \end{aligned}$$

are measurable, where the codomain is equipped with the Borel σ -algebra. (The measurability is to be understood in the same sense as in Section 6.2, by extending the objects arbitrarily on null sets if necessary.)

We now consider the following version of Assumption 7.12. As usual, we suppress the dependence on ω in the notation.

Assumption 7.15.

(i) *It holds that*

$$\begin{aligned} (S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n^1, p_n^2, \mathcal{L}_{X_n^1}, \mathcal{L}_{X_n^2}) &\xrightarrow{d} (S, d_S, \rho, \mu, a, p^1, p^2, \mathcal{L}_{X^1}, \mathcal{L}_{X^2}) \\ &\text{in } \mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau \times \tau_{\text{HK}}^{\otimes 2} \times \tau_{\text{SP}}^{\otimes 2}). \end{aligned}$$

(ii) *For all $r > 0$,*

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[\left(\sup_{x_1, x_2 \in S_n^{(r)}} \int_0^\delta \int_{S_n^{(r)}} p^1(t, x_1, y) p^2(t, x_2, y) \mu_n(dy) dt \right) \wedge 1 \right] = 0.$$

Under the above assumptions, by Lemma 6.13, μ^{diag} is in the local Kato class of $\hat{X}$, $\mathbf{P}$ -a.s. Denote by Π_n (resp. Π) the collision measure of X_n^1 and X_n^2 (resp. X^1 and X^2) associated with μ_n (resp. μ). We then obtain the following analogue of Theorem 7.13. The proof is similar to that of the deterministic case, with Theorem 6.14(ii) applied in place of Theorem 6.6(ii), and is therefore omitted.

Theorem 7.16. *Under Assumption 7.15, it holds that*

$$(S_n, d_{S_n}, \rho_n, \mu_n, a_n, p_n^1, p_n^2, \mathcal{L}_{(\hat{X}_n, \Pi_n)}) \xrightarrow{d} (S, d_S, \rho, \mu, a, p^1, p^2, \mathcal{L}_{(\hat{X}, \Pi)})$$

$$\text{in } \mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau \times \tau_{\text{HK}}^{\otimes 2} \times \tau_{\text{CM}}).$$

Remark 7.17. As noted in Remark 7.10, collision measures can also be defined for correlated processes. Analogous results for such processes, corresponding to Theorems 7.13 and 7.16, can be established under minor modifications of the assumptions. Indeed, the convergence of the product processes and their heat kernels plays a crucial role in the proofs. In the above discussions, these convergences are obtained directly from the individual convergences of X_n^1 and X_n^2 together with their independence, and independence is not required beyond that point.

8 Preliminary results on stochastic processes on resistance metric spaces

Typically, collisions between two independent and identically distributed (i.i.d.) stochastic processes occur only in low dimensions. The purpose of this section is to introduce a class of metric spaces called *resistance metric spaces*, which provides a natural framework for a unified study of stochastic processes in low-dimensional spaces. This framework was established by Kigami [49, 50] in the development of analysis on low-dimensional fractals, and recent research (cf. [24, 28, 61, 63, 64]) has demonstrated its usefulness, beyond deterministic fractals, in studying the scaling limits of random walks on critical random graphs.

We divide this section into four subsections. In Section 8.1, we review the theory of resistance metric spaces and the stochastic processes defined on them. In Section 8.2, we introduce a Gromov–Hausdorff-type space consisting of resistance metric spaces, which will serve as a foundation for stating our main results in Section 9. Sections 8.3 and 8.4 are devoted to consequences of the previous works [24, 63, 64], concerning the convergence of stochastic processes and their heat kernels on sequences of resistance metric spaces.

8.1 Preliminaries

In this subsection we recall the theory of resistance metric spaces and stochastic processes on them. This can be regarded as an extension of the theory of electrical networks and continuous-time Markov chains on them (e.g. [54]). The reader is referred to [50] for further background.

Definition 8.1 (Resistance form, [50, Definition 3.1]). Let F be a non-empty set. A pair $(\mathcal{E}, \mathcal{F})$ is called a *resistance form* on F if it satisfies the following conditions.

- (RF1) The symbol $\mathcal{F}$ is a linear subspace of the collection of functions $\{f : F \rightarrow \mathbb{R}\}$ containing constants, and $\mathcal{E}$ is a non-negative symmetric bilinear form on $\mathcal{F}$ such that $\mathcal{E}(f, f) = 0$ if and only if f is constant on F .
- (RF2) Let $\sim$ be the equivalence relation on $\mathcal{F}$ defined by saying $f \sim g$ if and only if $f - g$ is constant on F . Then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.
- (RF3) If $x \neq y$, then there exists a function $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.
- (RF4) For any $x, y \in F$,

$$R_{(\mathcal{E}, \mathcal{F})}(x, y) := \sup \left\{ \frac{|f(x) - f(y)|^2}{\mathcal{E}(f, f)} : f \in \mathcal{F}, \mathcal{E}(f, f) > 0 \right\} < \infty. \quad (8.1)$$

- (RF5) If $\bar{f} := (f \wedge 1) \vee 0$, then $\bar{f} \in \mathcal{F}$ and $\mathcal{E}(\bar{f}, \bar{f}) \leq \mathcal{E}(f, f)$ for any $f \in \mathcal{F}$.

For the following definition, recall the effective resistance on an electrical network with a finite vertex set from [54, Section 9.4] (see also [49, Section 2.1]).

Definition 8.2 (Resistance metric, [49, Definition 2.3.2]). A metric R on a non-empty set F is called a *resistance metric* if and only if, for any non-empty finite subset $V \subseteq F$, there exists an electrical network G (see Definition 8.7) with vertex set V such that the effective resistance on G coincides with $R|_{V \times V}$.

Theorem 8.3 ([49, Theorem 2.3.6]). *Fix a non-empty subset F . There exists a one-to-one correspondence between resistance forms $(\mathcal{E}, \mathcal{F})$ on F and resistance metrics R on F via $R = R_{(\mathcal{E}, \mathcal{F})}$. In other words, a resistance form $(\mathcal{E}, \mathcal{F})$ is characterized by $R_{(\mathcal{E}, \mathcal{F})}$ given in (RF4).*

Motivated by the variational formula (8.1), we define the resistance between sets as follows.

Definition 8.4. Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set F and let R be the corresponding resistance metric. For subsets $A, B \subseteq F$, we define

$$R(A, B) := (\inf\{\mathcal{E}(f, f) : f \in \mathcal{F}, f|_A \equiv 1, f|_B \equiv 0\})^{-1},$$

with the convention that it is zero if the infimum is taken over the empty set. Note that, by (RF4), we have $R(\{x\}, \{y\}) = R(x, y)$.

We will henceforth assume that we have a resistance metric R on a non-empty set F with corresponding resistance form $(\mathcal{E}, \mathcal{F})$. Furthermore, we assume that (F, R) is boundedly compact and recurrent, as described by the following definition.

Definition 8.5 (Recurrent resistance metric, [64, Definition 3.6]). We say that R is *recurrent* if and only if

$$\lim_{r \rightarrow \infty} R(\rho, B_R(\rho, r)^c) = \infty$$

for some (or, equivalently, any) $\rho \in F$.

We next introduce related Dirichlet forms and stochastic processes. First, suppose that we have a Radon measure m of full support on (F, R) . We define a bilinear form $\mathcal{E}_1$ on $\mathcal{F} \cap L^2(F, m)$ by setting $\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + \int_F fg \, dm$. Then $(\mathcal{F} \cap L^2(F, m), \mathcal{E}_1)$ is a Hilbert space (see [49, Theorem 2.4.1]). Let $\mathcal{D}$ denote the closure of $\mathcal{F} \cap C_c(F, \mathbb{R})$ with respect to $\mathcal{E}_1$, where $C_c(F, \mathbb{R})$ stands for the space of compactly supported continuous functions on F , equipped with the compact-convergence topology. The recurrence of the resistance metric R ensures that the resistance form $(\mathcal{E}, \mathcal{F})$ is regular in the sense of [50, Definition 6.2] (see [64, Corollary 3.22] for this fact). In particular, we can apply [50, Theorem 9.4], which implies that $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form on $L^2(F, m)$. Moreover, by [24, Lemma 2.3], this Dirichlet form is recurrent. (See [32] for the definitions of regular Dirichlet forms and recurrence.) Write X for the associated Hunt process. We refer to this Hunt process as the *(Hunt) process associated with (R, m)* . Note that such a process is, in general, only specified uniquely for starting points outside a set of zero capacity. However, in this setting, every point has strictly positive capacity (see [50, Theorem 9.9]), and so the process is defined uniquely everywhere. Since the Dirichlet form is recurrent, X is conservative (see [32, Exercise 4.5.1]). Moreover, X satisfies Hunt's hypothesis (H) by [32, Theorem 4.1.3].

From [50, Theorem 10.4], the Hunt process X admits a unique jointly continuous heat kernel $p_{(R, m)} : (0, \infty) \times F \times F \rightarrow [0, \infty)$ with respect to m , which we refer to as the *heat kernel associated with (R, m)* . When the context is clear, we simply write $p = p_{(R, m)}$. The heat kernel p is symmetric, i.e., $p(t, x, y) = p(t, y, x)$. In particular, the process X satisfies the DAC condition (Assumption 4.1). The following is useful for obtaining an upper bound on the heat kernel from lower volume estimates.

Lemma 8.6 ([50, Theorem 10.4]). *For any $x \in F$, $t > 0$, and $s > 0$,*

$$p(t, x, x) \leq \frac{2s}{t} + \frac{\sqrt{2}}{m(D_R(x, s))}.$$

To illustrate the above ideas more concretely, we now apply the resistance form framework to electrical networks,

Definition 8.7 (Electrical network). We say that a tuple (V, E, μ) is an *electrical network* if it satisfies the following conditions.

- (i) The pair (V, E) is a connected, simple, undirected graph with finitely or countably many vertices, where V denotes the vertex set and E denotes the edge set. (NB. A graph being simple means that it has no loops and no multiple edges.)

- (ii) The symbol μ denotes the *conductance*, that is, it is a function $\mu: V \times V \rightarrow \mathbb{R}_{\geq 0}$ such that $\mu(x, y) = \mu(y, x)$ for all $x, y \in V$, $\mu(x, y) > 0$ if and only if $\{x, y\} \in E$, and

$$\mu(x) := \sum_{y \in V} \mu(x, y) < \infty, \quad \forall x \in V.$$

Let (V, E, μ) be an electrical network. For functions $f, g: V \rightarrow \mathbb{R}$, we define

$$\mathcal{E}(f, g) := \frac{1}{2} \sum_{x, y \in V} \mu(x, y)(f(x) - f(y))(g(x) - g(y)),$$

whenever the right-hand side is well-defined. We set $\mathcal{F} := \{f: V \rightarrow \mathbb{R} \mid \mathcal{E}(f, f) < \infty\}$. Then the pair $(\mathcal{E}, \mathcal{F})$ is a regular resistance form on V . The associated resistance metric R induces the discrete topology on V . (See [64, Theorem 4.1]) If R is recurrent in the sense of Definition 8.5, then we say that the electrical network is *recurrent*.

Let m be a Radon measure on V with full support. The Hunt process X associated with (R, m) is the minimal continuous-time Markov chain on V with jump rates $w(x, y) = \mu(x, y)/m(\{x\})$. (See [64, Theorem 4.2]). Thus, we have the following consequences.

- (RW1) If m is the *counting measure* on V , i.e., $m(\{x\}) = 1$ for each $x \in V$, then the associated process X is the *variable speed random walk (VSRW)*.
- (RW2) If m is the *conductance measure* on V , i.e., $m(\{x\}) = \mu(x)$ for each $x \in V$, then the associated process X is the *constant speed random walk (CSRW)*.

8.2 Spaces of recurrent resistance metric spaces

In this subsection, we introduce a space of recurrent resistance metric spaces (Definition 8.8) and verify their measurability with respect to a Gromov-Hausdorff-type topology (Proposition 8.9). This serves as a fundamental result for studying stochastic processes on random resistance metric spaces, such as the continuum random tree.

Definition 8.8. Given a Polish structure τ , we define

$$\mathfrak{R}_\bullet(\tau) := \left\{ (F, R, \rho, m, a) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau) \left| \begin{array}{l} R \text{ is a recurrent resistance metric on } F, \\ m \text{ has full support} \end{array} \right. \right\}.$$

Similarly, $\mathfrak{R}_\bullet$ denotes the set of quadruples (F, R, ρ, m) obtained from the above definition by omitting a . We always equip $\mathfrak{R}_\bullet(\tau)$ with the relative topology induced from $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$.

Henceforth, we fix a Polish structure τ . The following is the main result of this subsection.

Proposition 8.9. *The set $\mathfrak{R}_\bullet(\tau)$ is a Borel subset of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$.*

By definition, $\mathfrak{R}_\bullet(\tau)$ is the intersection of the collection of recurrent resistance metric spaces and the collection of metric spaces equipped with a fully-supported measure. Thus, by showing that each collection is Borel, we establish the above proposition.

Lemma 8.10. *Define*

$$\mathfrak{N}_1 := \{(F, R, \rho, m, a) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau) \mid m \text{ is of full support}\}.$$

Then the set $\mathfrak{N}_1$ is a Borel subset of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$.

Proof. For each $N, k, l \in \mathbb{N}$, we define $\mathfrak{N}_{(N, k, l)}$ to be the collection of $(S, d, \rho, m, a) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$ such that, for all but countably many $r \in [N, N + 1]$,

$$\inf_{x \in S^{(r)}} m(D_S(x, 1/k)) \geq 1/l. \quad (8.2)$$

We then have that

$$\mathfrak{N}_1 = \bigcap_{N \geq 1} \bigcap_{k \geq 1} \bigcup_{l \geq 1} \mathfrak{N}_{(N,k,l)}.$$

Thus, it is enough to show that $\mathfrak{N}_{(N,k,l)}$ is a Borel subset of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$. We do this by showing that $\mathfrak{N}_{(N,k,l)}$ is closed. Let $G_n = (S_n, d_n, \rho_n, m_n, a_n) \in \mathfrak{N}_{(N,k,l)}$, $n \in \mathbb{N}$, be such that

$$G_n = (S_n, d_n, \rho_n, m_n, a_n) \rightarrow (S, d, \rho, m, a) = G$$

in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$. By Theorem 3.7, we may assume that (S_n, ρ_n) , $n \in \mathbb{N}$, and (S, ρ) are embedded isometrically into a common rooted bcm space (M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements of M , $S_n \rightarrow S$ in the Fell topology as closed subsets of M , $m_n \rightarrow m$ vaguely as measures on M , and $a_n \rightarrow a$ in $\tau(M)$. Let $r \in [N, N+1]$ be such that each G_n satisfies (8.2) and $S_n^{(r)} \rightarrow S^{(r)}$ in the Hausdorff topology. For an arbitrarily fixed $x \in S^{(r)}$, we choose elements $x_n \in S_n^{(r)}$ converging to x in M . Then, for any $\delta > 0$, we have that, for all sufficiently large n , $D_M(x, 1/k + \delta) \supseteq D_M(x_n, 1/k)$. This, combined with the Portmanteau theorem (cf. [43, Lemma 4.1]), yields that, for any $\delta > 0$,

$$m(D_S(x, 1/k + \delta)) \geq \limsup_{n \rightarrow \infty} m_n(D_S(x, 1/k + \delta)) \geq \limsup_{n \rightarrow \infty} m_n(D_S(x_n, 1/k)) \geq 1/l.$$

By letting $\delta \rightarrow 0$, we obtain that $m(D_S(x, 1/k)) \geq 1/l$, which implies $G \in \mathfrak{N}_{(N,k,l)}$. This completes the proof. $\square$

Lemma 8.11. *Define*

$$\mathfrak{N}_2 := \{(F, R, \rho, m, a) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau) \mid R \text{ is a recurrent resistance metric on } F\}.$$

Then the set $\mathfrak{N}_2$ is a Borel subset of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$.

Proof. We prove the result similarly to Lemma 8.10. For each $N, l \in \mathbb{N}$, we define $\mathfrak{N}_{(N,l)}$ to be the collection of $(F, R, \rho, m, a) \in \mathfrak{M}_\bullet(\tau)$ such that R is a resistance metric on F and, for all but countably many $r \in [N, N+1]$, it holds that $R(\rho, B_R(\rho, r)^c) \geq l$. We then have that

$$\mathfrak{N}_2 = \bigcap_{l \geq 1} \bigcup_{N \geq 1} \mathfrak{N}_{(N,l)}.$$

Thus, it suffices to show that each $\mathfrak{N}_{(N,l)}$ is closed in $\mathfrak{M}_\bullet(\tau)$. Let $G_n = (F_n, R_n, \rho_n, m_n, a_n) \in \mathfrak{N}_{(N,l)}$, $n \in \mathbb{N}$, be such that

$$G_n = (F_n, R_n, \rho_n, m_n, a_n) \rightarrow (F, R, \rho, m, a) = G$$

in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$. It then follows from [64, Theorem 5.1] that R is a resistance metric and

$$R(\rho, B_R(\rho, r)^c) \geq \limsup_{n \rightarrow \infty} R_n(\rho_n, B_{R_n}(\rho_n, r)^c).$$

The right-hand side of the above inequality is larger than or equal to l as $G_n \in \mathfrak{N}_{(N,l)}$ for all n . It follows that $G \in \mathfrak{N}_{(N,l)}$, which completes the proof. $\square$

Proof of Proposition 8.9. Since $\mathfrak{N}_\bullet(\tau) = \mathfrak{N}_1 \cap \mathfrak{N}_2$, the desired result follows from Lemmas 8.10 and 8.11. $\square$

8.3 Convergence of law map of stochastic process

In [24], Croydon proved that, when rooted-and-measured resistance metric spaces converge, the laws of associated stochastic processes started at the root also converge. In this subsection, we strengthen the result to convergence of law maps of the stochastic processes (see Theorems 8.17 and 8.24 below). This result will be used in the next subsection to establish the convergence of heat kernels.

Definition 8.12. Let (F, R, m) be a boundedly-compact recurrent resistance metric space equipped with a fully-supported Radon measure m . Write $((X_t)_{t \geq 0}, (P^x)_{x \in F})$ for the process associated with (R, m) . We then define a map $\Upsilon: F \rightarrow \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, F))$ by setting

$$\Upsilon_{(R,m)}(x) := \mathcal{L}_X(x) = P^x((X_t)_{t \geq 0} \in \cdot), \quad x \in F.$$

We refer to $\Upsilon_{(R,m)}$ as the *law map associated with (R, m)* . When the context is clear, we simply write $\Upsilon := \Upsilon_{(R,m)}$.

We recall the main result of [24].

Theorem 8.13 ([24, Proof of Theorem 1.2]). *Let $G_n = (F_n, R_n, \rho_n, m_n)$, $n \in \mathbb{N}$, and $G = (F, R, \rho, m)$ be elements of $\mathfrak{R}_\bullet(\tau)$. Assume that*

$$\lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} R_n(\rho_n, B_{R_n}(\rho_n, r)^c) = \infty. \quad (8.3)$$

Assume further that all the space F_n , $n \geq 1$, and F are isometrically embedded into a common bcm space M in such a way that $F_n \rightarrow F$ in the Fell topology as closed subsets of M , $\rho_n \rightarrow \rho$ as elements of M , and $m_n \rightarrow m$ vaguely as measures on M . It then holds that $\Upsilon_{(R_n, m_n)}(\rho_n) \rightarrow \Upsilon_{(R, m)}(\rho)$ as probability measures on $D_{J_1}(\mathbb{R}_{\geq 0}, M)$.

Remark 8.14. The condition (8.3) is called the *non-explosion condition* in [24]. Note that there is a typographical error in [24, Assumption 1.1(b)], where “ $\limsup_{n \rightarrow \infty}$ ” should read “ $\liminf_{n \rightarrow \infty}$ ”, as stated in the above theorem.

Using Theorem 8.13, we first prove the continuity of $\Upsilon_{(R,m)}$ with respect to the starting point. Recall that $\mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, F))$ is equipped with the weak topology induced from the usual J_1 -Skorohod topology.

Lemma 8.15. *The law map $\Upsilon: F \rightarrow \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, F))$ is continuous.*

Proof. Suppose that elements $x_n \in F$ converge to an element $x \in F$. By the recurrence of R , we can use Theorem 8.13 to obtain that $\Upsilon(x_n) \rightarrow \Upsilon(x)$ as probability measures on $D_{J_1}(\mathbb{R}_{\geq 0}, F)$. This completes the proof. $\square$

We are then interested in the continuity of $\Upsilon_{(R,m)}$ with respect to (R, m) . To discuss this within the framework of Gromov-Hausdorff-type topologies, we recall the structure τ_{SP} from (7.2). Henceforth, we fix a Polish structure τ .

Definition 8.16. For each $G = (F, R, \rho, m, a) \in \mathfrak{R}_\bullet(\tau)$, we set

$$\mathcal{X}_\tau(G) := (F, R, \rho, m, a, \Upsilon_{(R,m)}),$$

which is an element of $\mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}})$.

It is straightforward to verify that Theorem 8.13 implies the convergence of law maps of stochastic processes, as follows.

Theorem 8.17. *Let $G_n = (F_n, R_n, \rho_n, m_n, a_n)$, $n \in \mathbb{N}$, be elements of $\mathfrak{R}_\bullet(\tau)$, and $G = (F, R, \rho, m, a)$ be an element of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$ such that m is of full support. Assume that the following conditions are satisfied:*

- (i) $G_n \rightarrow G$ in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$;
- (ii) $\lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} R_n(\rho_n, B_{R_n}(\rho_n, r)^c) = \infty$.

Then $G \in \mathfrak{R}_\bullet(\tau)$ and $\mathcal{X}_\tau(G_n) \rightarrow \mathcal{X}_\tau(G)$ in $\mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}})$.

Proof. By [64, Theorem 5.1], we have $G \in \mathfrak{R}_\bullet(\tau)$. We write $\Upsilon_n := \Upsilon_{(R_n, m_n)}$ and $\Upsilon := \Upsilon_{(R, m)}$. By (i) and Theorem 3.7, we may assume that (F_n, R_n, ρ_n) and (F, R, ρ) are embedded isometrically into a common rooted bcm space (M, d_M, ρ_M) in such a way that $F_n \rightarrow F$ in the Fell topology as closed subsets of M , $\rho_n = \rho = \rho_M$ as elements of M , $m_n \rightarrow m$ vaguely as measures on M , and $a_n \rightarrow a$ in $\tau(M)$. Suppose that elements $x_n \in F_n$ converge to an element $x \in F$ in M . Then, using (ii), we can follow the proof of [64, Lemma 5.8] to obtain that

$$\lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} R_n(x_n, B_{R_n}(x_n, r)^c) = \infty.$$

By Theorem 8.13, we obtain that $\Upsilon_n(x_n) \rightarrow \Upsilon(x)$ as probability measures on $D_{J_1}(\mathbb{R}_{\geq 0}, M)$. Therefore, Υ_n converges to Υ in $\widehat{C}(M, \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, M)))$, which completes the proof. $\square$

To discuss a version of Theorem 8.17 for random resistance metric spaces, we prove the following result, which ensures that, if G is a random element of $\mathfrak{R}_\bullet(\tau)$, then $\mathcal{X}_\tau(G)$ is a random element of $\mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}})$.

Proposition 8.18. *The map $\mathcal{X}_\tau: \mathfrak{R}_\bullet(\tau) \rightarrow \mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}})$ is Borel measurable. Moreover, its image, that is,*

$$\{(F, R, \rho, m, a, \Upsilon_{(R, m)}) \mid (F, R, \rho, m, a) \in \mathfrak{R}_\bullet(\tau)\}$$

is a Borel subset of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau \times \tau_{\text{SP}})$.

We follow [64, Proof of Proposition 6.1] to show Proposition 8.18. For each $(F, R, \rho, m) \in \mathfrak{R}_\bullet$ and $r > 0$, we define $F^{[r]}$ to be the closure of $B_F(\rho, r)$ in F , and denote by $R^{[r]}$ and $m^{[r]}$ the restrictions of R and m to $F^{[r]}$ and $B_R(\rho, r)$, respectively. We note that $(F^{[r]}, R^{[r]}, \rho, m^{[r]}) \in \mathfrak{R}_\bullet$ by [50, Theorem 8.4]. We write

$$\Upsilon^{[r]}(x) = \Upsilon_{(R, m)}^{[r]}(x) := \Upsilon_{(R^{[r]}, m^{[r]})}(x), \quad x \in F^{[r]}.$$

Since $F^{[r]}$ is a subspace of F , we naturally regard $\Upsilon_{(R, m)}^{[r]}$ as an element of $\widehat{C}(F, \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, F)))$. Below, we present two lemmas corresponding to [64, Lemmas 6.2 and 6.3].

Lemma 8.19. *For each $(F, R, \rho, m) \in \mathfrak{R}_\bullet$, the following map*

$$(0, \infty) \ni r \mapsto \Upsilon^{[r]} \in \widehat{C}(F, \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, F)))$$

is left-continuous.

Proof. Fix $r > 0$ and a sequence $(r_n)_{n \geq 1}$ such that $r_n \uparrow r$. We regard each $(F^{[r_n]}, R^{[r_n]})$ and $(F^{[r]}, R^{[r]})$ as a subspace of (F, R) in the obvious way. It is then straightforward to check that $F^{[r_n]} \rightarrow F^{[r]}$ in the Fell topology on F and $m^{[r_n]} \rightarrow m^{[r]}$ vaguely as measures on F . Thus, by Theorem 8.13, we obtain that $\Upsilon^{[r_n]} \rightarrow \Upsilon^{[r]}$, which completes the proof. $\square$

Remark 8.20. In [61, Lemmas 2.14 and 6.2], one finds a similar left-continuity result. In that paper, the author considered the restriction of m to $F^{[r]}$. However, such a choice generally destroys the left-continuity in r of the map $r \mapsto m^{[r]} \in \mathcal{M}_{\text{fin}}(F)$. (For example, consider the case where $F = [0, 1]$ and $m = \text{Leb} + \delta_{\{1\}}$, which breaks the left-continuity at $r = 1$.) Thus, the restriction used in that paper should be replaced by the current definition. We note that this does not affect any of the results in that paper.

For each $r > 0$, we define a map $\mathcal{X}_\tau^{[r]}: \mathfrak{R}_\bullet(\tau) \rightarrow \mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}})$ as follows: for each $G = (F, R, \rho, m, a) \in \mathfrak{R}_\bullet(\tau)$,

$$\mathcal{X}_\tau^{[r]}(G) := (F, R, \rho, m, a, \Upsilon_{(R, m)}^{[r]}).$$

Lemma 8.21. *For each $r > 0$, the map $\mathcal{X}_\tau^{[r]}: \mathfrak{R}_\bullet(\tau) \rightarrow \mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}})$ is measurable.*

To prove the above lemma, we prepare an auxiliary result, which will also be used in the proof of Theorem 8.24 below.

Lemma 8.22. *Assume that elements $G_n \in \mathfrak{R}_\bullet(\tau)$ converge to an element $G \in \mathfrak{R}_\bullet(\tau)$. Then, for all but countably many $r > 0$,*

$$\mathcal{X}_\tau^{[r]}(G_n) \xrightarrow{n \rightarrow \infty} \mathcal{X}_\tau^{[r]}(G) \quad \text{in } \mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}}). \quad (8.4)$$

Proof. Write $G_n = (F_n, R_n, \rho_n, m_n, a_n)$ for $n \in \mathbb{N}$, and $G = (F, R, \rho, m, a)$. By Theorem 3.7, we may assume that (F_n, R_n, ρ_n) and (F, R, ρ) are embedded isometrically into a common rooted bcm space (M, d^M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements of M , $F_n \rightarrow F$ in the Fell topology as closed subsets of M , $m_n \rightarrow m$ vaguely as measures on M , and $a_n \rightarrow a$ in $\tau(M)$. It then follows that, for all but countably many $r > 0$, $F_n^{[r]} \rightarrow F^{[r]}$ in the Hausdorff topology, and $m_n^{[r]} \rightarrow m^{[r]}$ weakly. Following the proof of Theorem 8.17, one can verify that, for such radii $r > 0$, $\Upsilon_n^{[r]} \rightarrow \Upsilon^{[r]}$ in $\widehat{C}(M, \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, M)))$. Combining these convergences yields (8.4), which completes the proof. $\square$

Proof of Lemma 8.21. Fix $r > 0$. Let $f: \mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}}) \rightarrow [0, 1]$ be a continuous function. It suffices to show that $g := f \circ \mathcal{X}_\tau^{[r]}$ is measurable. For each $\varepsilon \in (0, r)$, define $g_\varepsilon: \mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}}) \rightarrow [0, 1]$ by

$$g_\varepsilon(G) := \frac{1}{\varepsilon} \int_{r-\varepsilon}^r f(\mathcal{X}_\tau^{[s]}(G)) \, ds, \quad G = (F, R, \rho, m, a) \in \mathfrak{R}_\bullet(\tau).$$

The above integral is well-defined since the integrand is measurable, which follows from Lemma 8.19. Moreover, by that lemma, $g_\varepsilon(G) \rightarrow f(\mathcal{X}_\tau^{[r]}(G))$ for each G . Hence, it suffices to show that g_ε is measurable. We verify this by showing that g_ε is continuous.

Indeed, let $G_n \rightarrow G$ in $\mathfrak{R}_\bullet(\tau)$. Then Lemma 8.22 implies that $\mathcal{X}_\tau^{[s]}(G_n) \rightarrow \mathcal{X}_\tau^{[s]}(G)$ for all but countably many $s > 0$. The continuity of f then gives $f(\mathcal{X}_\tau^{[s]}(G_n)) \rightarrow f(\mathcal{X}_\tau^{[s]}(G))$ for such s . Since f takes values in $[0, 1]$, the dominated convergence theorem yields $g_\varepsilon(G_n) \rightarrow g_\varepsilon(G)$. Hence g_ε is continuous, and therefore measurable. This completes the proof. $\square$

We next show that $\mathcal{X}_\tau$ is obtain as a (pointwise) limit of $\mathcal{X}_\tau^{[r]}$ as $r \rightarrow \infty$, which, combined with the above two lemmas, implies Proposition 8.18.

Lemma 8.23. *For any $G \in \mathfrak{R}_\bullet(\tau)$, it holds that $\mathcal{X}_\tau^{[r]}(G) \rightarrow \mathcal{X}_\tau(G)$ in $\mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}})$ as $r \rightarrow \infty$.*

Proof. Fix $G = (F, R, \rho, m, a) \in \mathfrak{R}_\bullet(\tau)$. It is easy to check that, as $r \rightarrow \infty$, $F^{[r]} \rightarrow F$ in the Fell topology as closed subsets of F and $m^{[r]} \rightarrow m$ vaguely as measures on F . Since R is a recurrent resistance metric, we can follow the proof of Theorem 8.17 to deduce that $\Upsilon^{[r]} \rightarrow \Upsilon$ as $r \rightarrow \infty$ in $\widehat{C}(F, \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, F)))$. This immediately implies the desired result. $\square$

Now, it is straightforward to show Proposition 8.18.

Proof of Proposition 8.18. The first assertion regarding the measurability of $\mathcal{X}_\tau$ is an immediate consequence of Lemmas 8.21 and 8.23. Clearly, the map $\mathcal{X}_\tau$ is injective. We recall from Proposition 8.9 that its domain $\mathfrak{R}_\bullet(\tau)$ is a Borel subset of the Polish space $\mathfrak{R}_\bullet(\tau_{\mathcal{M}} \times \tau)$. Thus, we can apply Lemma 2.14 to deduce that the image of $\mathcal{X}_\tau$ is a Borel subset of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau \times \tau_{\text{SP}})$, which completes the proof. $\square$

Thanks to Proposition 8.18, if G is a random element of $\mathfrak{R}_\bullet(\tau)$, then $\mathcal{X}_\tau(G)$ is a random element of $\mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}})$. Below, we provide a version of Theorem 8.17 for random resistance metric spaces.

Theorem 8.24. *Let $G_n = (F_n, R_n, \rho_n, m_n, a_n)$, $n \in \mathbb{N}$, be random elements of $\mathfrak{R}_\bullet(\tau)$, and $G = (F, R, \rho, m, a)$ be a random element of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau)$ such that m is of full support, almost surely. Assume that the following conditions are satisfied.*

- (i) *It holds that $G_n \xrightarrow{d} G$ in $\mathfrak{R}_\bullet(\tau)$;*
- (ii) *If we write $\mathbf{P}_n$ for the underlying probability measure of G_n , then, for all $\lambda > 0$,*

$$\lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbf{P}_n(R_n(\rho_n, B_{R_n}(\rho_n, r))^c > \lambda) = 1.$$

Then $G \in \mathfrak{R}_\bullet(\tau)$ almost surely, and $\mathcal{X}_\tau(G_n) \xrightarrow{d} \mathcal{X}_\tau(G)$ as random elements of $\mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}})$.

Proof. By [64, Proposition 6.1], we have $G \in \mathfrak{R}_\bullet(\tau)$ almost surely. For the second assertion, by [17, Theorem 2.1], it suffices to show that for all bounded and uniformly continuous functions $f: \mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}}) \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbf{E}_n[f(\mathcal{X}_\tau(G_n))] = \mathbf{E}[f(\mathcal{X}_\tau(G))],$$

where $\mathbf{E}$ denotes the expectation with respect to the randomness of G . Fix such a function f .

By Lemma 8.22 and Theorem 8.24 (i), we have that, for each $r > 1$,

$$\lim_{n \rightarrow \infty} \mathbf{E}_n \left[\int_{r-1}^r f(\mathcal{X}_\tau^{[s]}(G_n)) ds \right] = \mathbf{E} \left[\int_{r-1}^r f(\mathcal{X}_\tau^{[s]}(G)) ds \right].$$

(Note that the above integrals are well-defined by Lemma 8.19.) Moreover, Lemma 8.23 yields that

$$\lim_{r \rightarrow \infty} \mathbf{E} \left[\int_{r-1}^r f(\mathcal{X}_\tau^{[s]}(G)) ds \right] = \mathbf{E}[f(\mathcal{X}_\tau(G))].$$

Hence, it remains to show that

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[\int_{r-1}^r |f(\mathcal{X}_\tau^{[s]}(G_n)) - f(\mathcal{X}_\tau(G_n))| ds \right] = 0. \quad (8.5)$$

For each $\varepsilon > 0$, define the modulus of continuity of f by

$$w_f(\varepsilon) := \sup \{ |f(\mathcal{Y}_1) - f(\mathcal{Y}_2)| \mid \mathcal{Y}_1, \mathcal{Y}_2 \in \mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}}), d_{\mathfrak{M}_\bullet}^{\tau \times \tau_{\text{SP}}}(\mathcal{Y}_1, \mathcal{Y}_2) \leq \varepsilon \}.$$

Then it follows that

$$\begin{aligned} & \mathbf{E}_n \left[\int_{r-1}^r |f(\mathcal{X}_\tau^{[s]}(G_n)) - f(\mathcal{X}_\tau(G_n))| ds \right] \\ & \leq w_f(\varepsilon) + \|f\|_\infty \mathbf{P}_n \left(\sup_{r-1 < s < r} d_{\mathfrak{M}_\bullet}^{\tau \times \tau_{\text{SP}}}(\mathcal{X}_\tau^{[s]}(G_n), \mathcal{X}_\tau(G_n)) > \varepsilon \right). \end{aligned}$$

The uniform continuity of f implies $w_f(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore, it remains to prove that

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}_n \left(\sup_{r-1 < s < r} d_{\mathfrak{M}_\bullet}^{\tau \times \tau_{\text{SP}}}(\mathcal{X}_\tau^{[s]}(G_n), \mathcal{X}_\tau(G_n)) > \varepsilon \right) = 0, \quad \forall \varepsilon > 0. \quad (8.6)$$

We simply write $\Upsilon_n := \Upsilon_{(R_n, m_n)}$. Fix $r > 1$ and $r_0 > 0$ with $r-1 > r_0$, and take $s \in (r-1, r)$. By Lemma 3.9, we have

$$d_{\mathfrak{M}_\bullet}^{\tau \times \tau_{\text{SP}}}(\mathcal{X}_\tau(G_n), \mathcal{X}_\tau^{[s]}(G_n)) \leq d_{(F_n, \rho_n), \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, F_n))}^{\hat{C}}(\Upsilon_n, \Upsilon_n^{[s]}). \quad (8.7)$$

Since $s > r_0$, it holds that

$$\text{dom}(\Upsilon_n)^{(r_0)} = \text{dom}(\Upsilon_n^{[s]})^{(r_0)} = F_n^{(r_0)}.$$

Then Lemma 2.8 yields

$$d_{(F_n, \rho_n), \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, F_n))}^{\widehat{C}}(\Upsilon_n, \Upsilon_n^{[s]}) \leq e^{-r_0} + \sup_{x \in F_n^{(r_0)}} d_{D_{J_1}(\mathbb{R}_{\geq 0}, F_n)}^P(\Upsilon_n(x), \Upsilon_n^{[s]}(x)).$$

By the proof of [63, Lemma 6.7], for any $x \in F_n^{(r_0)}$ and $T > 0$,

$$\begin{aligned} d_{D_{J_1}(\mathbb{R}_{\geq 0}, F_n)}^P(\Upsilon_n(x), \Upsilon_n^{[s]}(x)) &\leq e^{-T} + P_n^x(\check{\sigma}_{B_{R_n}}(x, s/2) \leq T) \\ &\leq e^{-T} + P_n^x(\check{\sigma}_{B_{R_n}}(x, (r-1)/2) \leq T), \end{aligned}$$

where $((X_n(t))_{t \geq 0}, (P_n^x)_{x \in F_n})$ denotes the process associated with (R_n, m_n) , and $\check{\sigma}$ is the first exit time defined in (4.3). Taking the supremum over $x \in F_n^{(r_0)}$ gives

$$\sup_{x \in F_n^{(r_0)}} d_{D_{J_1}(\mathbb{R}_{\geq 0}, F_n)}^P(\Upsilon_n(x), \Upsilon_n^{[s]}(x)) \leq e^{-T} + \sup_{x \in F_n^{(r_0)}} P_n^x(\check{\sigma}_{B_{R_n}}(x, (r-1)/2) \leq T).$$

Combining this with (8.7), we have, for any $r-1 > r_0$, $T > 0$, and $s \in (r-1, r)$,

$$d_{\mathfrak{M}_\bullet}^{\tau \times \tau_{\text{SP}}}(\mathcal{X}_\tau(G_n), \mathcal{X}_\tau^{[s]}(G_n)) \leq e^{-r_0} + e^{-T} + \sup_{x \in F_n^{(r_0)}} P_n^x(\check{\sigma}_{B_{R_n}}(x, (r-1)/2) \leq T).$$

Taking the supremum over $s \in (r-1, r)$ yields

$$\sup_{r-1 < s < r} d_{\mathfrak{M}_\bullet}^{\tau \times \tau_{\text{SP}}}(\mathcal{X}_\tau(G_n), \mathcal{X}_\tau^{[s]}(G_n)) \leq e^{-r_0} + e^{-T} + \sup_{x \in F_n^{(r_0)}} P_n^x(\check{\sigma}_{B_{R_n}}(x, (r-1)/2) \leq T). \quad (8.8)$$

Finally, under the non-explosion condition (ii), by following [64, Proof of Lemma 6.3], one can verify that for each $r_0 > 0$ and $T > 0$,

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}_n \left(\sup_{x \in F_n^{(r_0)}} P_n^x(\check{\sigma}_{B_{R_n}}(x, (r-1)/2) \leq T) > \varepsilon \right) = 0, \quad \forall \varepsilon > 0.$$

Combining this with (8.8) proves (8.6), and hence (8.5). This completes the proof. $\square$

8.4 Local limit theorem

In this subsection, we establish a local limit theorem (see Theorem 8.30 below), that is, the convergence of heat kernels, under the assumption that measured resistance metric spaces and the law maps of the associated stochastic processes converge. Throughout this subsection, we fix a Polish structure τ .

Below, we modify the map $\mathcal{X}_\tau$ introduced in Definition 8.16 to incorporate the heat kernel. Recall the structure τ_{HK} from (6.1).

Definition 8.25. For each $G = (F, R, \rho, m, a) \in \mathfrak{R}_\bullet(\tau)$, we set

$$\mathcal{X}_\tau^{\text{HK}}(G) := (F, R, \rho, m, a, p_{(R, m)}, \Upsilon_{(R, m)}),$$

which is an element of $\mathfrak{R}_\bullet(\tau \times \tau_{\text{HK}} \times \tau_{\text{SP}})$.

We recall from [63] that the convergence of measured resistance metric spaces implies the precompactness of the associated heat kernels.

Lemma 8.26 ([63, Proposition 4.13]). Assume that $G_n = (F_n, R_n, \rho_n, m_n)$, $n \in \mathbb{N}$, and $G = (F, R, \rho, m)$ are elements of $\mathfrak{R}_\bullet$ such that $G_n \rightarrow G$ in $\mathfrak{R}_\bullet$. We simply write $p_n = p_{(R_n, m_n)}$. Then, for any $r > 0$ and $T > 0$,

$$\sup_{n \geq 1} \sup_{T \leq t < \infty} \sup_{x, y \in F_n^{(r)}} p_n(t, x, y) < \infty, \quad \lim_{\delta \rightarrow 0} \sup_{n \geq 1} w(p_n, T, r, \delta) = 0,$$

where $w(p_n, T, r, \delta)$ is defined to be

$$\sup \left\{ |p_n(t, x, y) - p_n(t', x', y')| \mid \begin{array}{l} (t, x, y), (t', x', y') \in [T, \infty) \times F_n^{(r)} \times F_n^{(r)} \\ |t - t'| \vee R_n(x, x') \vee R_n(y, y') \leq \delta \end{array} \right\}.$$

We now present the main result of this subsection, which states that the precompactness of heat kernels and the convergence of law maps of stochastic processes together imply the convergence of the heat kernels. A similar result for simple random walks on graphs can be found in [27].

Theorem 8.27. Let $G_n = (F_n, R_n, \rho_n, m_n, a_n)$, $n \in \mathbb{N}$ and $G = (F, R, \rho, m, a)$ are elements of $\mathfrak{R}_\bullet(\tau)$. If $\mathcal{X}_\tau(G_n) \rightarrow \mathcal{X}_\tau(G)$ in $\mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}})$ as $n \rightarrow \infty$, then, as $n \rightarrow \infty$,

$$\mathcal{X}_\tau^{\text{HK}}(G_n) \rightarrow \mathcal{X}_\tau^{\text{HK}}(G) \quad \text{in} \quad \mathfrak{R}_\bullet(\tau \times \tau_{\text{HK}} \times \tau_{\text{SP}}).$$

Proof. We simply write $p_n = p_{(R_n, m_n)}$ and $\Upsilon_n = \Upsilon_{(R_n, m_n)}$ for $n \in \mathbb{N}$, and $p = p_{(R, m)}$ and $\Upsilon = \Upsilon_{(R, m)}$. We let $((X_n(t))_{t \geq 0}, (P_n^x)_{x \in F_n})$ and $((X(t))_{t \geq 0}, (P^x)_{x \in F})$ be the processes associated with (R_n, m_n) and (R, m) , respectively. By Theorem 3.7, we may think that (F_n, R_n, ρ_n) and (F, R, ρ) are embedded isometrically into a common rooted bcm space (M, d_M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements of M , $F_n \rightarrow F$ in the Fell topology as closed subsets of M , $m_n \rightarrow m$ vaguely as measures on M , $a_n \rightarrow a$ in $\tau(M)$, and $\Upsilon_n \rightarrow \Upsilon$ in $\widehat{C}(M, \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, M)))$. We deduce from Lemma 8.26 and [62, Theorem 3.31] that the collection $\{p_n\}_{n \geq 1}$ is precompact in $\widehat{C}(\mathbb{R}_{>0} \times M \times M, \mathbb{R}_{\geq 0})$. Thus, it is enough to show that the limit of any convergent subsequence in $\{p_n\}_{n \geq 1}$ coincides with p . To avoid notational complexity, we assume that the full sequence p_n converges to q in $\widehat{C}(\mathbb{R}_{>0} \times M \times M, \mathbb{R}_{\geq 0})$. Since the domain $\text{dom}(p_n) = \mathbb{R}_{>0} \times F_n \times F_n$ converges to $\mathbb{R}_{>0} \times F \times F$ in the Fell topology, it follows from Lemma 2.6 that $\text{dom}(q) = \mathbb{R}_{>0} \times F \times F$. Fix $t \in (0, \infty)$ and $x \in F$. By the convergence of F_n to F , we can choose $x_n \in F_n$ such that $x_n \rightarrow x$ in M . It then follows from the convergence of Υ_n to Υ that

$$\Upsilon_n(x_n) = P_n^{x_n}((X_n(t))_{t \geq 0} \in \cdot) \rightarrow \Upsilon(x) = P^x((X(t))_{t \geq 0} \in \cdot)$$

weakly as probability measures on $D_{J_1}(\mathbb{R}_{\geq 0}, M)$. By its quasi-left continuity, X is continuous at each deterministic time $t \geq 0$, P^x -a.s. Thus, the above distributional convergence in $D_{J_1}(\mathbb{R}_{\geq 0}, M)$ implies that $P_n^{x_n}(X_n(t) \in \cdot) \xrightarrow{d} P^x(X(t) \in \cdot)$ for each $t \geq 0$. This yields that, for any compactly-supported continuous function f on M ,

$$\lim_{n \rightarrow \infty} \int_{F_n} p_n(t, x_n, y) f(y) m_n(dy) = \lim_{n \rightarrow \infty} E_n^{x_n}[f(X_t)] = E^x[f(X_t)] = \int_F p(t, x, y) f(y) m(dy).$$

On the other hand, since $p_n \rightarrow q$ in $\widehat{C}(\mathbb{R}_{>0} \times M \times M, \mathbb{R}_{\geq 0})$ and $f(y) m_n(dy) \rightarrow f(y) m(dy)$ weakly, we deduce from Lemma 2.9 that

$$\lim_{n \rightarrow \infty} \int_{F_n} p_n(t, x_n, y) f(y) m_n(dy) = \int_F q(t, x, y) f(y) m(dy).$$

Therefore, for all compactly-supported continuous functions f on M ,

$$\int_F p(t, x, y) f(y) m(dy) = \int_F q(t, x, y) f(y) m(dy).$$

In particular, $p(t, x, \cdot) = q(t, x, \cdot)$, m -a.e. Since m is of full support on F and both $p(t, x, \cdot)$ and $q(t, x, \cdot)$ are continuous functions on F , it follows that $p(t, x, y) = q(t, x, y)$ for all $y \in F$. Hence, we conclude that $p = q$, which completes the proof. $\square$

Below, we present a result analogous to Proposition 8.18. This enables us to discuss the convergence of heat kernels of stochastic processes on random resistance metric spaces.

Proposition 8.28. *The map $\mathcal{X}_\tau^{\text{HK}}: \mathfrak{R}_\bullet(\tau) \rightarrow \mathfrak{R}_\bullet(\tau \times \tau_{\text{HK}} \times \tau_{\text{SP}})$ is Borel measurable. Moreover, its image, that is,*

$$\{(F, R, \rho, m, a, p_{(R,m)}, \Upsilon_{(R,m)}) \mid (F, R, \rho, m, a) \in \mathfrak{R}_\bullet(\tau)\}$$

is a Borel subset of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau \times \tau_{\text{HK}} \times \tau_{\text{SP}})$.

Proof. If we define a map $\tilde{\mathcal{X}}_\tau^{\text{HK}}: \mathcal{X}_\tau(\mathfrak{R}_\bullet(\tau)) \rightarrow \mathfrak{R}_\bullet(\tau \times \tau_{\text{HK}} \times \tau_{\text{SP}})$ by setting $\tilde{\mathcal{X}}_\tau^{\text{HK}}(\mathcal{X}_\tau(G)) := \mathcal{X}_\tau^{\text{HK}}(G)$, then $\tilde{\mathcal{X}}_\tau^{\text{HK}}$ is continuous by Theorem 8.27. Moreover, by Proposition 8.18, the map $\mathcal{X}_\tau: \mathfrak{R}_\bullet(\tau) \rightarrow \mathcal{X}_\tau(\mathfrak{R}_\bullet(\tau))$ is Borel measurable (note that the codomain is restricted to its image). Since we have that $\mathcal{X}_\tau^{\text{HK}} = \tilde{\mathcal{X}}_\tau^{\text{HK}} \circ \mathcal{X}_\tau$, we obtain the first assertion. The second claim can be verified in the same way as in Proposition 8.18. $\square$

Corollary 8.29. *The set*

$$\{(F, R, \rho, m, a, p_{(R,m)}) \mid (F, R, \rho, m, a) \in \mathfrak{R}_\bullet(\tau)\}$$

is a Borel subset of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau \times \tau_{\text{HK}})$.

Proof. This is an immediate consequence of Lemma 2.14 and Proposition 8.18. $\square$

By Theorem 8.27, the results of Theorems 8.17 and 8.24 are extended to the convergence of not only the law maps of stochastic processes but also the associated heat kernels, as shown below.

Corollary 8.30. *Under the same conditions as Theorem 8.17, it holds that $\mathcal{X}_\tau^{\text{HK}}(G_n) \rightarrow \mathcal{X}_\tau^{\text{HK}}(G)$ in $\mathfrak{R}_\bullet(\tau \times \tau_{\text{HK}} \times \tau_{\text{SP}})$.*

Proof. This is immediate from Theorems 8.17 and 8.27. $\square$

Corollary 8.31. *Under the same conditions as Theorem 8.24, it holds that $\mathcal{X}_\tau^{\text{HK}}(G_n) \xrightarrow{d} \mathcal{X}_\tau^{\text{HK}}(G)$ as random elements of $\mathfrak{R}_\bullet(\tau \times \tau_{\text{HK}} \times \tau_{\text{SP}})$.*

Proof. Recall the continuous map $\tilde{\mathcal{X}}_\tau^{\text{HK}}: \mathcal{X}_\tau(\mathfrak{R}_\bullet(\tau)) \rightarrow \mathfrak{R}_\bullet(\tau \times \tau_{\text{HK}} \times \tau_{\text{SP}})$ from the proof of Proposition 8.28. By Theorem 8.24, we have that $\mathcal{X}_\tau(G_n) \xrightarrow{d} \mathcal{X}_\tau(G)$ as random elements of $\mathfrak{R}_\bullet(\tau \times \tau_{\text{SP}})$. Thus, we deduce from the continuity of $\tilde{\mathcal{X}}_\tau^{\text{HK}}$ that $\tilde{\mathcal{X}}_\tau^{\text{HK}}(\mathcal{X}_\tau(G_n)) = \mathcal{X}_\tau^{\text{HK}}(G_n)$ converges in distribution to $\tilde{\mathcal{X}}_\tau^{\text{HK}}(\mathcal{X}_\tau(G)) = \mathcal{X}_\tau^{\text{HK}}(G)$, which completes the proof. $\square$

9 Collisions of stochastic processes on resistance metric spaces

In Section 8.4, we showed that if measured resistance metric spaces converge and satisfy the non-explosion condition, then the law maps of the associated stochastic processes and heat kernels also converge. Building on this, and by applying Theorems 7.13 and 7.16, we establish convergence results for collision measures of stochastic processes on resistance metric spaces in this section.

9.1 Resistance metric spaces for the study of collisions

In this subsection, we introduce a space of resistance metric spaces and verify its measurability. This space will be used in the main results of the next subsection.

Definition 9.1. We define $\mathfrak{R}_\bullet^{\text{col}}$ to be the collection of $(F, R, \rho, m, \mu) \in \mathfrak{R}_\bullet(\tau_{\mathcal{M}})$ such that

$$\lim_{\delta \rightarrow 0} \sup_{x_1, x_2 \in F(r)} \int_0^\delta \int_{F(r)} p_{(R, m)}(t, x_1, y) p_{(R, m)}(t, x_2, y) \mu(dy) dt = 0, \quad \forall r > 0. \quad (9.1)$$

More generally, given a Polish structure τ , we write $\mathfrak{R}_\bullet^{\text{col}}(\tau)$ for the collection of $(F, R, \rho, m, \mu, a) \in \mathfrak{R}_\bullet(\tau_{\mathcal{M}} \times \tau)$ such that $(F, R, \rho, m, \mu) \in \mathfrak{R}_\bullet^{\text{col}}$. We always equip $\mathfrak{R}_\bullet^{\text{col}}(\tau)$ with the relative topology induced from $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2} \times \tau)$.

Let $(F, R, \rho, m, \mu) \in \mathfrak{R}_\bullet^{\text{col}}$, X^1 be the associated process, and X^2 be an independent copy of X^1 . Write $\hat{X} = (X^1, X^2)$ for their product process. Condition (9.1) implies that the Radon measure μ^{diag} on $F \times F$ belongs to the local Kato class of $\hat{X}$ (see Definition 4.8). Hence, the collision measure of X^1 and X^2 associated with μ is well defined (see Definition 7.3). We write

$$\hat{\Upsilon}^{\text{col}}(\mathbf{x}) = \hat{\Upsilon}_{(R, m, \mu)}^{\text{col}}(\mathbf{x}) := \mathcal{L}_{(\hat{X}, \Pi)}(\mathbf{x}), \quad \mathbf{x} \in F \times F.$$

The following is immediate from Proposition 7.7.

Lemma 9.2. For each $(F, R, \rho, m, \mu) \in \mathfrak{R}_\bullet$, the map

$$\hat{\Upsilon}_{(R, m, \mu)}^{\text{col}} : F \times F \rightarrow \mathcal{P}(D_{J_1}(\mathbb{R}_{\geq 0}, F) \times D_{J_1}(\mathbb{R}_{\geq 0}, F) \times \mathcal{M}(F \times \mathbb{R}_{\geq 0}))$$

is continuous.

Thus, the space $\mathfrak{R}_\bullet^{\text{col}}(\tau)$ forms a suitable class of resistance metric spaces for the study of collisions. Below, we verify its measurability.

Proposition 9.3. The set $\mathfrak{R}_\bullet^{\text{col}}(\tau)$ is a Borel subset of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2} \times \tau)$.

Proof. Define

$$\begin{aligned} \mathfrak{G}_1 &:= \{(F, R, \rho, m, \mu, a, p_{(R, m)}) \mid (F, R, \rho, m, \mu, a) \in \mathfrak{R}_\bullet(\tau_{\mathcal{M}} \times \tau)\}, \\ \mathfrak{G}_2 &:= \{(F, R, \rho, m, \mu, a, p_{(R, m)}) \mid (F, R, \rho, m, \mu, a) \in \mathfrak{R}_\bullet^{\text{col}}(\tau)\}. \end{aligned}$$

Since, given a measured resistance metric space (F, R, m) , the associated heat kernel $p_{(R, m)}$ is unique, the following map is injective:

$$\mathfrak{G}_2 \ni (F, R, \rho, m, \mu, a, p_{(R, m)}) \longmapsto (F, R, \rho, m, \mu, a) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2} \times \tau).$$

Moreover, by Theorem 3.7, the above map is continuous. Thus, once we establish that $\mathfrak{G}_2$ is a Borel subset of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2} \times \tau \times \tau_{\text{HK}})$, we can use Lemma 2.14 to obtain the desired result. Since $\mathfrak{G}_1$ is a Borel subset of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2} \times \tau \times \tau_{\text{HK}})$ by Corollary 8.29, it is enough to show that $\mathfrak{G}_2$ is a Borel subset of $\mathfrak{G}_1$.

For each $k, l, N \in \mathbb{N}$, we define $\mathfrak{G}^{(k, l, N)}(\tau)$ to be the collection of $(F, R, \rho, m, \mu, a, p_{(R, m)}) \in \mathfrak{G}_1$ such that, for all but countably many $r \in [k, k+1]$,

$$\sup_{x_1, x_2 \in F(r)} \int_0^{1/N} \int_{F(r)} p_{(R, m)}(t, x_1, y) p_{(R, m)}(t, x_2, y) \mu(dy) dt \leq 1/l. \quad (9.2)$$

Since it holds that

$$\mathfrak{G}_2 = \bigcap_{k \geq 1} \bigcap_{l \geq 1} \bigcup_{N \geq 1} \mathfrak{G}^{(k, l, N)}(\tau).$$

it is enough to show that each $\mathfrak{G}^{(k, l, N)}(\tau)$ is a Borel subset of $\mathfrak{G}_1$. We do this by proving that $\mathfrak{G}^{(k, l, N)}(\tau)$ is closed in $\mathfrak{G}_1$. Fix $G_n = (F_n, R_n, \rho_n, m_n, \mu_n, a_n, p_n) \in \mathfrak{G}^{(k, l, N)}(\tau)$, $n \in \mathbb{N}$, such that

$$G_n = (F_n, R_n, \rho_n, m_n, \mu_n, a_n, p_n) \rightarrow G = (F, R, \rho, m, \mu, a, p)$$

in $\mathfrak{G}_1$. Here, we simply write $p_n = p_{(R_n, m_n)}$, $n \in \mathbb{N}$, and $p = p_{(R, m)}$. By Theorem 3.7, we may think that (F_n, R_n, ρ_n) and (F, R, ρ) are embedded isometrically into a common rooted bcm space (M, d_M, ρ_M) in such a way that $\rho_n = \rho = \rho_M$ as elements of M , $F_n \rightarrow F$ in the Fell topology as closed subsets of M , $m_n \rightarrow m$ and $\mu_n \rightarrow \mu$ vaguely as measures on M , $a_n \rightarrow a$ in $\tau(M)$, and $p_n \rightarrow p$ in $\widehat{C}(\mathbb{R}_{>0} \times M \times M, \mathbb{R}_{\geq 0})$. Let $r \in [k, k+1]$ be such that $F_n^{(r)} \rightarrow F^{(r)}$ in the Hausdorff topology in M , $\mu_n^{(r)} \rightarrow \mu^{(r)}$ weakly in M , and all the pairs (μ_n, p_n) satisfy (9.2) with radius r . Fix $x, y \in F^{(r)}$. By the convergence of $F_n^{(r)}$ to $F^{(r)}$, we can find $x_n, y_n \in F_n^{(r)}$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$ in M . Since p_n converges to p , it holds that

$$p_n(t, x_n, z)p_n(t, y_n, z) \rightarrow p(t, x, z)p(t, y, z)$$

in $\widehat{C}(\mathbb{R}_{>0} \times M, \mathbb{R}_{\geq 0})$ with respect to $(t, z) \in \mathbb{R}_{>0} \times M$. This leads to the following vague convergence as measures on $\mathbb{R}_{>0} \times M$:

$$p_n(t, x_n, z)p_n(t, y_n, z)\mu_n^{(r)}(dz)dt \rightarrow p(t, x, z)p(t, y, z)\mu^{(r)}(dz)dt.$$

By the Portmanteau theorem for vague convergence (see [43, Lemma 4.1]), it follows that, for any $\varepsilon \in (0, 1/N)$,

$$\begin{aligned} \int_{\varepsilon}^{1/N} \int_F p(t, x, z)p(t, y, z)\mu^{(r)}(dz)dt &\leq \liminf_{n \rightarrow \infty} \int_{\varepsilon}^{1/N} \int_{F_n} p_n(t, x_n, z)p_n(t, y_n, z)\mu_n^{(r)}(dz)dt \\ &\leq 1/l, \end{aligned} \quad (9.3)$$

where the last inequality follows from (9.2). Letting $\varepsilon \rightarrow 0$ and taking the supremum over $x, y \in F^{(r)}$, we obtain that

$$\sup_{x, y \in F^{(r)}} \int_0^{1/N} \int_{F^{(r)}} p(t, x, z)p(t, y, z)\mu(dz)dt \leq 1/l.$$

This implies that $G \in \mathfrak{G}^{(k, l, N)}(\tau)$, which completes the proof. $\square$

9.2 Convergence of collision measures of stochastic processes on resistance metric spaces

The main results of this subsection are stated in Theorems 9.6 and 9.8, which establish the convergence of collision measures of stochastic processes on resistance metric spaces varying along a sequence, as an application of Theorems 7.13 and 7.16. Throughout this subsection, we fix a Polish structure τ .

We first introduce the map of interest. Recall the structure τ_{CM} defined in (7.3).

Definition 9.4. For each $G = (F, R, \rho, m, \mu, a) \in \mathfrak{R}_{\bullet}^{\text{col}}(\tau)$, we set

$$\mathcal{X}_{\tau}^{\text{col}}(G) := \left(F, R, \rho, m, \mu, a, p_{(R, m)}, \hat{\Upsilon}_{(R, m, \mu)}^{\text{col}} \right),$$

which is an element of $\mathfrak{R}_{\bullet}^{\text{col}}(\tau \times \tau_{\text{HK}} \times \tau_{\text{CM}})$.

We first state a convergence result for deterministic resistance metric spaces.

Assumption 9.5. Let $G_n = (F_n, R_n, \rho_n, m_n, \mu_n, a_n)$, $n \in \mathbb{N}$, be elements of $\mathfrak{R}_{\bullet}^{\text{col}}(\tau)$, and $G = (F, R, \rho, m, \mu, a)$ be an element of $\mathfrak{M}_{\bullet}(\tau_{\mathcal{M}}^{\otimes 2} \times \tau)$ such that m is of full support. Assume that the following conditions hold.

- (i) The sequence satisfies $G_n \rightarrow G$ in $\mathfrak{M}_{\bullet}(\tau_{\mathcal{M}}^{\otimes 2} \times \tau)$.
- (ii) We have $\lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} R_n(\rho_n, B_{R_n}(\rho_n, r)^c) = \infty$.

(iii) For all $r > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{x_1, x_2 \in F_n^{(r)}} \int_0^\delta \int_{F_n^{(r)}} p_n(t, x_1, y) p_n(t, x_2, y) \mu_n(dy) dt = 0,$$

where $p_n := p_{(R_n, m_n)}$.

Theorem 9.6. Under Assumption 9.5, it holds that $G \in \mathfrak{R}_\bullet^{\text{col}}(\tau)$, and $\mathcal{X}_\tau^{\text{col}}(G_n) \rightarrow \mathcal{X}_\tau^{\text{col}}(G)$ in $\mathfrak{R}_\bullet^{\text{col}}(\tau \times \tau_{\text{HK}} \times \tau_{\text{CM}})$.

Proof. By Theorem 8.17, $G \in \mathfrak{R}_\bullet(\tau_{\mathcal{M}} \times \tau)$. Moreover, we deduce from (9.3) and Assumption 9.5(iii) that, for all $r > 0$,

$$\lim_{\delta \rightarrow 0} \sup_{x, y \in F^{(r)}} \int_0^\delta \int_{F^{(r)}} p(t, x, z) p(t, y, z) \mu(dz) dt = 0.$$

Thus, $G \in \mathfrak{R}_\bullet^{\text{col}}(\tau)$. Now the desired convergence follows from Theorem 7.13 and Corollary 8.30. $\square$

We next provide a version of Theorem 7.13 for random resistance metric spaces.

Assumption 9.7. Let $G_n = (F_n, R_n, \rho_n, m_n, \mu_n, a_n)$, $n \in \mathbb{N}$, be random elements of $\mathfrak{R}_\bullet^{\text{col}}(\tau)$, and $G = (F, R, \rho, m, \mu, a)$ be an random element of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2} \times \tau)$ such that m is of full support, almost surely. Assume that the following conditions are satisfied.

(i) It holds that $G_n \xrightarrow{d} G$ in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2} \times \tau)$.

(ii) If we write $\mathbf{P}_n$ for the underlying probability measure of G_n , then, for all $\lambda > 0$,

$$\lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbf{P}_n(R_n(\rho_n, B_{R_n}(\rho_n, r)^c) > \lambda) = 1.$$

(iii) For all $r > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[\left(\sup_{x_1, x_2 \in F_n^{(r)}} \int_0^\delta \int_{F_n^{(r)}} p_n(t, x_1, y) p_n(t, x_2, y) \mu_n(dy) dt \right) \wedge 1 \right] = 0,$$

where we write $p_n := p_{(R_n, m_n)}$.

Theorem 9.8. Under Assumption 9.7, it holds that $G \in \mathfrak{R}_\bullet^{\text{col}}(\tau)$ almost surely, and $\mathcal{X}_\tau^{\text{col}}(G_n) \rightarrow \mathcal{X}_\tau^{\text{col}}(G)$ in $\mathfrak{R}_\bullet^{\text{col}}(\tau \times \tau_{\text{HK}} \times \tau_{\text{CM}})$.

Proof. We have from Theorem 8.24 that $G \in \mathfrak{R}_\bullet(\tau_{\mathcal{M}} \times \tau)$ almost surely. By the Skorohod representation theorem, we may assume that $G_n \rightarrow G$ almost surely on some probability space. Denote by $\mathbb{P}$ the underlying probability measure. From Fatou's lemma and Assumption 9.7(iii), we deduce that, for all $r > 0$ and $\varepsilon > 0$,

$$\mathbb{E} \left[\lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \left(\sup_{x_1, x_2 \in F_n^{(r)}} \int_0^\delta \int_{F_n^{(r)}} p_n(t, x_1, y) p_n(t, x_2, y) \mu_n(dy) dt \right) \wedge 1 \right] = 0.$$

This implies that

$$\liminf_{n \rightarrow \infty} \sup_{x_1, x_2 \in F_n^{(r)}} \int_0^\delta \int_{F_n^{(r)}} p_n(t, x_1, y) p_n(t, x_2, y) \mu_n(dy) dt = 0, \quad \mathbb{P}\text{-a.s.}$$

Therefore, by the same argument as in (9.3), we obtain

$$\lim_{\delta \rightarrow 0} \sup_{x, y \in F^{(r)}} \int_0^\delta \int_{F^{(r)}} p(t, x, z) p(t, y, z) \mu(dz) dt = 0, \quad \mathbb{P}\text{-a.s.}$$

Hence, $G \in \mathfrak{R}_\bullet^{\text{col}}(\tau)$ almost surely. Now the desired result follows from Theorem 7.16 and Corollary 8.31. $\square$

9.3 Scaling limits of collision measures of VSRWs

The heat kernel conditions in Assumption 9.5(iii) and Assumption 9.7(iii) can be verified through uniform lower bounds on the volumes of balls. In this subsection, we focus on uniform collision measures of VSRWs on electrical networks and provide sufficient lower-volume conditions for these heat kernel assumptions; see Theorems 9.10 and 9.13 below. These results apply to a broad class of low-dimensional fractals and critical random graphs, as discussed in Examples 9.11 and 9.14 below.

We first clarify the setting for the main results. For each $n \geq 1$, let (V_n, E_n, μ_n) be a recurrent electrical network, as recalled from Definition 8.7. Write R_n for the associated resistance metric and m_n for the counting measure on V_n , i.e., $m(\{x\}) = 1$ for each $x \in V_n$. Let $a_n, b_n > 0$ be constants, serving as scaling factors for the metric and measure, respectively. Let X_n^1 be the process associated with $(a_n^{-1}R_n, b_n^{-1}m_n)$ and X_n^2 be an independent copy of X_n^1 . In other words, X_n^1 and X_n^2 are i.i.d. VSRWs on the electrical network, rescaled in time by the factor $a_n b_n$. We write $\hat{X}_n = (X_n^1, X_n^2)$ for the product process. Let Π_n denote the collision measure of X_n^1 and X_n^2 associated with $b_n^{-1}m_n$, i.e.,

$$\Pi_n(dx dt) = b_n \sum_{y \in V_n} \mathbf{1}_{\{X_n^1(t)=X_n^2(t)=y\}} \delta_y(dx) dt.$$

This identity follows from Proposition 7.8. In particular, the scaling factor b_n arises from the following computation:

$$\frac{b_n^{-1}m_n(\{x\})}{(b_n^{-1}m_n(\{x\}))^2} = \frac{b_n^{-1}}{b_n^{-2}} = b_n, \quad \forall x \in V_n.$$

Assumption 9.9.

- (i) As $n \rightarrow \infty$, $b_n \rightarrow \infty$.
- (ii) There exists an element $G = (F, R, \rho, m) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}})$ such that

$$(V_n, a_n^{-1}R_n, \rho_n, b_n^{-1}m_n) \rightarrow (F, R, \rho, m) \quad \text{in } \mathfrak{M}_\bullet(\tau_{\mathcal{M}}).$$

- (iii) It holds that

$$\lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} a_n^{-1}R_n(\rho_n, B_{R_n}(\rho_n, a_n r)^c) = \infty.$$

- (iv) For each $r > 0$, there exist constants $\alpha_r, \beta_r, C_r > 0$ such that, for all sufficiently large $n \geq 1$,

$$\inf_{x \in B_{R_n}(\rho_n, a_n r)} b_n^{-1}m_n(B_{R_n}(x, a_n s)) \geq C_r s^{\alpha_r}, \quad \forall s \in \left(\frac{1}{(\log b_n)(\log \log b_n)^{(1+\beta_r)}}, 1 \right).$$

(NB. By condition (i), for all sufficiently large n , $b_n > e$ and thus $\log \log b_n > 0$.)

Under Assumption 9.9, by Theorem 9.6, we have $(F, R, \rho, m) \in \mathfrak{R}_\bullet$. Let X^1 be the process associated with (R, m) and X^2 be an independent copy of X^1 . Write $\hat{X} = (X^1, X^2)$ for their product process.

Theorem 9.10. *Suppose that Assumption 9.9 is satisfied. Then the measure m^{diag} belongs to the local Kato class of $\hat{X}$. Moreover, if we write Π for the collision measure of X^1 and X^2 associated with m , then*

$$(V_n, a_n^{-1}R_n, \rho_n, b_n^{-1}m_n, \mathcal{L}_{(\hat{X}_n, \Pi_n)}) \rightarrow (F, R, \rho, m, \mathcal{L}_{(\hat{X}, \Pi)})$$

in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau_{\text{CM}})$.

In order to prove the above theorem, we define, for each $\beta > 0$, a bijective function $\text{trf}_\beta: (0, 1/e) \rightarrow (0, \infty)$ by

$$\text{trf}_\beta(t) := \frac{1}{\log(1/t) (\log \log(1/t))^{1+\beta}}.$$

We denote by trf_β^{-1} its inverse. The properties of trf_β used in the proof below are collected in Appendix A.3.

Proof. It is enough to check that Assumption 9.5(iii) is satisfied. Let p_n denote the heat kernel of X_n^1 with respect to $b_n^{-1}m_n$. Fix $r > 0$ for the remainder of this proof. By condition (i), we have $b_n > e$ for all sufficiently large n . Henceforth, we restrict our attention to such large values of n , since the subsequent arguments concern the limit as $n \rightarrow \infty$. Set

$$\delta_n^{(r)} := \frac{1}{(\log b_n)(\log \log b_n)^{(1+\beta_r)}}.$$

For each $n \geq 1$, define

$$v_n^{(r)}(s) := \begin{cases} C_r s^{\alpha_r}, & s \in (\delta_n^{(r)}, 1), \\ b_n^{-1}, & s \in (0, \delta_n^{(r)}). \end{cases} \quad (9.4)$$

Since m_n is the counting measure, Assumption 9.9(iv) yields

$$\inf_{x \in B_{R_n}(\rho_n, a_n r)} b_n^{-1} m_n(B_{R_n}(x, a_n s)) \geq v_n^{(r)}(s), \quad \forall s \in (0, 1).$$

Since $\text{trf}_{\beta_r/2}$ is increasing, we can find $t_0^{(r)} \in (0, 1/e)$ such that $\text{trf}_{\beta_r/2}(t) < 1$ for all $t \in (0, t_0)$. By Lemma 8.6, for any $t \in (0, t_0^{(r)})$ and $x \in B_{R_n}(\rho_n, a_n r)$,

$$p_n(t, x, x) \leq \frac{2 \text{trf}_{\beta_r/2}(t)}{t} + \frac{\sqrt{2}}{v_n^{(r)}(\text{trf}_{\beta_r/2}(t))}.$$

The Chapman–Kolmogorov equation (4.2) and the Cauchy–Schwarz inequality yield

$$p_n(t, x, y) \leq p_n(t, x, x)^{1/2} p_n(t, y, y)^{1/2}, \quad \forall x, y \in V_n \quad (9.5)$$

(see [65, Lemma 4.4(i)]). Thus, for any $x, y \in B_{R_n}(\rho_n, a_n r)$ and $t \in (0, t_0^{(r)})$,

$$p_n(t, x, y) \leq \frac{2 \text{trf}_{\beta_r/2}(t)}{t} + \frac{\sqrt{2}}{v_n^{(r)}(\text{trf}_{\beta_r/2}(t))}. \quad (9.6)$$

By the Chapman–Kolmogorov equation again, for any $x_1, x_2 \in B_{R_n}(\rho_n, a_n r)$ and $\delta \in (0, t_0^{(r)})$,

$$\begin{aligned} & \int_0^\delta \int_{B_{R_n}(\rho_n, a_n r)} p_n(t, x_1, y) p_n(t, x_2, y) b_n^{-1} m_n(dy) dt \\ & \leq \int_0^\delta p_n(2t, x_1, x_2) dt \\ & \leq \int_0^\delta \left(\frac{\text{trf}_{\beta_r/2}(2t)}{t} + \frac{\sqrt{2}}{v_n^{(r)}(\text{trf}_{\beta_r/2}(2t))} \right) dt. \end{aligned} \quad (9.7)$$

By the definition of $v_n^{(r)}$, the last expression is bounded by

$$\begin{aligned} & \int_0^\delta \frac{\text{trf}_{\beta_r/2}(2t)}{t} dt + \int_0^\delta \frac{\sqrt{2}}{C_r \text{trf}_{\beta_r/2}(2t)^{\alpha_r}} dt + \int_0^{\text{trf}_{\beta_r/2}^{-1}(\delta_n^{(r)})/2} \sqrt{2} b_n dt \\ & \leq \int_0^\delta \frac{\text{trf}_{\beta_r/2}(2t)}{t} dt + \int_0^\delta \frac{\sqrt{2}}{C_r \text{trf}_{\beta_r/2}(2t)^{\alpha_r}} dt + \frac{\sqrt{2}}{2} b_n \text{trf}_{\beta_r/2}^{-1}(\delta_n^{(r)}). \end{aligned} \quad (9.8)$$

By Lemma A.4(i) and (ii), the first and second terms in the last expression converge to 0 as $\delta \rightarrow 0$. Thus, it remains to show that the third term vanishes as $n \rightarrow \infty$. Using Lemma A.4(iii), we obtain

$$\begin{aligned} \log(b_n \operatorname{trf}_{\beta_r/2}^{-1}(\delta_n^{(r)})) &= \log b_n - \frac{1 + o(1)}{\delta_n^{(r)} (\log(\delta_n^{(r)})^{-1})^{1+\beta_r/2}} \\ &= \log b_n \left(1 - \frac{(1 + o(1))(\log \log b_n)^{1+\beta_r}}{(\log(\log b_n)(\log \log b_n)^{1+\beta_r})^{1+\beta_r/2}} \right), \end{aligned}$$

which tends to $-\infty$ as $n \rightarrow \infty$ by Assumption 9.9(i). This implies that $b_n \operatorname{trf}_{\beta_r/2}^{-1}(\delta_n^{(r)}) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, we conclude that Assumption 9.5(iii) is satisfied. $\square$

Example 9.11. In [28], it is shown that, under Assumption 9.9(ii) and the uniform volume doubling (UVD) condition (see [28, Definition 1.1]), the associated processes and their local times converge. The UVD condition implies the non-explosion condition, Assumption 9.9(iii), see [24, Remark 1.3(b)]. Moreover, it can be readily verified that the UVD condition implies Assumption 9.9(iv). Hence, Theorem 9.10 applies to all the examples treated in [28]. The following is a list of such applicable examples.

- (a) A sequence of scaled one-dimensional lattices converging to $\mathbb{R}$.
- (b) A sequence of graphs approximating a uniform finitely ramified fractal, such as the two-dimensional Sierpiński gasket (see [28, Section 6.2] for details).
- (c) A (sub)sequence of graphical approximations of the two-dimensional Sierpiński carpet (see [28, Example 4.5(iv)] for details).

In particular, by applying Theorem 9.10 to (a), we obtain the scaling limit of uniform collision measures of two i.i.d. VSRWs on $\mathbb{Z}$, which provides a continuous analogue of Nguyen's result [59] (recall that Nguyen considered discrete-time random walks, see Remarks 7.4 and 7.5).

We next state a version of Theorem 9.10 for random electrical networks. We use the same notation as before, but now assume that the recurrent electrical network (V_n, E_n, μ_n) is random, so that (V_n, R_n, ρ_n, m_n) is a random element of $\mathfrak{R}_\bullet$. We also assume that the scaling factors a_n and b_n are random variables defined on the same probability space as (V_n, E_n, μ_n) . (They may be taken to be deterministic.) We write $\mathbf{P}_n$ for the underlying complete probability measure for the random electrical network (V_n, E_n, μ_n) .

Assumption 9.12.

- (i) As $n \rightarrow \infty$, $b_n \rightarrow \infty$ in probability, i.e.,

$$\lim_{\lambda \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbf{P}_n(b_n > \lambda) = 1.$$

- (ii) There exists a random element $G = (F, R, \rho, m)$ of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}})$ with the underlying complete probability measure $\mathbf{P}$ such that

$$(V_n, a_n^{-1} R_n, \rho_n, b_n^{-1} m_n) \xrightarrow{d} (F, R, \rho, m) \quad \text{in } \mathfrak{M}_\bullet(\tau_{\mathcal{M}}).$$

- (iii) For all $\lambda > 0$,

$$\lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbf{P}_n(a_n^{-1} R_n(\rho_n, B_{R_n}(\rho_n, a_n r)^c) > \lambda) = 1.$$

- (iv) For each $r > 0$ and $\varepsilon > 0$, there exist deterministic constants $\alpha_{r,\varepsilon}, \beta_{r,\varepsilon}, C_{r,\varepsilon} > 0$ such that, for all sufficiently large n , with probability at least $1 - \varepsilon$, we have $b_n > e$ and

$$\inf_{x \in B_{R_n}(\rho_n, a_n r)} b_n^{-1} m_n(B_{R_n}(x, a_n s)) \geq C_{r,\varepsilon} s^{\alpha_{r,\varepsilon}}, \quad \forall s \in \left(\frac{1}{(\log b_n)(\log \log b_n)^{1+\beta_{r,\varepsilon}}}, 1 \right).$$

(NB. By condition (i), for all sufficiently large n , we have $b_n > e$ with probability at least $1 - \varepsilon$.)

Theorem 9.13. *Suppose that Assumption 9.12 is satisfied. Then the measure m^{diag} belongs to the local Kato class of $\hat{X}$, $\mathbf{P}$ -a.s. Moreover, if we write Π for the collision measure of X^1 and X^2 associated with m , then*

$$\left(V_n, a_n^{-1} R_n, \rho_n, b_n^{-1} m_n, \mathcal{L}_{(\hat{X}_n, \Pi_n)} \right) \xrightarrow{d} \left(F, R, \rho, m, \mathcal{L}_{(\hat{X}, \Pi)} \right)$$

in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}} \times \tau_{\text{CM}})$.

Proof. It is sufficient to verify Assumption 9.7(iii). The proof proceeds in the same way as that of Theorem 9.10. Fix $r > 0$ for the remainder of this proof. For each $n \geq 1$ and $\varepsilon > 0$, define

$$\delta_{n,\varepsilon}^{(r)} := \begin{cases} \frac{1}{(\log b_n)(\log \log b_n)^{(1+\beta_{r,\varepsilon})}}, & \text{if } b_n > e, \\ 0, & \text{if } b_n \leq e. \end{cases}$$

and

$$v_{n,\varepsilon}^{(r)}(s) := \begin{cases} C_{r,\varepsilon} s^{\alpha_{r,\varepsilon}}, & s \in (\delta_{n,\varepsilon}^{(r)}, 1), \\ b_n^{-1}, & s \in (0, \delta_{n,\varepsilon}^{(r)}). \end{cases}$$

If we write $E_{n,\varepsilon}$ for the event that $b_n > e$ and

$$\inf_{x \in B_{R_n}(\rho_n, a_n r)} b_n^{-1} m_n(B_{R_n}(x, a_n s)) \geq v_{n,\varepsilon}^{(r)}(s), \quad \forall s \in (0, 1),$$

then Assumption 9.12(iv) implies that

$$\liminf_{n \rightarrow \infty} \mathbf{P}_n(E_{n,\varepsilon}) \geq 1 - \varepsilon. \quad (9.9)$$

Write $B_n(r) := B_{R_n}(\rho_n, a_n r)$. Using (9.7) and (9.8), we obtain

$$\begin{aligned} & \mathbf{E}_n \left[\left(\sup_{x_1, x_2 \in B_n(r/2)} \int_0^\delta \int_{B_n(r/2)} p_n(t, x_1, y) p_n(t, x_2, y) b_n^{-1} m_n(dy) dt \right) \wedge 1 \right] \\ & \leq \mathbf{P}_n(E_{n,\varepsilon}^c) \\ & \quad + \mathbf{E}_n \left[\left(\int_0^\delta \left(\frac{\text{trf}_{\beta_{r/2}}(2t)}{t} + \frac{\sqrt{2}}{C_{r,\varepsilon} \text{trf}_{\beta_{r,\varepsilon}/2}(2t)^{\alpha_{r,\varepsilon}}} \right) dt + \frac{\sqrt{2}}{2} b_n \text{trf}_{\beta_{r,\varepsilon}/2}^{-1}(\delta_{n,\varepsilon}^{(r)}) \right) \wedge 1 \right] \quad (9.10) \end{aligned}$$

Following the argument in the proof of Theorem 9.10, we obtain

$$\begin{aligned} & \int_0^\delta \left(\frac{\text{trf}_{\beta_{r/2}}(2t)}{t} + \frac{\sqrt{2}}{C_{r,\varepsilon} \text{trf}_{\beta_{r,\varepsilon}/2}(2t)^{\alpha_{r,\varepsilon}}} \right) dt \xrightarrow{\delta \rightarrow 0} 0, \\ & b_n \text{trf}_{\beta_{r,\varepsilon}/2}^{-1}(2\delta_{n,\varepsilon}^{(r)}) \xrightarrow[n \rightarrow \infty]{\mathbf{P}} 0. \end{aligned}$$

Combining these with (9.9) and (9.10), we obtain

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbf{E}_n \left[\left(\sup_{x_1, x_2 \in B_n(r/2)} \int_0^\delta \int_{B_n(r/2)} p_n(t, x_1, y) p_n(t, x_2, y) b_n^{-1} m_n(dy) dt \right) \wedge 1 \right] \leq \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we conclude that Assumption 9.7(iii) holds. $\square$

Example 9.14. In [61], it is shown that the volume condition (iv) of Assumption 9.12 is satisfied for many random graph models. In particular, all the examples considered in [61, Section 8] satisfy Assumption 9.12, and hence Theorem 9.13 applies to them, see the list below.

- (a) A random recursive Sierpiński gasket (cf. [38]).
- (b) Critical Galton–Watson trees (cf. [4, 6, 29]).
- (c) Uniform spanning trees on $\mathbb{Z}^d$ with $d = 2, 3$, and on high-dimensional tori (cf. [8, 9, 11]).
- (d) The critical Erdős–Rényi random graph (cf. [2]).
- (e) The critical configuration model (cf. [16]).

9.4 Scaling limits of collision measures of CSRWs

In this subsection, we provide versions of Theorems 9.10 and 9.13 for CSRWs on electrical networks. In this case, verifying the assumptions for convergence of collision measures is more difficult than in the VSRW case, which has been discussed in Section 9.3 above. This is because the canonical weighting measure (see Remark 7.9) of i.i.d. CSRWs does not coincide with the invariant measure of the walk. The main results of this subsection are Theorems 9.16 and 9.20 below. In the following subsubsection, Section 9.4.1, we apply the results to critical Galton–Watson trees conditioned on their size.

We clarify the setting for the main results. For each $n \geq 1$, let (V_n, E_n, μ_n) be a recurrent electrical network. Write R_n for the associated resistance metric. We identify μ_n with the associated conductance measure on V_n , that is,

$$\mu_n(A) := \sum_{x \in A} \mu_n(x), \quad A \subseteq V_n.$$

More generally, for each $q > 0$, we define the q -power conductance measure μ_n^q on V_n by

$$\mu_n^q(A) := \sum_{x \in A} \mu_n(x)^q, \quad A \subseteq V_n.$$

Let $a_n, b_n > 0$ be constants, serving as scaling factors for the metric and measure, respectively. Let X_n^1 be the process associated with $(a_n^{-1}R_n, b_n^{-1}\mu_n)$ and X_n^2 be an independent copy of X_n^1 . In other words, X_n^1 and X_n^2 are i.i.d. CSRWs on the electrical network, rescaled in time by the factor a_nb_n . We write $\hat{X}_n = (X_n^1, X_n^2)$ for their product process. Let Π_n be the collision measure of X_n^1 and X_n^2 associated with $b_n^{-1}\mu_n^2$, i.e.,

$$\Pi_n(dx dt) = b_n \sum_{y \in V_n} \mathbf{1}_{\{X_n^1(t)=X_n^2(t)=y\}} \delta_y(dx) dt.$$

Below, we state the assumption for the convergence of Π_n , which requires stronger volume conditions than that of the VSRW case, Assumption 9.9. In particular, we assume the precompactness of the $(1+q)$ -power measures for suitable $q > 1$.

Assumption 9.15.

- (i) There exists an element $G = (F, R, \rho, \mu, \nu) \in \mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2})$ such that

$$(V_n, a_n^{-1}R_n, \rho_n, b_n^{-1}\mu_n, b_n^{-1}\mu_n^2) \rightarrow (F, R, \rho, \mu, \nu) \quad \text{in} \quad \mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2}).$$

- (ii) It holds that

$$\lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} a_n^{-1}R_n(\rho_n, B_{R_n}(\rho_n, a_nr)^c) = \infty.$$

- (iii) For each $r > 0$, there exist constants $\alpha_r, C_r > 0$ such that

$$\inf_{x \in B_{R_n}(\rho_n, a_nr)} b_n^{-1}\mu_n(B_{R_n}(x, a_ns)) \geq C_r s^{\alpha_r}, \quad \forall s \in (0, 1)$$

for each $n \geq 1$, and, for some $q_r \in (1, \infty)$ with $q_r > \alpha_r$,

$$\limsup_{n \rightarrow \infty} b_n^{-1}\mu_n^{1+q_r}(B_{R_n}(\rho_n, a_nr)) < \infty. \quad (9.11)$$

Under Assumption 9.15, by Theorem 9.6, we have $(F, R, \rho, m) \in \mathfrak{R}_\bullet$. Let X^1 be the process associated with (R, m) and X^2 be an independent copy of X^1 . Write $\hat{X} = (X^1, X^2)$ for their product process.

Theorem 9.16. *Suppose that Assumption 9.15 is satisfied. Then the measure ν^{diag} belongs to the local Kato class of $\hat{X}$. Moreover, if we write Π for the collision measure of X^1 and X^2 associated with ν , then*

$$\left(V_n, a_n^{-1} R_n, \rho_n, b_n^{-1} \mu_n, b_n^{-1} \mu_n^2, \mathcal{L}_{(\hat{X}_n, \Pi_n)} \right) \rightarrow \left(F, R, \rho, \mu, \nu, \mathcal{L}_{(\hat{X}, \Pi)} \right)$$

in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2} \times \tau_{\text{CM}})$.

Proof. It is sufficient to verify that Assumption 9.5(iii) holds. Denote by p_n the heat kernel of the scaled walk X_n^1 with respect to $b_n^{-1} \mu_n$. Fix $r > 0$ for the remainder of this proof, and set $B_n(r) := B_{R_n}(\rho_n, a_n r)$. Since $q_r > \alpha_r$ by Assumption 9.19(iii), we can find $\gamma_r \in (0, 1)$ such that

$$\frac{1}{q_r + 1} < \gamma_r < \frac{q_r}{\alpha_r(q_r + 1)}. \quad (9.12)$$

Following the argument that yields (9.6), with $v_n^{(r)}(s)$ in (9.4) replaced by $C_r s^{\gamma_r}$, we obtain

$$\sup_{x, y \in B_n(r)} p_n(t, x, y) \leq \frac{2}{t^{1-\gamma_r}} + \frac{\sqrt{2}}{C_r t^{\alpha_r \gamma_r}}, \quad \forall t \in (0, 1). \quad (9.13)$$

Let $p_r > 1$ be such that $p_r^{-1} + q_r^{-1} = 1$. By Hölder's inequality, for any $x_1, x_2 \in B_n(r)$,

$$\begin{aligned} & \int_{B_n(r)} p_n(t, x_1, y) p_n(t, x_2, y) b_n^{-1} \mu_n^2(dy) \\ &= \int_{B_n(r)} p_n(t, x_1, y) p_n(t, x_2, y) \mu_n(y) b_n^{-1} \mu_n(dy) \\ &\leq \left(\int_{B_n(r)} p_n(t, x_1, y) p_n(t, x_2, y) b_n^{-1} \mu_n(dy) \right)^{1/p_r} \\ &\quad \left(\int_{B_n(r)} p_n(t, x_1, y) p_n(t, x_2, y) \mu_n(y)^{q_r} b_n^{-1} \mu_n(dy) \right)^{1/q_r} \\ &\leq p_n(2t, x_1, x_2)^{1/p_r} \sup_{y \in B_n(r)} p_n(t, y, y)^{2/q_r} \{b_n^{-1} \mu_n^{1+q_r}(B_{R_n}(\rho_n, a_n r))\}^{1/q_r}, \end{aligned} \quad (9.14)$$

where we have used the Chapman–Kolmogorov equation (4.2) together with (9.5). Using (9.13) and the monotonicity of its right-hand side in t , we deduce that

$$\begin{aligned} p_n(2t, x_1, y)^{1/p_r} \sup_{y \in B_n(r)} p_n(t, y, y)^{2/q_r} &\leq \left(\frac{2}{t^{1-\gamma_r}} + \frac{\sqrt{2}}{C_r t^{\alpha_r \gamma_r}} \right)^{p_r^{-1} + 2q_r^{-1}} \\ &= \left(\frac{2}{t^{1-\gamma_r}} + \frac{\sqrt{2}}{C_r t^{\alpha_r \gamma_r}} \right)^{1+q_r^{-1}}. \end{aligned}$$

Combining this with (9.14), we obtain

$$\begin{aligned} & \sup_{x_1, x_2 \in B_n(r)} \int_{B_n(r)} p_n(t, x_1, y) p_n(t, x_2, y) b_n^{-1} \mu_n^2(dy) \\ &\leq \{b_n^{-1} \mu_n^{1+q_r}(B_{R_n}(\rho_n, a_n r))\}^{1/q_r} \left(\frac{2}{t^{1-\gamma_r}} + \frac{\sqrt{2}}{C_r t^{\alpha_r \gamma_r}} \right)^{1+q_r^{-1}}. \end{aligned}$$

Hence, by (9.11), it suffices to show that

$$\lim_{\delta \rightarrow 0} \int_0^\delta \left(\frac{2}{t^{1-\gamma_r}} + \frac{\sqrt{2}}{C_r t^{\alpha_r \gamma_r}} \right)^{1+q_r^{-1}} dt = 0. \quad (9.15)$$

Let $c_1 > 0$ be a constant such that

$$(a+b)^{1+q_r^{-1}} \leq c_1 (a^{1+q_r^{-1}} + b^{1+q_r^{-1}}), \quad \forall a, b > 0.$$

Then

$$\left(\frac{2}{t^{1-\gamma_r}} + \frac{\sqrt{2}}{C_r t^{\alpha_r \gamma_r}} \right)^{1+q_r^{-1}} \leq \frac{c_1 2^{1+q_r^{-1}}}{t^{(1-\gamma_r)(1+q_r^{-1})}} + \frac{c_1 2^{\frac{1}{2} + \frac{1}{2q_r}}}{C_r^{1+q_r^{-1}} t^{\alpha_r \gamma_r (1+q_r^{-1})}}.$$

By (9.12), we have

$$(1-\gamma_r)(1+q_r^{-1}) < 1, \quad \alpha_r \gamma_r (1+q_r^{-1}) < 1.$$

Hence (9.15) follows, which completes the proof. $\square$

When the conductance weights on vertices are uniformly bounded away from 0 and ∞ , the proof of Theorem 9.10 can be applied, and hence the same volume condition as Assumption 9.9(iv) is sufficient, as shown below.

Proposition 9.17. *In addition to Assumption 9.15(i) and (ii), assume that the following conditions hold.*

(iv) *For each $r > 0$,*

$$\liminf_{n \rightarrow \infty} \inf_{x \in B_{R_n}(\rho_n, a_n r)} \mu_n(x) > 0, \quad \limsup_{n \rightarrow \infty} \sup_{x \in B_{R_n}(\rho_n, a_n r)} \mu_n(x) < \infty. \quad (9.16)$$

(v) *As $n \rightarrow \infty$, $b_n \rightarrow \infty$.*

(vi) *For each $r > 0$, there exist constants $\alpha_r, \beta_r, C_r > 0$ such that, for each $n \geq 1$,*

$$\inf_{x \in B_{R_n}(\rho_n, a_n r)} b_n^{-1} \mu_n(B_{R_n}(x, a_n s)) \geq C_r s^{\alpha_r}, \quad \forall s \in \left(\frac{1}{(\log b_n)(\log \log b_n)^{(1+\beta_r)}}, 1 \right).$$

Then the conclusion of Theorem 9.16 holds.

Proof. We use the same notation as in the proof of Theorem 9.16. Fix $r > 0$. By (9.16), we can find a constant $M_r \in (0, \infty)$ satisfying

$$M_r > \limsup_{n \rightarrow \infty} \sup_{x \in B_{R_n}(\rho_n, a_n r)} \mu_n(x).$$

Using the Chapman–Kolmogorov equation, for all sufficiently large n , we obtain that, for any $x_1, x_2 \in B_n(r)$,

$$\begin{aligned} \int_{B_n(r)} p_n(t, x_1, y) p_n(t, x_2, y) b_n^{-1} \mu_n^2(dy) &\leq M_r \int_{B_n(r)} p_n(t, x_1, y) p_n(t, x_2, y) b_n^{-1} \mu_n(dy) \\ &\leq M_r p_n(2t, x_1, x_2). \end{aligned}$$

Thus, it suffices to show that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{x_1, x_2 \in B_n(r)} \int_0^\delta p_n(2t, x_1, x_2) dt = 0. \quad (9.17)$$

Henceforth, we consider only sufficiently large n for which $b_n > e$. Set

$$\delta_n^{(r)} := \frac{1}{(\log b_n)(\log \log b_n)^{(1+\beta_r)}}.$$

By (9.16), we can find a constant $m_r \in (0, \infty)$ satisfying

$$m_r < \liminf_{n \rightarrow \infty} \inf_{x \in B_{R_n}(\rho_n, a_n r)} \mu_n(x).$$

For each n , define

$$v_n^{(r)}(s) := \begin{cases} C_r s^{\alpha_r}, & s \in (\delta_n^{(r)}, 1), \\ m_r b_n^{-1}, & s \in (0, \delta_n^{(r)}). \end{cases}$$

Condition (vi) yields that, for all sufficiently large n ,

$$\inf_{x \in B_{R_n}(\rho_n, a_n r)} b_n^{-1} m_n(B_{R_n}(x, a_n s)) \geq v_n^{(r)}(s), \quad \forall s \in (0, 1).$$

Therefore, by the same argument as in the proof of Theorem 9.10, we obtain (9.17). $\square$

Example 9.18. The examples (a), (b), and (c) listed in Example 9.11 all have uniformly bounded degrees. Hence, the preceding proposition applies. In particular, Theorem 9.16 then yields the scaling limits of the (uniform) collision measures of two i.i.d. CSRWs on these models.

We next provide a version of the above theorem for random electrical networks. Similarly to Theorem 9.10, we now assume that the recurrent electrical network (V_n, E_n, μ_n) is random, so that $(V_n, R_n, \rho_n, \mu_n)$ is a random element of $\mathfrak{R}_\bullet$. We also assume that the scaling factors a_n and b_n are random variables defined on the same probability space as (V_n, E_n, μ_n) . (They may be taken to be deterministic.) We write $\mathbf{P}_n$ for the underlying complete probability measure for the random electrical network (V_n, E_n, μ_n) .

Assumption 9.19.

- (i) *There exists a random element $G = (F, R, \rho, \mu, \nu)$ of $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2})$ with the underlying complete probability measure $\mathbf{P}$ such that*

$$(V_n, a_n^{-1} R_n, \rho_n, b_n^{-1} \mu_n, b_n^{-1} \mu_n^2) \xrightarrow{d} (F, R, \rho, \mu, \nu) \quad \text{in } \mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2}).$$

- (ii) *For all $\lambda > 0$,*

$$\lim_{r \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbf{P}_n(a_n^{-1} R_n(\rho_n, B_{R_n}(\rho_n, a_n r)^c) > \lambda) = 1.$$

- (iii) *For each $r > 0$ and $\varepsilon > 0$, there exist deterministic constants $\alpha_{r,\varepsilon}, \beta_{r,\varepsilon}, C_{r,\varepsilon} > 0$ such that, for all sufficiently large n , with probability at least $1 - \varepsilon$,*

$$\inf_{x \in B_{R_n}(\rho_n, a_n r)} b_n^{-1} \mu_n(B_{R_n}(x, a_n s)) \geq C_{r,\varepsilon} s^{\alpha_{r,\varepsilon}}, \quad \forall s \in (0, 1),$$

for some $q_{r,\varepsilon} > 1 \vee \alpha_{r,\varepsilon}$,

$$\limsup_{\lambda \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}_n(b_n^{-1} \mu_n^{1+q_{r,\varepsilon}}(B_{R_n}(\rho_n, a_n r)) > \lambda) = 0.$$

Theorem 9.20. *Suppose that Assumption 9.12 is satisfied. Then the measure ν^{diag} belongs to the local Kato class of $\hat{X}$, $\mathbf{P}$ -a.s. Moreover, if we write Π for the collision measure of X^1 and X^2 associated with ν , then*

$$(V_n, a_n^{-1} R_n, \rho_n, b_n^{-1} \mu_n, b_n^{-1} \mu_n^2, \mathcal{L}_{(\hat{X}_n, \Pi_n)}) \xrightarrow{d} (F, R, \rho, \mu, \nu, \mathcal{L}_{(\hat{X}, \Pi)})$$

in $\mathfrak{M}_\bullet(\tau_{\mathcal{M}}^{\otimes 2} \times \tau_{\text{CM}})$.

Proof. This is verified by following the arguments in the proof of Theorems 9.13 and 9.16, and therefore we omit the proof. $\square$

The next proposition is a counterpart of Proposition 9.17 in the setting of random electrical networks. As its proof follows the same idea as in Theorem 9.13, we state the result without proof.

Proposition 9.21. *In addition to Assumption 9.15(i) and (ii), assume that the following conditions hold.*

(vii) *For each $r > 0$,*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbf{P}_n \left(\inf_{x \in B_{R_n}(\rho_n, a_n r)} \mu_n(x) > \varepsilon \right) &= 1, \\ \lim_{\lambda \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}_n \left(\sup_{x \in B_{R_n}(\rho_n, a_n r)} \mu_n(x) > \lambda \right) &= 0. \end{aligned}$$

(viii) *As $n \rightarrow \infty$, $b_n \rightarrow \infty$ in probability.*

(ix) *For each $r > 0$ and $\varepsilon > 0$, there exist deterministic constants $\alpha_{r,\varepsilon}, \beta_{r,\varepsilon}, C_{r,\varepsilon} > 0$ such that, for all sufficiently large n , with probability at least $1 - \varepsilon$, we have $b_n > e$ and*

$$\inf_{x \in B_{R_n}(\rho_n, a_n r)} b_n^{-1} \mu_n(B_{R_n}(x, a_n s)) \geq C_{r,\varepsilon} s^{\alpha_{r,\varepsilon}}, \quad \forall s \in \left(\frac{1}{(\log b_n)(\log \log b_n)^{1+\beta_{r,\varepsilon}}}, 1 \right).$$

Then the conclusion of Theorem 9.20 holds.

9.4.1 Critical Galton–Watson trees

In this subsection, as an application of Theorem 9.20, we establish the convergence of the uniform collision measures of i.i.d. CSRWs on critical Galton–Watson trees conditioned on their size, see Theorem 9.24 below. For definitions and further details concerning trees, we follow [53].

Fix a probability measure $\boldsymbol{\pi} = (\pi_k)_{k \geq 0}$ on $\mathbb{Z}_+$ such that $\sum_{k \geq 1} k\pi_k = 1$, $p_1 < 1$, and whose variance is σ^2 . We do not assume that $\boldsymbol{\pi}$ is aperiodic. Let d denote the period of $\boldsymbol{\pi}$, that is, the greatest common divisor of the set $\{k \geq 1 \mid \pi_k > 0\}$. Then $d = 1$ corresponds to the aperiodic case, and when $d \geq 2$, all the statements below should be understood along the subsequence $n \in d\mathbb{N}$. Let T^{GW} be the Galton–Watson tree with offspring distribution $\boldsymbol{\pi}$ (see [53, Section 3] for its definition). We write T_n for a random plane tree having the same distribution as T^{GW} conditioned on having exactly n vertices, which is well defined for all sufficiently large n . We denote by $\mathbf{P}_n$ the underlying probability measure for T_n .

We begin by fixing some notation for T_n . Let ρ_n denote the root of T_n , and let d_n denote the graph metric on T_n . For each $x \in T_n$, we denote by $\deg(x)$ its degree, that is, the number of vertices adjacent to x . We then define the degree measure μ_n on T_n by

$$\mu_n(A) := \sum_{x \in A} \deg(x), \quad A \subseteq T_n.$$

More generally, for each $q > 1$, we define the q -th power degree measure μ_n^q on T_n by

$$\mu_n^q(A) := \sum_{x \in A} \deg(x)^q, \quad A \subseteq T_n.$$

The scaling limit of T_n is the continuum random tree. To recall it, we define $\mathbf{e} = (\mathbf{e}(t))_{t \in [0,1]}$ to be the normalized Brownian excursion, built on a probability space with probability measure $\mathbf{P}$. We write $(T_{\mathbf{e}}, d_{\mathbf{e}})$ for the real tree coded by $\mathbf{e}$ and $p_{\mathbf{e}} : [0, 1] \rightarrow T_{\mathbf{e}}$ for the canonical projection

(see [53, Section 2]). The canonical Radon measure on $T_{\mathbf{e}}$ is given by $\mu_{\mathbf{e}} := \text{Leb} \circ (p_{\mathbf{e}})^{-1}$, where Leb stands for the one-dimensional Lebesgue measure. We define the root $\rho_{\mathbf{e}}$ by setting $\rho_{\mathbf{e}} := p_{\mathbf{e}}(0)$. It then holds that

$$\left(T_n, \frac{\sigma}{2\sqrt{n}}d_n, \rho_n, \frac{1}{n}\mu_n\right) \rightarrow (T_{\mathbf{e}}, d_{\mathbf{e}}, \rho_{\mathbf{e}}, \mu_{\mathbf{e}}) \quad \text{in } \mathfrak{K}_{\bullet}(\tau_{\mathcal{M}_{\text{fin}}}),$$

where we recall that $\mathfrak{K}_{\bullet}(\cdot)$ was introduced at the end of Section 3.1. This is an immediate consequence of [4, Theorem 23] and [1, Proposition 3.3]. See also [61, Corollary 8.7] for this extension to offspring distribution with infinite variance.

In order to apply Theorem 9.13, we need to prove the convergence of the squared degree measures μ_n^2 . As a first step, we establish their tightness. The following proposition in fact provides a stronger statement.

Proposition 9.22. *Fix $q > 1$. Assume that the offspring distribution has a finite q -th moment, i.e., $\sum_{k \geq 1} k^q \pi_k < \infty$. Then, for any $k_0 \geq 0$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E}_n \left[\sum_{x \in T_n} \deg(x)^q \mathbf{1}_{\{\deg(x) \geq k_0\}} \right] = \sum_{k=k_0}^{\infty} (k+1)^q \pi_k.$$

Proof. Let $v_0 = \rho_T, v_1, \dots, v_{n-1}$ be the vertices of T_n in lexicographical order. Write $\xi_{n,i}$ for the number of children of v_i . Let $(Y_i)_{i=1}^{\infty}$ be i.i.d. random variables with distribution π , with underlying probability measure P . By [41, Lemma 17.1], the multiset $\{\xi_{n,i}\}_{i=0}^{n-1}$ has the same distribution as $\{Y_i\}_{i=1}^n$ conditioned on $\sum_{i=1}^n Y_i = n-1$. Thus, writing $S_n = \sum_{i=1}^n Y_i$, we have

$$\begin{aligned} \frac{1}{n} \mathbf{E}_n \left[\sum_{i=0}^{n-1} (\xi_{n,i} + 1)^q \mathbf{1}_{\{\xi_{n,i} + 1 \geq k_0\}} \right] &= \frac{1}{n} E \left[\sum_{i=1}^n (Y_i + 1)^q \mathbf{1}_{\{Y_i + 1 \geq k_0\}} \middle| S_n = n-1 \right] \\ &= E[(Y_1 + 1)^q \mathbf{1}_{\{Y_1 + 1 \geq k_0\}} \mid S_n = n-1]. \end{aligned}$$

The last conditional expectation equals

$$\frac{E[(Y_1 + 1)^q \mathbf{1}_{\{Y_1 + 1 \geq k_0\}} \mathbf{1}_{\{S_n = n-1\}}]}{P(S_n = n-1)} = \sum_{k=k_0}^{\infty} (k+1)^q \pi_k \frac{P(S_{n-1} = n-1-k)}{P(S_n = n-1)},$$

where we used the Markov property of $(S_n)_{n \geq 1}$. By the local central limit theorem,

$$\sup_{l \in \mathbb{Z}} \left| \sqrt{2\pi\sigma^2 n} P(S_n - n = l) - \exp\left(-\frac{l^2}{2\sigma^2 n}\right) \right| \xrightarrow{n \rightarrow \infty} 0$$

(cf. [69, Chapter VII, Theorem 1]). It follows that

$$\begin{aligned} \sup_{n \geq 1} \sup_{k \geq 0} \frac{P(S_{n-1} = n-1-k)}{P(S_n = n-1)} &< \infty, \\ \lim_{n \rightarrow \infty} \frac{P(S_{n-1} = n-1-k)}{P(S_n = n-1)} &= 1, \quad \forall k \geq 0. \end{aligned}$$

Hence, by the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{E}_n \left[\sum_{i=0}^{n-1} (\xi_{n,i} + 1)^q \mathbf{1}_{\{\xi_{n,i} + 1 \geq k_0\}} \right] = \sum_{k=k_0}^{\infty} (k+1)^q \pi_k. \quad (9.18)$$

Except for the root, the degree of a vertex equals its number of children plus one. Therefore,

$$\begin{aligned} & \left| \frac{1}{n} \mathbf{E}_n \left[\sum_{x \in T_n} \deg(x)^q \mathbf{1}_{\{\deg(x) \geq k_0\}} \right] - \frac{1}{n} \mathbf{E}_n \left[\sum_{i=0}^{n-1} (\xi_{n,i} + 1)^q \mathbf{1}_{\{\xi_{n,i} + 1 \geq k_0\}} \right] \right| \\ &= \frac{1}{n} \mathbf{E}_n \left[((\xi_{n,0} + 1)^q - \xi_{n,0}^q) \mathbf{1}_{\{\xi_{n,0} + 1 \geq k_0\}} \right] \\ &\leq \frac{q}{n} \mathbf{E}_n \left[(\xi_{n,0} + 1)^{q-1} \mathbf{1}_{\{\xi_{n,0} + 1 \geq k_0\}} \right], \end{aligned} \quad (9.19)$$

where, for the last inequality, we used the elementary bound $(x+1)^q - x^q \leq q(x+1)^{q-1}$ for all $x \geq 0$. By [41, Lemma 15.7],

$$\mathbf{P}_n(\xi_{n,0} = k) = \frac{n}{n-1} k P(Y_1 = k | S_n = n-1), \quad \forall k \geq 0.$$

Hence,

$$\begin{aligned} \mathbf{E}_n \left[(\xi_{n,0} + 1)^{q-1} \mathbf{1}_{\{\xi_{n,0} + 1 \geq k_0\}} \right] &= \frac{n}{n-1} \sum_{k=k_0}^{\infty} (k+1)^{q-1} k P(Y_1 = k | S_n = n-1) \\ &\leq \frac{n}{n-1} E[(Y_1 + 1)^q \mathbf{1}_{\{Y_1 + 1 \geq k_0\}} | S_n = n-1], \end{aligned}$$

which converges to the same limit as (9.18), as $n \rightarrow \infty$. Combining (9.18) and (9.19) yields the desired result. $\square$

Combining the above proposition with the results of [63, Section 7], we obtain the convergence of the squared degree measures as follows.

Theorem 9.23. *It holds that*

$$\left(T_n, \frac{\sigma^2}{2\sqrt{n}} d_n, \rho_n, \frac{1}{n} \mu_n, \frac{1}{n} \mu_n^2 \right) \rightarrow (T_{\mathbf{e}}, d_{\mathbf{e}}, \rho_{\mathbf{e}}, \mu_{\mathbf{e}}, (\sigma^2 + 4) \mu_{\mathbf{e}}).$$

in $\mathfrak{R}_{\bullet}(\tau_{\mathcal{M}}^{\otimes 2})$.

Proof. Define a Radon measure $\dot{\mu}_n$ on $T_n \times \mathbb{N}$ by setting

$$\dot{\mu}_n(\{x\} \times \{k\}) = \mathbf{1}_{\{\deg(x)=k\}}, \quad x \in T_n, \quad k \in \mathbb{N}.$$

Define a probability measure $\tilde{\pi} = (\tilde{\pi}_k)_{k \geq 1}$ on $\mathbb{N}$ by setting $\tilde{\pi}_k := \pi_{k-1}$ for each $k \geq 1$. By following the proof of [63, Corollary 7.4], we deduce that

$$\left(T_n, \frac{\sigma^2}{2\sqrt{n}} d_n, \rho_n, \frac{1}{n} \mu_n, \frac{1}{n} \dot{\mu}_n \right) \xrightarrow{d} (T_{\mathbf{e}}, d_{\mathbf{e}}, \rho_{\mathbf{e}}, \mu_{\mathbf{e}}, \mu_{\mathbf{e}} \otimes \tilde{\pi}) \quad (9.20)$$

in $\mathfrak{R}_{\bullet}(\tau_{\mathcal{M}_{\text{fin}}} \times \tau_{\mathcal{M}_{\text{fin}}}(\Psi_{\text{id} \times \mathbb{R}_{\geq 0}}))$. For each $M \in \mathbb{N}$, we define a Radon measure $\mu_{n,M}^2$ on T_n by

$$\mu_{n,M}^2(\{x\}) := (\deg(x) \wedge M)^2.$$

For any function f on T_n , we have

$$\int_{T_n} f(x) \mu_{n,M}^2(dx) = \sum_{x \in T_n} f(x) (\deg(x) \wedge M)^2 = \int_{T_n \times \mathbb{N}} f(x) (k \wedge M)^2 \dot{\mu}_n(dx dk).$$

Thus, if we define a Radon measure $\mu_{\mathbf{e},M}$ on $T_{\mathbf{e}}$ by

$$\mu_{\mathbf{e},M}(dx) := \left(\sum_{k=1}^{\infty} (k \wedge M)^2 \tilde{\pi}_k \right) \mu_{\mathbf{e}}(dx),$$

then we deduce from (9.20) that

$$\left(T_n, \frac{\sigma^2}{2\sqrt{n}}d_n, \rho_n, \frac{1}{n}\mu_n, \frac{1}{n}\mu_{n,M}^2\right) \xrightarrow{d} (T_e, d_e, \rho_e, \mu_e, \mu_{e,M}). \quad (9.21)$$

For any subset $A \subseteq T_n$, it holds that

$$\frac{1}{n} |\mu_{n,M}^2(A) - \mu_n^2(A)| \leq \frac{1}{n} \sum_{x \in T_n} \deg(x)^2 \mathbf{1}_{\{\deg(x) > M\}}. \quad (9.22)$$

Since

$$\sum_{k=1}^{\infty} k^2 \tilde{\pi}_k = \sum_{k=0}^{\infty} (k+1)^2 p_k = \sigma^2 + 4,$$

we have that, for any Borel subset A of T_e ,

$$|(\sigma^2 + 4)\mu_e(A) - \mu_{e,M}(A)| \leq \sum_{k > M} k^2 \tilde{\pi}_k. \quad (9.23)$$

Now the result is immediate from Proposition 9.22, (9.21), (9.22), and (9.23). $\square$

We are ready to prove the main result, Theorem 9.24 below. Let Y_n^1 be the CSRW on T_n and Y_n^2 be an independent copy of Y_n^1 . For each $i \in \{1, 2\}$, we define the scaled walk X_n^i by

$$X_n^i(t) := Y_n^i(2\sigma^{-1}n^{3/2}t), \quad t \geq 0.$$

We write $\hat{X}_n = (X_n^1, X_n^2)$ for their product process. Define a collision measure Π_n by

$$\Pi_n(dxdt) := n \sum_{y \in T_n} \mathbf{1}_{\{X_n^1(t)=X_n^2(t)=y\}} \delta_y(dx) dt.$$

By [61, Corollary 8.11], $(T_e, d_e, \rho_e, \mu_e) \in \mathfrak{R}_\bullet$, $\mathbf{P}$ -a.s. Given a realization of $(T_e, d_e, \rho_e, \mu_e)$, we write X^1 for the associated process and X^2 for an independent copy of X^1 . We then set $\hat{X} = (X^1, X^2)$ for their product process.

Theorem 9.24. *Assume that the offspring distribution has a finite q -th moment for some $q > 3$. Then μ_e^{diag} belongs to the local Kato class of $\hat{X}$, $\mathbf{P}$ -a.s. If we write Π for the collision measure of X^1 and X^2 associated with $(\sigma^2 + 4)\mu_e$, then*

$$\left(T_n, \frac{\sigma}{2\sqrt{n}}d_n, \rho_n, \frac{1}{n}\mu_n, \frac{1}{n}\mu_n^2, \mathcal{L}_{(\hat{X}_n, \Pi_n)}\right) \rightarrow \left(T_e, d_e, \rho_e, \mu_e, (\sigma^2 + 4)\mu_e, \mathcal{L}_{(\hat{X}, \Pi)}\right).$$

in $\mathfrak{R}_\bullet(\tau_{\mathcal{M}}^{\otimes 2} \times \tau_{\text{CM}})$.

Proof. By placing conductance 1 on each edge of T_n , we regard T_n as an electrical network. Then the graph metric d_n coincides with the resistance metric on T_n (see [48, Proposition 5.1]). Thus, it is enough to verify that Assumption 9.19 is satisfied.

By Theorem 9.23, Assumption 9.19(i) holds. Moreover, the same theorem implies that the diameters of the metric spaces $(T_n, \frac{\sigma}{2\sqrt{n}}d_n)$ converge in distribution to the diameter of (T_e, d_e) . (This follows from the continuity of the diameter with respect to the Gromov–Hausdorff topology; see [21, Exercise 7.3.14].) From this, one can verify Assumption 9.19(ii). Furthermore, by [61, Proposition 8.10], we deduce that for every $\varepsilon > 0$ there exists a constant $c_{q,\varepsilon} > 0$ such that

$$\liminf_{n \rightarrow \infty} \mathbf{P}_n \left(\inf_{x \in T_n} n^{-1} \mu_n(B_{d_n}(x, 2\sigma^{-1}\sqrt{nr})) \geq c_{q,\varepsilon} r^{q-1}, \quad r \in (0, 1) \right) \geq 1 - \varepsilon.$$

Combining this with Proposition 9.22, we obtain Assumption 9.19(iii). Therefore, the desired result follows from Theorem 9.20. $\square$

Remark 9.25. The above theorem is proved under the finite $(3 + \varepsilon)$ -moment assumption, but the convergence of the collision measures is in fact expected to hold under the weaker assumption that the offspring distribution has finite variance. Indeed, in this case the squared degree measures still converge to $(\sigma^2 + 4)\mu_e$ by Theorem 9.23, so the result should follow once the heat-kernel condition (Assumption 7.15(ii)) is verified. However, verifying this condition appears to require finer heat-kernel estimates beyond the argument based on Hölder's inequality and lower volume bounds used in Theorem 9.20. In the infinite-variance regime, by contrast, the squared degree measures are expected to become singular with respect to the invariant measures, suggesting that collisions would concentrate on vertices of exceptionally large degrees, which would be an intriguing direction for future study.

A Omitted proofs and technical results

A.1 Proof of Proposition 4.6

Recall the setting of Section 4.2. In particular, we consider a standard process X satisfying the DAC condition (Assumption 4.1), with dual process $\check{X}$, reference measure m , and heat kernel p . In this appendix, we prove Proposition 4.6. To this end, we recall several notions from the potential theory of Markov processes.

Recall that, given a Borel measure ν on S and $\alpha > 0$, its α -potential is given by

$$\mathcal{R}^\alpha \nu(x) := \int r^\alpha(x, y) \nu(dy), \quad x \in S.$$

When $\nu(dx) = f(x) m(dx)$ for some non-negative Borel measurable function f , we simply write $\mathcal{R}^\alpha f := \mathcal{R}^\alpha \nu$. The mapping $f \mapsto \mathcal{R}^\alpha f$ is called the α -potential (or resolvent) operator.

A measurable function $f: S \rightarrow [0, \infty)$ is said to be α -excessive if it satisfies

$$e^{-\alpha t} E^x[f(X_t)] \leq f(x), \quad t \geq 0, \quad x \in S,$$

and

$$\lim_{t \rightarrow 0} e^{-\alpha t} E^x[f(X_t)] = f(x), \quad x \in S,$$

(see [18, Definition 2.1 in Chapter III]). A σ -finite Borel measure ν on S is said to be α -excessive if and only if it satisfies

$$\int_S e^{-\alpha t} E^x[f(X_t)] \nu(dx) \leq \int_S f(x) \nu(dx), \quad \forall t \geq 0,$$

for all non-negative Borel measurable functions f on S (see [18, Definition 1.10 in Chapter VI]).

Let m' be a Borel measure on S . A Borel subset A of S is called m' -polar if and only if

$$\int_S P^x(\sigma_A < \infty) m'(dx) = 0$$

(see [22, Definition A.2.12]).

Concepts corresponding to the dual process $\check{X}$ are prefixed with “co-”; for example, a function that is excessive for $\check{X}$ is called a *coexcessive function*. We also attach the symbol $\check{\cdot}$ to the corresponding notation; for instance, $\check{\mathcal{R}}^\alpha$ denotes the α -potential operator associated with $\check{X}$.

Proposition 4.6 readily follows from the next proposition.

Proposition A.1. *Let μ be a Borel measure on S . If $\|\mathcal{R}^\alpha \mu\|_\infty < \infty$ for some $\alpha > 0$, then μ charges no m -copolar sets, i.e., $\mu(E) = 0$ for any m -copolar Borel set E .*

Proof. The claim follows by applying [15, Theorem 3.4.1] to the shifted resolvent operators $\{\check{\mathcal{R}}^{\alpha+\beta}\}_{\beta>0}$ of the dual process $\check{X}$. Our argument below follows the proof of [15, Theorem 3.4.2].

Let $g: S \rightarrow (0, 1]$ be a Borel measurable function such that $\int g(x) m(dx) < \infty$, which exists by the σ -finiteness of m . Define Borel measures m' and μ' on S by

$$m' := (g \cdot m) \circ \check{\mathcal{R}}^\alpha, \quad \mu' := \mu \circ \check{\mathcal{R}}^\alpha.$$

Here, $\check{\mathcal{R}}^\alpha$ is regarded as an operator acting on measures, that is, for a Borel measure ν on S , the measure $\nu \circ \check{\mathcal{R}}^\alpha$ is defined by

$$(\nu \circ \check{\mathcal{R}}^\alpha)(f) := \int_S \check{\mathcal{R}}^\alpha f(x) \nu(dx)$$

for each Borel measurable function $f: S \rightarrow \mathbb{R}_{\geq 0}$. By duality, for each non-negative Borel measurable function f on S we have

$$\begin{aligned} \int f(x) m'(dx) &= \int \check{\mathcal{R}}^\alpha f(x) g(x) m(dx) = \int f(y) \mathcal{R}^\alpha g(y) m(dy), \\ \int f(x) \mu'(dx) &= \int \check{\mathcal{R}}^\alpha f(x) \mu(dx) = \int f(y) \mathcal{R}^\alpha \mu(y) m(dy). \end{aligned}$$

Hence,

$$m'(dx) = \mathcal{R}^\alpha g(x) m(dx), \quad \mu'(dx) = \mathcal{R}^\alpha \mu(x) m(dx).$$

Since $\mathcal{R}^\alpha g$ and $\mathcal{R}^\alpha \mu$ are α -excessive, both m' and μ' are α -coexcessive (cf. [18, Proposition 1.11 in Chapter VI]).

Noting that $g(x) > 0$ for all $x \in S$, we have $\mathcal{R}^\alpha g(x) > 0$ for all $x \in S$, and hence m is absolutely continuous with respect to m' . Let $h := \frac{dm}{dm'}$ denote the Radon–Nikodym derivative. For each $n \in \mathbb{N}$, define a Borel subset $E_n := \{x \in S \mid h(x) \leq n\}$, and let $\mu'_n := \mu'|_{E_n}$. Then, for any Borel subset $B \subseteq S$,

$$\mu'_n(B) \leq \|\mathcal{R}^\alpha \mu\|_\infty m(E_n \cap B) = \|\mathcal{R}^\alpha \mu\|_\infty \int_{E_n \cap B} h(x) m'(dx) \leq n \|\mathcal{R}^\alpha \mu\|_\infty m'(B).$$

By assumption, $\|\mathcal{R}^\alpha \mu\|_\infty < \infty$. Therefore, μ' is m' -quasi bounded in the sense of [15, p. 88]. By [15, Corollary 1.8.6 and Theorem 3.4.1], it follows that $\mu(E) = 0$ for any Borel set E that is both m' -copolar and m -copolar. Since m' is absolutely continuous with respect to m , every m -copolar set is also m' -copolar. Hence, we have $\mu(E) = 0$ for any m -copolar Borel set E , which completes the proof. $\square$

Proof of Proposition 4.6. By [18, Proposition 1.19 in Chapter VI], every polar set is copolar (and hence m -copolar). Thus, the desired result immediately follows from Proposition A.1. $\square$

A.2 Proof of Lemmas 5.22 and 5.27

Recall the setting of Section 5.3, and in particular the maps $\mathfrak{P}^{(\delta, R)}$ and $\mathfrak{S}^{(\delta, R)}$ introduced in Definitions 5.4 and 5.9, respectively. We provide the proofs of Lemmas 5.22 and 5.27, which are recalled below for convenience.

Lemma A.2. Fix $\delta > 0$ and $R > 1$.

- (i) The map $\mathfrak{P}^{(\delta, R)}$ is Borel measurable.
- (ii) Fix an arbitrary sequence $(q_n, \nu_n, \eta_n)_{n \geq 1}$ converging to (q, ν, η) in the domain of $\mathfrak{P}^{(\delta, R)}$. Assume that

$$\sup_{n \geq 1} \sup_{y \in S} \int_\delta^{2\delta} \int_S q_n(u, y, z) \nu_n^{(R)}(dz) du < \infty. \quad (\text{A.1})$$

Then $\mathfrak{P}^{(\delta, R)}(q_n, \nu_n, \eta_n)$ converges to $\mathfrak{P}^{(\delta, R)}(q, \nu, \eta)$ in $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$.

Lemma A.3. Fix $\delta > 0$ and $R > 1$.

- (i) The map $\mathfrak{S}^{(\delta, R)}$ is Borel measurable.
- (ii) Fix an arbitrary sequence $(q_n, \nu_n, \eta_n)_{n \geq 1}$ converging to (q, ν, η) in the domain of $\mathfrak{S}^{(\delta, R)}$. Assume that

$$\sup_{n \geq 1} \sup_{y \in S} \int_{\delta}^{2\delta} \int_S q_n(u, y, z) \nu_n^{(R)}(dz) du < \infty. \quad (\text{A.2})$$

Then $\mathfrak{S}^{(\delta, R)}(q_n, \nu_n, \eta_n)$ converges to $\mathfrak{S}^{(\delta, R)}(q, \nu, \eta)$ in $\mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0})$.

We first prove Lemma A.3.

Proof of Lemma A.3. Since the argument is rather long and technical, we first give an outline. We begin by constructing a family of continuous maps $\{\mathfrak{S}^{(\delta, R, r)}\}_{r > 0}$. Condition (i) is then verified by showing that $\mathfrak{S}^{(\delta, R, r)}$ converges pointwise to $\mathfrak{S}^{(\delta, R)}$ as $r \rightarrow \infty$. Furthermore, condition (ii) follows from the fact that, under (A.1), the family $\mathfrak{S}^{(\delta, R, r)}$ uniformly approximates $\mathfrak{S}^{(\delta, R)}$.

To define $\mathfrak{P}^{(\delta, R, r)}$, we use the cutoff function introduced in (2.4), that is,

$$\chi^{(r)}(x) := \int_{r-1}^r \mathbf{1}_{S(v)}(x) dv, \quad x \in S, \quad r > 1.$$

Recall that $\chi^{(r)}$ is continuous. Moreover,

$$\chi^{(r)} \nearrow \mathbf{1}_S \quad \text{pointwise as } r \rightarrow \infty. \quad (\text{A.3})$$

For each $r > 1$, we define a map $\mathfrak{S}^{(\delta, R, r)}$

$$\mathfrak{S}^{(\delta, R, r)} : C(\mathbb{R}_{>0} \times S \times S, \mathbb{R}_{\geq 0}) \times \mathcal{M}(S) \times D_{L^0}(\mathbb{R}_{\geq 0}, S) \rightarrow C_{\uparrow}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$$

by

$$(q, \nu, \eta) \mapsto \frac{1}{\delta} \int_0^\infty \mathbf{1}_{(\eta_t, t)}(\cdot) \int_S \int_{\delta}^{2\delta} \chi^{(r)}(\eta_t) q(u, \eta_t, x) du \tilde{\nu}^{(R)}(dx) dt.$$

Using this map $\mathfrak{S}^{(\delta, R, r)}$, we prove the results simultaneously. Fix an arbitrary sequence (q_n, ν_n, η_n) , $n \geq 1$, converging to (q, ν, η) in the domain of $\mathfrak{S}^{(\delta, R, r)}$. Fix a bounded continuous function F on $S \times \mathbb{R}_{\geq 0}$ whose support is contained in $\mathbb{R}_{\geq 0} \times [0, T]$ for some $T \in \mathbb{R}_{\geq 0}$. Write

$$\begin{aligned} G_n^{(r)} &:= \frac{1}{\delta} \int_0^T F(\eta_n(t), t) \int_S \int_{\delta}^{2\delta} \chi^{(r)}(\eta_n(t)) q_n(u, \eta_n(t), x) du \tilde{\nu}_n^{(R)}(dx) dt, \\ G^{(r)} &:= \frac{1}{\delta} \int_0^T F(\eta(t), t) \int_S \int_{\delta}^{2\delta} \chi^{(r)}(\eta(t)) q(u, \eta(t), x) du \tilde{\nu}^{(R)}(dx) dt, \\ G_n &:= \frac{1}{\delta} \int_0^T F(\eta_n(t), t) \int_S \int_{\delta}^{2\delta} q_n(u, \eta_n(t), x) du \tilde{\nu}_n^{(R)}(dx) dt, \\ G &:= \frac{1}{\delta} \int_0^T F(\eta(t), t) \int_S \int_{\delta}^{2\delta} q(u, \eta(t), x) du \tilde{\nu}^{(R)}(dx) dt. \end{aligned}$$

We first show that $\mathfrak{S}^{(\delta, R, r)}$ is continuous for each $r > 0$. Fix $r > 0$. It suffices to show that

$$G_n^{(r)} \xrightarrow{n \rightarrow \infty} G^{(r)}. \quad (\text{A.4})$$

By the convergence of q_n , we have

$$\sup_{n \geq 1} \left\{ q_n(u, y, z) \mid u \in [\delta, 2\delta], \ y \in S^{(r)}, \ x \in S^{(R)} \right\} < \infty. \quad (\text{A.5})$$

For each $n \geq 1$, define

$$f_n(x) := \frac{1}{\delta} \int_0^T \int_\delta^{2\delta} F(\eta_n(t), t) \chi^{(r)}(\eta_n(t)) q_n(u, \eta_n(t), x) du dt, \quad x \in S.$$

Similarly, define

$$f(x) := \frac{1}{\delta} \int_0^T \int_\delta^{2\delta} F(\eta(t), t) \chi^{(r)}(\eta(t)) q(u, \eta(t), x) du dt, \quad x \in S.$$

By the dominated convergence theorem, one readily verifies that all the functions f_n and f are continuous. By definition,

$$G_n^{(r)} = \int_S f_n(x) \tilde{\nu}_n^{(R)}(dx), \quad G^{(r)} = \int_S f(x) \tilde{\nu}^{(R)}(dx).$$

Hence, it suffices to show that

$$\lim_{n \rightarrow \infty} \int_S f_n(x) \tilde{\nu}_n^{(R)}(dx) = \int_S f(x) \tilde{\nu}^{(R)}(dx).$$

Since $\tilde{\nu}_n^{(R)} \rightarrow \tilde{\nu}^{(R)}$ weakly, Lemma 2.9 implies that the above convergence holds once we show $f_n \rightarrow f$ in the compact-convergence topology.

Let $(x_n)_{n \geq 1}$ be a sequence converging to x in S , and fix an arbitrary subsequence $(n_k)_{k \geq 1}$. By the L^0 -convergence of η_n to η , we can find a further subsequence $(n_{k(l)})_{l \geq 1}$ such that $\eta_{n_{k(l)}}(t) \rightarrow \eta(t)$ for Lebesgue-almost every $t \geq 0$. For every such t , by the convergence of q_n to q and the continuity of $\chi^{(r)}$, we have that, for any $u > 0$,

$$\chi^{(r)}(\eta_{n_{k(l)}}(t)) q_{n_{k(l)}}(u, \eta_{n_{k(l)}}(t), x_{n_{k(l)}}) \xrightarrow{l \rightarrow \infty} \chi^{(r)}(\eta(t)) q(u, \eta(t), x).$$

This, together with (A.5) and the dominated convergence theorem, yields

$$\begin{aligned} f_{n_{k(l)}}(x_{n_{k(l)}}) &= \frac{1}{\delta} \int_0^T \int_\delta^{2\delta} F(\eta_{n_{k(l)}}(t), t) \chi^{(r)}(\eta_{n_{k(l)}}(t)) q_{n_{k(l)}}(u, \eta_{n_{k(l)}}(t), x_{n_{k(l)}}) du dt \\ &\xrightarrow{l \rightarrow \infty} \frac{1}{\delta} \int_0^T \int_\delta^{2\delta} F(\eta(t), t) \chi^{(r)}(\eta(t)) q(u, \eta(t), x) du dt \\ &= f(x). \end{aligned}$$

Since the subsequence $(n_k)_{k \geq 1}$ was arbitrary, we conclude that $f_n(x_n) \rightarrow f(x)$, which implies that $f_n \rightarrow f$ in the compact-convergence topology. Therefore, the map $\mathfrak{S}^{(\delta, R, r)}$ is continuous for each $r > 1$.

By (A.3) and the monotone convergence theorem, one readily verifies that

$$G^{(r)} \xrightarrow{r \rightarrow \infty} G, \tag{A.6}$$

which implies that

$$\mathfrak{S}^{(\delta, R, r)}(q, \nu, \eta) \xrightarrow{r \rightarrow \infty} \mathfrak{S}^{(\delta, R)}(q, \nu, \eta) \quad \text{in } \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0}).$$

In particular, the map $\mathfrak{S}^{(\delta, R)}$ is the pointwise limit of continuous maps, and hence we obtain (i).

It remains to verify (ii). Assume that the above sequence $(q_n, \nu_n, \eta_n)_{n \geq 1}$ satisfies (A.2), that is,

$$M := \sup_{n \geq 1} \sup_{y \in S} \int_\delta^{2\delta} \int_S q_n(u, y, z) \nu_n^{(R)}(dz) du < \infty.$$

By definition,

$$|\chi^{(r)}(x) - 1| \leq \mathbf{1}_{S \setminus S^{(r-1)}}(x), \quad \forall x \in S, \quad r > 1.$$

This yields

$$\begin{aligned} & |G_n^{(r)} - G_n| \\ & \leq \frac{\|F\|_\infty}{\delta} \int_0^T \int_S \int_\delta^{2\delta} \mathbf{1}_{S \setminus S^{(r-1)}}(\eta_n(t)) q_n(u, \eta_n(t), x) du \tilde{\nu}_n^{(R)}(dx) dt \\ & \leq \frac{\|F\|_\infty M}{\delta} \text{Leb}(\{t \in [0, T] \mid \eta_n(t) \notin S^{(r-1)}\}). \end{aligned}$$

By Lemma 2.11, the last expression converges to 0 in the successive limits $n \rightarrow \infty$ and then $r \rightarrow \infty$, that is,

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} |G_n^{(r)} - G_n| = 0.$$

Combining this with (A.4) and (A.6), we obtain that $G_n \rightarrow G$, which yields the conclusion of (ii). $\square$

Lemma A.2 is then readily verified by Lemma A.2.

Proof of Lemma A.2. Define a map

$$\mathfrak{T}: \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0}) \ni \Sigma \mapsto (t \mapsto \Sigma(S \times [0, t])) \in L^0(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}).$$

It is readily verified that the map $\mathfrak{T}$ is measurable (in fact, continuous). By definition, we have $\mathfrak{P}^{(\delta, R)} = \mathfrak{T} \circ \mathfrak{S}^{(\delta, R)}$. Hence the map $\mathfrak{P}^{(\delta, R)}$ is measurable when its codomain is taken to be $L^0(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$. Moreover, as in Proposition 2.13, the space $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ is a Borel subset of $L^0(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$. Therefore, $\mathfrak{P}^{(\delta, R)}$ is Borel measurable with respect to its original codomain $C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$.

Fix an arbitrary sequence $(q_n, \nu_n, \eta_n)_{n \geq 1}$ converging to (q, ν, η) in the domain of $\mathfrak{P}^{(\delta, R)}$. Then, by Lemma A.3(ii), we have

$$\mathfrak{S}^{(\delta, R)}(q_n, \nu_n, \eta_n) \xrightarrow{n \rightarrow \infty} \mathfrak{S}^{(\delta, R)}(q, \nu, \eta) \quad \text{in } \mathcal{M}_{(*, \text{fin})}(S \times \mathbb{R}_{\geq 0}).$$

By the same argument as in the proof of Proposition 2.25(i), we deduce that

$$\mathfrak{S}^{(\delta, R)}(q_n, \nu_n, \eta_n)(S \times [0, \cdot]) \xrightarrow{n \rightarrow \infty} \mathfrak{S}^{(\delta, R)}(q, \nu, \eta)(S \times [0, \cdot]) \quad \text{in } C_\uparrow(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}),$$

which completes the proof. $\square$

A.3 The function trf_β

Here, we prove some technical results that are used in Section 9.3. For each $\beta > 0$, define a bijective function $\text{trf}_\beta: (0, 1/e) \rightarrow (0, \infty)$ by setting

$$\text{trf}_\beta(t) := \frac{1}{\log(1/t) (\log \log(1/t))^{1+\beta}}.$$

We write $\text{trf}_\beta^{-1}: (0, \infty) \rightarrow (0, 1/e)$ for its inverse.

Lemma A.4. *For each $\beta > 0$, the following statements hold.*

(i) *It holds that*

$$\lim_{\delta \rightarrow 0} \int_0^\delta \frac{\text{trf}_\beta(t)}{t} dt = 0.$$

(ii) For any $\alpha > 0$,

$$\lim_{\delta \rightarrow 0} \int_0^\delta \frac{1}{(\text{trf}_\beta(t))^\alpha} dt = 0.$$

(iii) As $t \downarrow 0$,

$$\text{trf}_\beta^{-1}(t) = \exp\left(-\frac{1+o(1)}{t(\log(1/t))^{1+\beta}}\right).$$

Proof. (i). By the change of variables $s = \log(1/t)$,

$$\int_0^\delta \frac{\text{trf}_\beta(t)}{t} dt = \int_{\log(1/\delta)}^\infty \frac{1}{s(\log s)^{1+\beta}} ds = \left[-\frac{1}{\beta} (\log s)^{-\beta}\right]_{\log(1/\delta)}^\infty = \frac{1}{\beta(\log \log(1/\delta))^\beta},$$

which yields the desired result.

(ii). Using the same substitution $s = \log(1/t)$,

$$\int_0^\delta \frac{1}{(\text{trf}_\beta(t))^\alpha} dt = \int_{\log(1/\delta)}^\infty e^{-s} s^\alpha (\log s)^{\alpha(1+\beta)} ds.$$

The integrand is dominated by the exponential factor e^{-s} , hence the tail integral tends to 0 as $\delta \rightarrow 0$, which yields the claim.

(iii). Define three bijective functions:

$$\begin{aligned} g_1: (0, 1/e) &\rightarrow (0, \infty), & t &\mapsto \frac{1}{1+\beta} \log(\log(1/t)), \\ g_2: (0, \infty) &\rightarrow (0, \infty), & u &\mapsto u e^u, \\ g_3: (0, \infty) &\rightarrow (0, \infty), & v &\mapsto \frac{1}{((1+\beta)v)^{1+\beta}}. \end{aligned}$$

Then $\text{trf}_\beta = g_3 \circ g_2 \circ g_1$, hence $\text{trf}_\beta^{-1} = g_1^{-1} \circ g_2^{-1} \circ g_3^{-1}$. A direct computation gives

$$g_1^{-1}(y) = \exp(-e^{(1+\beta)y}), \quad g_3^{-1}(t) = \frac{1}{1+\beta} t^{-1/(1+\beta)}.$$

The inverse of g_2 is the Lambert W function, and as $u \rightarrow \infty$,

$$g_2^{-1}(u) = \log\left(\frac{u}{\log u}\right) + o(1)$$

(cf. [23, Equation (4.20)]). Therefore, as $t \downarrow 0$,

$$\begin{aligned} -\log(\text{trf}_\beta^{-1}(t)) &= \exp\left((1+\beta) g_2^{-1}(g_3^{-1}(t))\right) \\ &= \exp\left((1+\beta) \log\left(\frac{g_3^{-1}(t)}{\log g_3^{-1}(t)}\right) + o(1)\right) \\ &= e^{o(1)} \left(\frac{g_3^{-1}(t)}{\log g_3^{-1}(t)}\right)^{1+\beta} \\ &= e^{o(1)} \left(\frac{\frac{1}{1+\beta} t^{-1/(1+\beta)}}{\log\left(\frac{1}{1+\beta} t^{-1/(1+\beta)}\right)}\right)^{1+\beta} \\ &= e^{o(1)} \left(\frac{t^{-1/(1+\beta)}}{\log(1/t) (1+o(1))}\right)^{1+\beta} \\ &= \frac{1+o(1)}{t(\log(1/t))^{1+\beta}}, \end{aligned}$$

which proves the desired asymptotic. □

B An inverse version of the continuous mapping theorem

Let S_1 and S_2 be bcm spaces and let $f: S_1 \rightarrow S_2$ be a continuous map. By the continuous mapping theorem, the pushforward map $\mu \mapsto \mu \circ f^{-1}$ from $\mathcal{M}(S_1)$ to $\mathcal{M}(S_2)$ is continuous. When f is injective, the Lusin–Souslin theorem (Lemma 2.14) ensures that $f(B)$ is a Borel subset of S_2 for every Borel set $B \subseteq S_1$. Hence, one can define the map in the inverse direction $\nu \mapsto \nu \circ f$ from $\mathcal{M}(S_2)$ to $\mathcal{M}(S_1)$. Although this map is not continuous in general, its continuity can be ensured under certain additional assumptions. Below, we provide a result on this point, which is used in Proposition 7.7.

Proposition B.1. *Let S_1 and S_2 be bcm spaces and let $\iota: S_1 \rightarrow S_2$ be an injective Borel measurable map such that its image $\iota(S_1)$ is closed in S_2 and its inverse is continuous. Let $(\Sigma_n)_{n \geq 1}$ be a sequence of Radon measures on S_2 . Assume that $\text{supp}(\Sigma_n) \subseteq \iota(S_1)$ for all $n \geq 1$, and that $\Sigma_n \rightarrow \Sigma$ vaguely. Then $\Sigma_n \circ \iota \rightarrow \Sigma \circ \iota$ vaguely as measures on S_1 .*

The idea of the proof is to show that the restrictions of the measures to $\iota(S_1)$ still converge vaguely. Once this is obtained, the desired result follows from the continuous mapping theorem. We use the following lemma to verify the convergence of such restrictions.

Lemma B.2. *Let S be a bcm space, and let $(\Sigma_n)_{n \geq 1}$ be a sequence of Radon measures on S converging vaguely to Σ . If a closed subset F of S satisfies $\text{supp}(\Sigma_n) \subseteq F$ for all $n \geq 1$, then $\Sigma_n|_F \rightarrow \Sigma|_F$ vaguely as measures on F .*

Note that this is not a consequence of the standard fact that vague convergence is preserved under restriction to continuity sets. Indeed, we allow that the limiting measure Σ has positive mass on the boundary of F .

Proof of Lemma B.2. Fix a compactly supported continuous function $f: F \rightarrow \mathbb{R}$, and set $K := \text{supp}(f)$. Choose a continuous function $\chi: S \rightarrow [0, 1]$ with compact support such that $\chi \equiv 1$ on K . Then $\chi(x) \Sigma_n(dx) \rightarrow \chi(x) \Sigma(dx)$ weakly. By Lemma 2.9, it follows that

$$\lim_{n \rightarrow \infty} \int_F f(x) \chi(x) \Sigma_n(dx) = \int_F f(x) \chi(x) \Sigma(dx).$$

Since f vanishes outside K and $\chi \equiv 1$ on K , we conclude that

$$\lim_{n \rightarrow \infty} \int_F f(x) \Sigma_n(dx) = \int_F f(x) \Sigma(dx).$$

This completes the proof. □

Now we can prove Proposition B.1.

Proof of Proposition B.1. By Lemma B.2, we have $\Sigma_n|_{\iota(S_1)} \rightarrow \Sigma|_{\iota(S_1)}$ vaguely as measures on $\iota(S_1)$. Applying the continuous mapping theorem to the inverse $\iota^{-1}: \iota(S_1) \rightarrow S_1$, we obtain

$$\Sigma_n \circ \iota = \Sigma_n|_{\iota(S_1)} \circ \iota \rightarrow \Sigma|_{\iota(S_1)} \circ \iota = \Sigma \circ \iota$$

vaguely as measures on S_1 . This completes the proof. □

C Replacement of the L^0 topology by stronger topologies

Throughout this paper, we have primarily worked with the L^0 topology on path spaces of stochastic processes, and most of our main results are formulated and proved under this topology. This choice allows us to establish the results under minimal assumptions and to cover a wide range of applications. In many concrete situations, however, the convergence of stochastic processes

is known to hold under stronger topologies such as the compact-convergence topology, the J_1 -Skorohod topology, or the M_1 topology. In this appendix, we provide a result which ensures that whenever such stronger convergence holds, the L^0 topology appearing in the statements of the main results can be replaced with the corresponding stronger topology.

Proposition C.1. *Let $\mathfrak{X}_1$, $\mathfrak{X}_2$, and $\mathfrak{Y}$ be Polish spaces. Assume that the following conditions hold.*

- (i) *The set $\mathfrak{X}_1$ is a Borel subset of $\mathfrak{X}_2$.*
- (ii) *We have $\mathcal{B}(\mathfrak{X}_1) = \{B \cap \mathfrak{X}_1 \mid B \in \mathcal{B}(\mathfrak{X}_2)\}$.*
- (iii) *The inclusion map $\mathfrak{X}_1 \hookrightarrow \mathfrak{X}_2$ is continuous.*

Let (X, Y) and (X_n, Y_n) , $n \geq 1$, be random elements of $\mathfrak{X}_1 \times \mathfrak{Y}$, where each pair (X_n, Y_n) and (X, Y) is defined on its own probability space. If $X_n \xrightarrow[n \rightarrow \infty]{d} X$ in $\mathfrak{X}_1$ and $(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{d} (X, Y)$ in $\mathfrak{X}_2 \times \mathfrak{Y}$, then $(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{d} (X, Y)$ in $\mathfrak{X}_1 \times \mathfrak{Y}$.

Proof. By the assumption, the family $\{(X_n, Y_n)\}_{n \geq 1}$ is tight as random elements of $\mathfrak{X}_1 \times \mathfrak{Y}$. Assume that, along a subsequence $(n_k)_{k \geq 1}$, the law of (X_{n_k}, Y_{n_k}) converges to a probability measure P on $\mathfrak{X}_1 \times \mathfrak{Y}$. By (iii), we can apply the continuous mapping theorem and deduce that the law of (X_{n_k}, Y_{n_k}) also converges to P as probability measures on $\mathfrak{X}_2 \times \mathfrak{Y}$. Then, by the assumption, we deduce that P coincides with the law of (X, Y) as probability measure on $\mathfrak{X}_2 \times \mathfrak{Y}$. Conditions (i) and (ii) yield that P coincides with the law of (X, Y) as probability measures on $\mathfrak{X}_1 \times \mathfrak{Y}$. This implies the uniqueness of the limit P , which completes the proof. $\square$

Remark C.2. By the Lusin–Souslin theorem (Lemma 2.14), condition (i) follows from the continuity of the inclusion map in condition (iii).

We illustrate an application of Proposition C.1. Consider the setting of Section 5.3. Suppose that the standard process X satisfies the weak J_1 -continuity condition (Assumption 5.17) so that the map $\mathcal{L}_X$ is continuous with respect to the weak topology induced by the J_1 -Skorohod topology. Let A be a PCAF in the local Kato class. Then, by Theorem 5.25, the map $\mathcal{L}_{(X,A)}$ is continuous with respect to the weak topology induced by the product of the L^0 topology and the compact-convergence topology. Applying Proposition C.1, we see that this continuity remains valid when the L^0 topology is replaced by the J_1 -Skorohod topology.

D The product process of independent standard processes

In this appendix, we prove that the pair of two independent standard processes is again a standard process.

Fix two locally compact separable metrizable topological spaces S_1 and S_2 . For each $i \in \{1, 2\}$, we write $S_i \cup \{\Delta\}$ for the one-point compactification of S_i . We fix, for each $i \in \{1, 2\}$, a standard process on S_i , denoted by

$$X^i = (\Omega_i, \mathcal{M}_i, (X_t^i)_{t \in [0, \infty]}, (P_i^x)_{x \in S_\Delta}, (\theta_t^i)_{t \in [0, \infty]}).$$

We first recall the product process of X^1 and X^2 from Section 7.1. Let $(S_1 \times S_2) \cup \{\Delta\}$ be the one-point compactification of the product space $S_1 \times S_2$. For each $\mathbf{x} = (x_1, x_2) \in S \times S$, we define $\hat{P}^{\mathbf{x}} := P_1^{x_1} \otimes P_2^{x_2}$, which is a probability measure on the product measurable space $(\hat{\Omega}, \hat{\mathcal{M}}) := (\Omega_1 \times \Omega_2, \mathcal{M}_1 \otimes \mathcal{M}_2)$. Also, we define $\hat{P}^\Delta := P_1^\Delta \otimes P_2^\Delta$. For each $\omega = (\omega_1, \omega_2) \in \hat{\Omega}$ and $t \geq 0$, we set

$$\hat{X}_t(\omega) := \begin{cases} (X_t^1(\omega_1), X_t^2(\omega_2)), & t < \zeta_1(\omega_1) \wedge \zeta_2(\omega_2), \\ \Delta, & t \geq \zeta_1(\omega_1) \wedge \zeta_2(\omega_2), \end{cases}$$

where ζ_i denotes the lifetime of X^i for each $i = 1, 2$. Finally, we define, for each $\omega = (\omega_1, \omega_2) \in \hat{\Omega}$ and $t \geq 0$,

$$\hat{\theta}_t(\omega) := (\theta_t^1(\omega_1), \theta_t^2(\omega_2)).$$

The main result is the following.

Theorem D.1. *In the above setting,*

$$\hat{X} := (\hat{\Omega}, \hat{\mathcal{M}}, (\hat{X}_t)_{t \in [0, \infty]}, (\hat{P}^x)_{x \in (S_1 \times S_2) \cup \{\Delta\}}, (\hat{\theta}_t)_{t \in [0, \infty]})$$

is a standard process on $S_1 \times S_2$.

Let $(\mathcal{F}_t^{X^1, 0})_{t \geq 0}$ be the filtration generated by X^1 , that is,

$$\mathcal{F}_t^{X^1, 0} := \sigma(X_s^1; s \leq t).$$

We similarly define $(\mathcal{F}_t^{X^2, 0})_{t \geq 0}$ and $(\mathcal{F}_t^{\hat{X}, 0})_{t \geq 0}$ to be the filtrations generated by X^2 and $\hat{X}$, respectively. Below, we state a basic result regarding $(\mathcal{F}_t^{X^1, 0})_{t \geq 0}$. In the following lemma, for sets A and B , and a subset $C \subseteq A \times B$, we write $C|_{(a, *)} := \{b \in B \mid (a, b) \in C\}$ for each $a \in A$ and $C|_{(*, b)} := \{a \in A \mid (a, b) \in C\}$ for each $b \in B$.

Lemma D.2. *Fix $t \in [0, \infty)$ and $\Lambda \in \mathcal{F}_t^{\hat{X}, 0}$, $\Lambda|_{(*, \omega_2)} \in \mathcal{F}_t^{X^1, 0}$ for any $\omega_2 \in \Omega_2$, and $\Lambda|_{(\omega_1, *)} \in \mathcal{F}_t^{X^2, 0}$ for any $\omega_1 \in \Omega_1$.*

Proof. By the π - λ theorem, it is enough to consider the case where $\Lambda = \bigcap_{k=1}^n \hat{X}_{s_k}^{-1}(A_k)$, where $n \in \mathbb{N}$, $s_k \in [0, t]$, and $A_k \in \mathcal{B}((S_1 \times S_2) \cup \{\Delta\})$ for each k . Fix $\omega_2 \in \Omega_2$ and write $B_k := A_k|_{(*, X_{s_k}^2(\omega_2))}$, which is a Borel subset of S . We then have that

$$\begin{aligned} \Lambda|_{(*, \omega_2)} &= \{\omega_1 \in \Omega_1 \mid (X_{s_k}^1(\omega_1), X_{s_k}^2(\omega_2)) \in A_k, \forall k \in \{1, \dots, n\}\} \\ &= \{\omega_1 \in \Omega_1 \mid X_{s_k}^1(\omega_1) \in B_k, \forall k \in \{1, \dots, n\}\}. \end{aligned}$$

From the last expression, we see that $\Lambda|_{(*, \omega_2)} \in \mathcal{F}_t^{X^1, 0}$. Similarly, one can prove the second assertion. $\square$

We next verify that $\hat{X}$ is a Markov process. Recall that, given a filtration $(\mathcal{G}_t)_{t \geq 0}$, its right-continuous version $(\mathcal{G}_{t+})_{t \geq 0}$ is defined by $\mathcal{G}_{t+} = \bigcap_{s > t} \mathcal{G}_s$.

Lemma D.3. *The process $\hat{X}$ is a normal Markov process, and it has the Markov property with respect to $(\mathcal{F}_{t+}^{\hat{X}, 0})_{t \geq 0}$.*

Proof. Recall the definition of a normal Markov process from [32, p. 385]. We briefly check conditions (M1)–(M5) in [32, p. 385]. All these conditions, except for (M4), are readily verified since each X^1 and X^2 satisfies the corresponding conditions. To verify (M4), we show the Markov property of $\hat{X}$ with respect to $(\mathcal{F}_{t+}^{\hat{X}, 0})_{t \geq 0}$.

Fix $s, t \in [0, \infty)$, $\mathbf{x} = (x_1, x_2) \in S_1 \times S_2$, $E \in \mathcal{B}(S_1 \times S_2)$, and $\Lambda \in \mathcal{F}_{t+}^{\hat{X}, 0}$. It suffices to show that

$$\hat{E}^{\mathbf{x}}[\mathbf{1}_E(\hat{X}_{s+t}) \cdot \mathbf{1}_\Lambda] = \hat{E}^{\mathbf{x}}[\hat{P}^{\hat{X}_t}(\hat{X}_s \in E) \cdot \mathbf{1}_\Lambda]. \quad (\text{D.1})$$

We first consider the case where $E = E_1 \times E_2$ for some $E_1 \in \mathcal{B}(S_1)$ and $E_2 \in \mathcal{B}(S_2)$. By Lemma D.2, we have $\Lambda|_{(*, \omega_2)} \in \mathcal{F}_{t+}^{X^1, 0}$ for each $\omega_2 \in \Omega_2$. Thus, using the Markov property of X^1 with respect to $(\mathcal{F}_{u+}^{X^1, 0})_{u \geq 0}$, we deduce that, for each $\omega_2 \in \Omega_2$,

$$\int_{\Omega_1} \mathbf{1}_{E_1}(X_{s+t}^1(\omega_1)) \mathbf{1}_\Lambda(\omega_1, \omega_2) P_1^{x_1}(d\omega_1) = \int_{\Omega_1} P_1^{X_t^1(\omega_1)}(X_s^1 \in E_1) \mathbf{1}_\Lambda(\omega_1, \omega_2) P_1^{x_1}(d\omega_1).$$

Similarly, for each $\omega_1 \in \Omega_1$,

$$\int_{\Omega_2} \mathbf{1}_{E_2}(X_{s+t}^2(\omega_2)) \mathbf{1}_\Lambda(\omega_1, \omega_2) P_2^{x_2}(d\omega_2) = \int_{\Omega_2} P_1^{X_t^1(\omega_2)}(X_s^2 \in E_2) \mathbf{1}_\Lambda(\omega_1, \omega_2) P_2^{x_2}(d\omega_2).$$

Therefore, using Fubini's theorem, we deduce that

$$\begin{aligned} & \hat{E}^x[\mathbf{1}_{E_2}(\hat{X}_{s+t}) \cdot \mathbf{1}_\Lambda] \\ &= \int_{\Omega_1} \int_{\Omega_2} \mathbf{1}_{E_1}(X_{s+t}^1(\omega_1)) \mathbf{1}_{E_2}(X_{s+t}^2(\omega_2)) \mathbf{1}_\Lambda(\omega_1, \omega_2) P_1^{x_1}(d\omega_1) P_2^{x_2}(d\omega_2) \\ &= \int_{\Omega_1} \int_{\Omega_2} P_1^{X_t^1(\omega_1)}(X_s^1 \in E_1) P_1^{X_t^1(\omega_2)}(X_s^2 \in E_2) \mathbf{1}_\Lambda(\omega_1, \omega_2) P_1^{x_1}(d\omega_1) P_2^{x_2}(d\omega_2) \\ &= \int_{\Omega_1 \times \Omega_2} \hat{P}^{\hat{X}_t(\omega_1, \omega_2)}(\hat{X}_s \in E) \mathbf{1}_\Lambda(\omega_1, \omega_2) \hat{P}^x(d(\omega_1, \omega_2)) \\ &= \hat{E}^x[\hat{P}^{\hat{X}_t}(\hat{X}_s \in E) \cdot \mathbf{1}_\Lambda]. \end{aligned}$$

Thus, (D.1) holds for $E = E_1 \times E_2$. It is now a standard argument to extend the result to general Borel subsets $E \subseteq S_1 \times S_2$. $\square$

We next prove that $\hat{X}$ is a Borel right process (see [22, Definition A.1.17] for the definition). To this end, we introduce some notation for filtrations of Markov processes. Recall that $(\mathcal{F}_t^{X^1, 0})_{t \geq 0}$ denotes the natural filtration of X^1 , and set $\mathcal{F}_\infty^{X^1, 0} := \bigcup_{t \geq 0} \mathcal{F}_t^{X^1, 0}$. Given a probability measure μ on $S_1 \cup \{\Delta\}$, define

$$P^\mu(\cdot) := \int_{S_1 \cup \{\Delta\}} P_1^x(\cdot) \mu(dx),$$

which is a probability measure on $(\Omega_1, \mathcal{F}_\infty^{X^1, 0})$. We then let $\mathcal{F}_\infty^{X^1, \mu}$ denote the P^μ -completion of $\mathcal{F}_\infty^{X^1, 0}$. Let $\mathcal{N}^\mu$ be the collection of all P^μ -null sets in $\mathcal{F}_\infty^{X^1, \mu}$, and set $\mathcal{F}_t^{X^1, \mu} := \sigma(\mathcal{F}_t^{X^1, 0}, \mathcal{N}^\mu)$ for each $t \in [0, \infty)$. When μ is the Dirac measure δ_x at x , we simply write $\mathcal{F}_t^{X^1, x} := \mathcal{F}_t^{X^1, \delta_x}$, $t \in [0, \infty]$. The minimum augmented admissible filtration $(\mathcal{F}_t^{X^1})_{t \in [0, \infty]}$ is then defined by

$$\mathcal{F}_t^{X^1} := \bigcap_{\mu \in \mathcal{P}(S_1 \cup \{\Delta\})} \mathcal{F}_t^{X^1, \mu}, \quad t \in [0, \infty].$$

The same notation applies to X^2 and $\hat{X}$.

Lemma D.4. *The process $\hat{X}$ is a Borel right process.*

Proof. Fix a probability measure μ on $(S \times S) \cup \{\Delta\}$. By Lemma D.3 and [32, Lemma A.2.2], the filtration $(\mathcal{F}_t^{\hat{X}, \mu})_{t \geq 0}$ is right-continuous. Fix $s > 0$, an $(\mathcal{F}_t^{\hat{X}, \mu})$ -stopping time σ , and $\Lambda \in \mathcal{F}_\sigma^{\hat{X}, \mu}$. By [22, Theorem A.1.18], it suffices to show that

$$\hat{E}^\mu[\mathbf{1}_{E_2}(\hat{X}_{\sigma+s}) \cdot \mathbf{1}_\Lambda \cdot \mathbf{1}_{\{\sigma < \infty\}}] = \hat{E}^\mu[\hat{P}^{\hat{X}_\sigma}(\hat{X}_s \in E) \cdot \mathbf{1}_\Lambda \cdot \mathbf{1}_{\{\sigma < \infty\}}], \quad \forall E \in \mathcal{B}(S_1 \times S_2).$$

By the monotone class theorem, it is enough to prove the following: for any bounded and continuous function $f_i: S_i \rightarrow [0, \infty)$, $i \in \{1, 2\}$,

$$\hat{E}^\mu[f_1(X_{\sigma+s}^1) f_2(X_{\sigma+s}^2) \cdot \mathbf{1}_\Lambda \cdot \mathbf{1}_{\{\sigma < \infty\}}] = \hat{E}^\mu[\hat{E}^{\hat{X}_\sigma}[f_1(X_s^1) f_2(X_s^2)] \cdot \mathbf{1}_\Lambda \cdot \mathbf{1}_{\{\sigma < \infty\}}].$$

From [18, Chapter II, Exercise 4.14], for each $i \in \{1, 2\}$ the map $x \mapsto E_i^x[f_i(X_s^i)]$ is continuous in the fine topology of X^i . By [18, Chapter II, Theorem 4.8], for any $x \in S_i$ we then have that

$E_i^{X_t^i}[f_i(X_s^i)]$ is right-continuous in $t \geq 0$, P_i^x -a.s. Using the independence of X^1 and X^2 , we obtain

$$\hat{E}^{\hat{X}_t}[f_1(X_s^1)f_2(X_s^2)] = E_1^{X_t^1}[f_1(X_s^1)] E_2^{X_t^2}[f_2(X_s^2)],$$

and hence

$$\hat{E}^{\hat{X}_t}[f_1(X_s^1)f_2(X_s^2)] \text{ is right-continuous in } t \geq 0, \hat{P}^\mu\text{-a.s.} \quad (\text{D.2})$$

For each $n \geq 1$, define

$$\sigma_n := \frac{\lfloor 2^n \sigma \rfloor + 1}{2^n}.$$

Then σ_n is an $(\mathcal{F}_t^{\hat{X}, \mu})$ -stopping time, and $\sigma_n(\omega) \downarrow \sigma(\omega)$ for all $\omega \in \hat{\Omega}$ (see [55, Proof of Lemma 2.2.5]). By the right-continuity of $\hat{X}$ and (D.2),

$$\begin{aligned} \hat{E}^\mu[f_1(X_{\sigma_n+s}^1)f_2(X_{\sigma_n+s}^2) \cdot \mathbf{1}_\Lambda \cdot \mathbf{1}_{\{\sigma < \infty\}}] &= \lim_{n \rightarrow \infty} \hat{E}^\mu[f_1(X_{\sigma_n+s}^1)f_2(X_{\sigma_n+s}^2) \cdot \mathbf{1}_\Lambda \cdot \mathbf{1}_{\{\sigma_n < \infty\}}], \\ \hat{E}^\mu[\hat{E}^{\hat{X}_\sigma}[f_1(X_s^1)f_2(X_s^2)] \cdot \mathbf{1}_\Lambda \cdot \mathbf{1}_{\{\sigma < \infty\}}] &= \lim_{n \rightarrow \infty} \hat{E}^\mu[\hat{E}^{\hat{X}_{\sigma_n}}[f_1(X_s^1)f_2(X_s^2)] \cdot \mathbf{1}_\Lambda \cdot \mathbf{1}_{\{\sigma_n < \infty\}}]. \end{aligned}$$

Thus it suffices to prove that, for each fixed n ,

$$\hat{E}^\mu[f_1(X_{\sigma_n+s}^1)f_2(X_{\sigma_n+s}^2) \cdot \mathbf{1}_\Lambda \cdot \mathbf{1}_{\{\sigma_n < \infty\}}] = \hat{E}^\mu[\hat{E}^{\hat{X}_{\sigma_n}}[f_1(X_s^1)f_2(X_s^2)] \cdot \mathbf{1}_\Lambda \cdot \mathbf{1}_{\{\sigma_n < \infty\}}]. \quad (\text{D.3})$$

Fix $n \geq 1$. By [32, Lemma A.2.3], there exist an $(\mathcal{F}_{t+}^{\hat{X}, 0})$ -stopping time σ'_n and $\Lambda' \in \mathcal{F}_{\infty}^{\hat{X}, 0}$ such that

$$\begin{aligned} \hat{P}^\mu(\sigma_n \neq \sigma'_n) &= 0, \quad \hat{E}^\mu[\|\mathbf{1}_\Lambda - \mathbf{1}_{\Lambda'}\|] = 0, \\ \Lambda' \cap \{\sigma'_n \leq t\} &\in \mathcal{F}_{t+}^{\hat{X}, 0}, \quad \forall t \geq 0. \end{aligned} \quad (\text{D.4})$$

Hence it suffices to prove (D.3) with σ_n and Λ replaced by σ'_n and Λ' , respectively. Since σ'_n takes values in $\{j2^{-n} \mid j \in \mathbb{N}\}$,

$$\begin{aligned} &\hat{E}^\mu[f_1(X_{\sigma'_n+s}^1)f_2(X_{\sigma'_n+s}^2) \cdot \mathbf{1}_{\Lambda'} \cdot \mathbf{1}_{\{\sigma'_n < \infty\}}] \\ &= \sum_{j=1}^{\infty} \hat{E}^\mu[f_1(X_{j2^{-n}+s}^1)f_2(X_{j2^{-n}+s}^2) \cdot \mathbf{1}_{\{\Lambda' \cap \{\sigma'_n = j2^{-n}\}\}}] \\ &= \sum_{j=1}^{\infty} \hat{E}^\mu[\mathbf{1}_{\{\Lambda' \cap \{\sigma'_n = j2^{-n}\}\}} \hat{E}^\mu[f_1(X_{j2^{-n}+s}^1)f_2(X_{j2^{-n}+s}^2) \mid \mathcal{F}_{j2^{-n}}^{\hat{X}, 0}]], \end{aligned}$$

where we used (D.4). Applying the Markov property of $\hat{X}$ with respect to $(\mathcal{F}_{t+}^{\hat{X}, 0})$ (Lemma D.3), this equals

$$\begin{aligned} &\sum_{j=1}^{\infty} \hat{E}^\mu[\mathbf{1}_{\{\Lambda' \cap \{\sigma'_n = j2^{-n}\}\}} \hat{E}^{\hat{X}_{j2^{-n}}} [f_1(X_s^1)f_2(X_s^2)]] \\ &= \hat{E}^\mu[\hat{E}^{\hat{X}_{\sigma'_n}} [f_1(X_s^1)f_2(X_s^2)] \cdot \mathbf{1}_{\Lambda'} \cdot \mathbf{1}_{\{\sigma'_n < \infty\}}]. \end{aligned}$$

This completes the proof. $\square$

We can now conclude the proof of Theorem D.1.

Proof of Theorem D.1. Write ζ for the lifetime of $\hat{X}$. By the definition of the product process $\hat{X}$, we have

$$\zeta = \zeta_1 \wedge \zeta_2, \quad (\text{D.5})$$

where we recall that ζ_i denotes the lifetime of X^i for each $i \in \{1, 2\}$. By Lemma D.4, it remains to show the quasi-left-continuity of $\hat{X}$ on $(0, \zeta)$. We note that, by Lemma D.3 and [32, Lemma A.2.2], the filtration $(\mathcal{F}_t^{\hat{X}})_{t \geq 0}$ is right-continuous.

Fix $\mathbf{x} = \mathbf{x} \in S_1 \times S_2$, and an increasing sequence $(\sigma_n)_{n \geq 1}$ of $(\mathcal{F}_t^{\hat{X}})_{t \geq 0}$ -stopping times such that $\sigma_n(\omega)$ converges to $\sigma(\omega) \in [0, \infty]$ for each $\omega \in \hat{\Omega}$. Note that σ is also an $(\mathcal{F}_t^{\hat{X}})_{t \geq 0}$ -stopping time thanks to the right-continuity of $(\mathcal{F}_t^{\hat{X}})_{t \geq 0}$. Using [32, Lemma A.2.3], we can find $(\mathcal{F}_{t+}^{\hat{X}, 0})_{t \geq 0}$ -stopping times $\tilde{\sigma}_n$, $n \in \mathbb{N}$, and $\tilde{\sigma}$ such that

$$\hat{P}^{\mathbf{x}}(\Lambda) = 1, \quad \text{where } \Lambda := \{\sigma = \tilde{\sigma}, \text{ and } \tilde{\sigma}_n = \sigma_n \text{ for all } n\} \in \mathcal{F}_\infty^{\hat{X}}. \quad (\text{D.6})$$

By the definition of $\mathcal{F}_\infty^{\hat{X}}$, we have $\Lambda \in \mathcal{F}_\infty^{\hat{X}, \mathbf{x}}$. Thus, we can find $\Lambda', \Lambda'' \in \mathcal{F}_\infty^{\hat{X}, 0}$ such that

$$\Lambda' \subseteq \Lambda \subseteq \Lambda'', \quad \hat{P}^{\mathbf{x}}(\Lambda'' \setminus \Lambda') = 0. \quad (\text{D.7})$$

It follows from (D.6), (D.7), and Fubini's theorem that

$$\hat{P}^{\mathbf{x}}(\Lambda') = \int_{\Omega_2} P_1^{x_1}(\Lambda'|_{(*, \omega_2)}) P_2^{x_2}(d\omega_2) = 1.$$

In particular, there exists a set $\Omega_2^0 \in \mathcal{F}_\infty^{X^2, 0}$ such that

$$P_2^{x_2}(\Omega_2^0) = 1, \quad (\text{D.8})$$

$$P_1^{x_1}(\Lambda'|_{(*, \omega_2)}) = 1, \quad \forall \omega_2 \in \Omega_2^0. \quad (\text{D.9})$$

The second property yields that

$$\Lambda'|_{(*, \omega_2)} \in \mathcal{F}_0^{X^1, x_1}, \quad \forall \omega_2 \in \Omega_2^0. \quad (\text{D.10})$$

Fix $\omega_2 \in \Omega_2^0$. We define

$$\tilde{\sigma}^{\omega_2}(\omega_1) := \begin{cases} \tilde{\sigma}(\omega_1, \omega_2), & \omega_1 \in \Lambda'|_{(*, \omega_2)}, \\ 0, & \omega_1 \notin \Lambda'|_{(*, \omega_2)}. \end{cases} \quad (\text{D.11})$$

Similarly, for each $n \in \mathbb{N}$, we define $\tilde{\sigma}_n^{\omega_2}$. We deduce from Lemma D.2 and (D.10) that $\tilde{\sigma}^{\omega_2}$ is an $(\mathcal{F}_{t+}^{X^1, x_1})_{t \geq 0}$ -stopping time. The same holds for $\tilde{\sigma}_n^{\omega_2}$ for each $n \in \mathbb{N}$. By the inclusion $\Lambda' \subseteq \Lambda$ given in (D.7) and the definition of Λ stated at (D.6), we have that $\tilde{\sigma}_n^{\omega_2}(\omega_1) \uparrow \tilde{\sigma}^{\omega_2}(\omega_1)$ for each $\omega \in \Omega_1$. Hence, by the quasi-left-continuity of X^1 on $(0, \zeta_1)$, we obtain that

$$P_1^{x_1} \left(\lim_{n \rightarrow \infty} X_{\tilde{\sigma}_n^{\omega_2}}^1 = X_{\tilde{\sigma}^{\omega_2}}^1, \tilde{\sigma}^{\omega_2} < \zeta_1 \right) = P_1^{x_1}(\tilde{\sigma}^{\omega_2} < \zeta_1), \quad \forall \omega_2 \in \Omega_2^0.$$

It then follows from (D.9) and (D.11) that

$$P_1^{x_1} \left(\lim_{n \rightarrow \infty} X_{\tilde{\sigma}_n(\cdot, \omega_2)}^1 = X_{\tilde{\sigma}(\cdot, \omega_2)}^1, \tilde{\sigma}(\cdot, \omega_2) < \zeta_1 \right) = P_1^{x_1}(\tilde{\sigma}(\cdot, \omega_2) < \zeta_1), \quad \forall \omega_2 \in \Omega_2^0.$$

Integrating both sides of the above equation with respect to $P_2^{x_2}(d\omega_2)$ and using (D.8), we obtain that

$$\hat{P}^{\mathbf{x}} \left(\lim_{n \rightarrow \infty} X_{\tilde{\sigma}_n}^1 = X_{\tilde{\sigma}}^1, \tilde{\sigma} < \zeta_1 \right) = \hat{P}^{\mathbf{x}}(\tilde{\sigma} < \zeta_1). \quad (\text{D.12})$$

By the same argument, we also deduce that

$$\hat{P}^{\mathbf{x}} \left(\lim_{n \rightarrow \infty} X_{\tilde{\sigma}_n}^2 = X_{\tilde{\sigma}}^2, \tilde{\sigma} < \zeta_2 \right) = \hat{P}^{\mathbf{x}}(\tilde{\sigma} < \zeta_2). \quad (\text{D.13})$$

Therefore, we conclude from (D.5), (D.6), (D.12), and (D.13) that

$$\hat{P}^{\mathbf{x}} \left(\lim_{n \rightarrow \infty} \hat{X}_{\sigma_n} = \hat{X}_\sigma, \sigma < \zeta \right) = \hat{P}^{\mathbf{x}}(\sigma < \zeta),$$

which implies the quasi-left-continuity of $\hat{X}$ on $(0, \zeta)$. This completes the proof. $\square$

Remark D.5. By the same argument, one can verify that if both X^1 and X^2 are Hunt processes, then so is $\hat{X}$.

Below, we state that the DAC condition (Assumption 4.1) is inherited by the product process.

Proposition D.6. *In the setting of Theorem D.1, if, for each $i \in \{1, 2\}$, the process X^i satisfies Assumption 4.1 with a reference measure m^i and the heat kernel $p^i: \mathbb{R}_{>0} \times S_i \times S_i \rightarrow [0, \infty]$, then the product process $\hat{X}$ also satisfies Assumption 4.1 with the reference measure $m^1 \otimes m^2$ and the heat kernel $\hat{p}$ given by*

$$\hat{p}(t, (x_1, x_2), (y_1, y_2)) = p^1(t, x_1, y_1) p^2(t, x_2, y_2), \quad t > 0, (x_1, y_1), (x_2, y_2) \in S_1 \times S_2,$$

where we use the convention $0 \cdot \infty = 0$.

Proof. For each $i \in \{1, 2\}$, let Y^i denote the dual process of X^i . Then one can readily check that the product process $\hat{Y}$ of Y^1 and Y^2 is the dual process of $\hat{X}$. Verifying the remaining conditions regarding heat kernel is straightforward, so we omit the details. $\square$

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