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Olsen's inequality and its applications to Schrödinger equations

By

Yoshihiro SAWANO * and Satoko SUGANO **and Hitoshi TANAKA ***

Abstract

Morrey spaces turned out to be useful in that it grasps the subtle property of the fractional integral operators.

In 1995 Olsen obtained a bilinear estimate. Olsen applied his estimate called the Olsen inequality to the Schrödiger equation. We improve this estimate.

This Olsen inequality is a part of trace inequality that has kin relation with potential theory. Here we present some applications of our results to PDEs and the potential theory.

§ 1. Introduction

The aim of this note is to establish the following inequality.

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C \, \|g\|_{\mathcal{M}^{p_0}_p} \|f\|_{\mathcal{M}^{q_0}_q}, \tag{1.1}$$

where $\| \cdot \|_{\mathcal{M}^{p_0}_p}$ denotes the Morrey (quasi-)norm given by

$$\|f\|_{\mathcal{M}^{p_0}_p} = \sup_{Q \in \mathcal{D}} |Q|^{1/p_0 - 1/p} \left(\int_Q |f(y)|^p \, dy \right)^{1/p}, \; 0 < p \le p_0 < \infty \tag{1.2}$$

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*Department of Mathematics, Kyoto University, Kyoto University, Kyoto 606-8502, Japan.
e-mail: yosihiro@math.kyoto-u.ac.jp
**Kobe City College of Technology, 8-3 Gakuen-higashimachi, Nishi-ku, Kobe 651-2194, Japan.
e-mail: sugano@kobe-kosen.ac.jp
***Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan.
e-mail: htanaka@ms.u-tokyo.ac.jp

and I_α denotes the fractional integral operator defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,\ 0<\alpha<n. \tag{1.3}$$

Here $\mathcal{D}$ denotes the set of all dyadic cubes in $\mathbb{R}^n$. It is well known that I_2 is the inverse of $-\Delta$ modulo multiplicative constants. In this note we shall prove the following theorem on the fractional integral operator I_α.

Here is a series of equivalent definitions of Morrey norms.

$$\|f\|^{(1)}_{\mathcal{M}^p_q} = \sup_{Q \in \mathcal{Q}} |Q|^{1/p_0-1/p} \left(\int_Q |f(y)|^p\, dy \right)^{1/p}$$
$$\|f\|^{(2)}_{\mathcal{M}^p_q} = \sup_{Q \in \mathcal{Q}^*} |Q|^{1/p_0-1/p} \left(\int_Q |f(y)|^p\, dy \right)^{1/p},$$

where $\mathcal{Q}$ denotes the set of all cubes whose edges are parallel to the coordinate axis and $\mathcal{Q}^*$ denotes the set of all cubes whose edges are not always parallel to the coordinate axis. In the present paper we identify $\|f\|^{(1)}_{\mathcal{M}^p_q}$ with $\|f\|_{\mathcal{M}^p_q}$ but we do not use $\|f\|^{(2)}_{\mathcal{M}^p_q}$. If we formally let $p_0=\infty$, then we obtain the L^∞ space.

Theorem 1.1. *Let $0<\alpha<n$, $1<p\le p_0<\infty$, $1<q\le q_0<\infty$ and $1<r\le r_0<\infty$. Suppose that*

$$q>r,\ \frac{1}{p_0}>\frac{\alpha}{n},\ \frac{1}{q_0}\le\frac{\alpha}{n}, \tag{1.4}$$

and that

$$\frac{1}{r_0}=\frac{1}{q_0}+\frac{1}{p_0}-\frac{\alpha}{n},\ \frac{r}{r_0}=\frac{p}{p_0}. \tag{1.5}$$

Then

$$\|g\cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C\|g\|_{\mathcal{M}^{q_0}_q}\cdot\|f\|_{\mathcal{M}^{p_0}_p},$$

where the constant C is independent of f and g.

Remark. Hölder's inequality yields

$$L^{p_0}=\mathcal{M}^{p_0}_{p_0}\hookrightarrow\mathcal{M}^{p_0}_{p_1}\hookrightarrow\mathcal{M}^{p_0}_{p_2} \tag{1.6}$$

for all $p_0\ge p_1\ge p_2\ge 1$.

Here is a precise result by Olsen.

Theorem 1.2 ([10, Theorem 2]). *Let $0<\alpha<n$, $1<p\le p_0<\infty$, $1<q\le q_0<\infty$ and $1<r\le r_0<\infty$. Suppose that*

$$q>r,\ \frac{1}{p_0}>\frac{\alpha}{n},\ \frac{1}{q_0}\le\frac{\alpha}{n}, \tag{1.7}$$

and that

$$\frac{1}{r_0}=\frac{1}{q_0}+\frac{1}{p_0}-\frac{\alpha}{n},\ \frac{1}{r}=\frac{1}{q_0}+\frac{1}{p}-\frac{\alpha}{n}. \tag{1.8}$$

Then

$$\|g\cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r}\le C\|g\|_{\mathcal{M}^{q_0}_q}\cdot\|f\|_{\mathcal{M}^{p_0}_p},$$

where the constant C is independent of f and g.

However, this result is not sharp as the following calculation shows.

Remark. Using naively the Adams theorem [1] and Hölder's inequality, one can prove a minor part of q in Theorem 1.1. That is, the proof of Theorem 1.1 is fundamental provided $\frac{p}{p_0}q_0\le q\le q_0$. Indeed, by virtue of the Adams theorem we have, for any cube $Q\in\mathcal{Q}$,

$$|Q|^{1/s_0}\left(\frac{1}{|Q|}\int_Q|I_\alpha f(x)|^s\,dx\right)^{1/s}\le C\|f\|_{\mathcal{M}^{p_0}_p},\quad \frac{1}{s}=\frac{p_0}{p}\frac{1}{s_0},\ \frac{1}{s_0}=\frac{1}{p_0}-\frac{\alpha}{n}. \tag{1.9}$$

The condition $\frac{r}{r_0}=\frac{p}{p_0}$, $\frac{1}{r_0}=\frac{1}{q_0}+\frac{1}{p_0}-\frac{\alpha}{n}$ reads

$$\frac{1}{r}=\frac{p_0}{p}\left(\frac{1}{q_0}+\frac{1}{p_0}-\frac{\alpha}{n}\right)=\frac{p_0}{p}\frac{1}{q_0}+\frac{1}{s}.$$

These yield

$$|Q|^{1/q_0+1/s_0}\left(\frac{1}{|Q|}\int_Q|g(x)I_\alpha f(x)|^r\ dx\right)^{1/r}\le C\|g\|_{\mathcal{M}^{q_0}_q}\|f\|_{\mathcal{M}^{p_0}_p}$$

if $\frac{r}{r_0}=\frac{p}{p_0}=\frac{q}{q_0}$. In view of the inclusion (1.6), the same can be said when

$$\frac{p}{p_0}q_0\le q\le q_0.$$

Also observe that $\frac{1}{r_0}=\frac{1}{q_0}+\frac{1}{p_0}-\frac{\alpha}{n}>\frac{1}{q_0}$. Hence we have $q_0>r_0$. Thus, since the condition $q>r$, Theorem 1.1 is significant only when $\frac{p}{p_0}r_0<q<\frac{p}{p_0}q_0$.

We generalized Theorem 1.1 in our subsequent papers [12, 13]. The motivation stemed from the earlier works due to Sugano and Tanaka [17, 18]. To prove them we used some auxiliary results of maximal operators, which strenghten those by Nakai [8, 9] (see Lemmas 8.1 and 8.2).

As we have established in [14], the fractional maximal operater I_α is not surjective from $\mathcal{M}^p_q$ to $\mathcal{M}^s_t$ if

$$1 < q \le p < \infty,\ 1 < t \le s < \infty,\ \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n},\ \frac{q}{p} = \frac{t}{s}. \tag{1.10}$$

Therefore, one can expect such a strange phenomenon like Theorem 1.1.

§ 2. Fundamentals on Morrey spaces

Here we shall present several examples of the functions in Morrey spaces. Here and below given a cube Q and $\kappa > 0$, we denote by κQ the cube concentric to Q and having volume κ^n.

Example Let $1 < q < q_0 < \infty$. Then $|x|^{-n/q_0} \in \mathcal{M}^{q_0}_q$.

The following proposition is due to Pérez [11]. It seems that we can construct counterexamples by using the following proposition.

Proposition 2.1. *Let $p_0 > p > 0$ and $R > 1$ be fixed so that*

$$(R+1)^{-1/p_0} = 2^{1/p}(1+R)^{-1/p}. \tag{2.1}$$

For $\varepsilon = \{\varepsilon_a\}_{a=1}^n \in \{0,1\}^n$, we define an Affine transformation T_ε by

$$T_\varepsilon(x) = \frac{1}{R+1}x + \frac{R}{R+1}\varepsilon \quad (x \in \mathbb{R}^n). \tag{2.2}$$

Let $E_0 = [0,1]^n$. Suppose that we have defined $E_0, E_1, E_2, \cdots, E_j$. Define

$$E_{j+1} = \bigcup_{\varepsilon \in \{0,1\}^n} T_\varepsilon(E_j). \tag{2.3}$$

Then we have

$$\|\chi_{E_j}\|_{\mathcal{M}^{p_0}_p} \sim (1+R)^{-jn/p_0}, \tag{2.4}$$

where the implicit constants in $\sim$ does not depend on j but can depend on p, p_0.

Before the proof we make a preparatory observation. Note that

$$\|E_{j,0}\|_{L^{p_0}} = \|E_j\|_{L^p}(1+R)^{-jn/p_0}, \tag{2.5}$$

where $E_{j,0} = [0,(1+R)^{-j}]^n$.

Proof. Let us calculate

$$\|\chi_{E_j}\|_{\mathcal{M}^{p_0}_p} \sim \sup_{S \in \mathcal{Q}} |S|^{1/p_0 - 1/p}|S \cap E_j|^{1/p}, \tag{2.6}$$

where $\mathcal{Q}$ denotes the set of all cubes.

Let us temporally say that $Q \in \mathcal{Q}$ is wasteful, if there exists a cube $S \in \mathcal{Q}$ such that

$$|Q|^{1/p_0-1/p}|Q \cap E_j|^{1/p} < |S|^{1/p_0-1/p}|S \cap E_j|^{1/p}. \tag{2.7}$$

Thus, by definition, any cube is clearly wasteful unless it is contained in $[0,1]^n$. Also, if the side-length of a cube Q is less than $(R+1)^{-j}$, then it is wasteful. Indeed, then the equality

$$\begin{aligned}\sup_{Q\,:\,cube\,:\,|Q|\le(R+1)^{-jn}} |Q|^{1/p_0-1/p}|Q\cap E_j|^{1/p} &= \sup_{Q\,:\,cube\,:\,Q\subset E_j} |Q|^{1/p_0-1/p}|Q\cap E_j|^{1/p}\\ &= \sup_{Q\,:\,cube\,:\,Q\subset E_j} |Q|^{1/p_0}\\ &= |E_{j,0}|^{1/p_0}\end{aligned}$$

holds. Thus, if the cube Q is not wasteful and $(R+1)^{-kn} \le |Q| \le (R+1)^{-(k-1)n}$, then $\dfrac{R+1}{R-1}Q$ contains a connected component of E_k. Hence it follows that

$$\begin{aligned}&\|\chi_{E_j}\|_{\mathcal{M}^{p_0}_p}\\ &\sim \sup\{|Q|^{1/p_0-1/p}|Q\cap E_j|^{1/p}\,:\, Q \text{ contains a connected component of } E_k\,\}.\end{aligned}$$

Let us write

$$\alpha = \sup\left\{\frac{|S\cap E_j|^{1/p}}{|S|^{1/p-1/p_0}} \,:\, S \text{ contains a connected component of } E_k\right\}. \tag{2.8}$$

Let S be a cube which contains a connected component of E_j. We define

$$S^* = \operatorname{co}\left(\bigcup\{W \,:\, W \text{ is a connected component of } E_j \text{ intersecting } S\}\right), \tag{2.9}$$

where $\operatorname{co}(A)$ denotes the smallest convex set containing a set A. Then a geometric observation shows that S^* engulfs k^n connected component of E_j for some $1 \le k \le 2^j$. Take an integer l so that $2^{l-1} \le k \le 2^l$. Then we have

$$|S^*\cap E_j| = k^n(1+R)^{-jn},\ |S^*| = (1+R)^{-jn+ln}. \tag{2.10}$$

Consequently, we obtain

$$\begin{aligned}|S^*|^{1/p_0-1/p}|S^*\cap E_j|^{1/p} &\sim 2^{ln/p}(1+R)^{-jn/p}(1+R)^{(-j+l)(n/p_0-n/p)}\\ &\sim 2^{ln/p}(1+R)^{-jn/p_0+l(n/p_0-n/p)}\\ &\sim (1+R)^{-jn/p_0}.\end{aligned}$$

Thus, we obtain (2.4). $\square$

We remark that this example is essentially employed in our paper [12] to prove the sharpness of our result.

Proposition 2.2. *Let $0 < \alpha < n$, $1 < q \le p < \infty$ and $1 < t \le s < \infty$. If there exists a constant $C > 0$*

$$\|I_\alpha f\|_{\mathcal{M}^s_t} \le C\|f\|_{\mathcal{M}^p_q} \tag{2.11}$$

for all positive measurable functions f, it is necessary and sufficient that

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n},\ \frac{t}{s} \le \frac{q}{p}. \tag{2.12}$$

Proof. The sufficiency is well known as Adam's theorem and we will use in this lecture. For the necessity, we first obtain

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n} \tag{2.13}$$

by the dilation

$$\|f(\lambda\cdot)\|_{\mathcal{M}^p_q} = \lambda^{-\frac{n}{p}}\|f\|_{\mathcal{M}^p_q},\ \|f(\lambda\cdot)\|_{\mathcal{M}^s_t} = \lambda^{-\frac{n}{s}}\|f\|_{\mathcal{M}^s_t},\ I_\alpha[f(\lambda\cdot)] = \lambda^{-\alpha}I_\alpha f(\lambda\cdot) \tag{2.14}$$

for $\lambda > 0$. Consequently, we obtain

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}. \tag{2.15}$$

We may assume that $q < p$ for the purpose of establishing $\dfrac{t}{s} = \dfrac{q}{p}$. Let $R > 1$ be the solution of $(1+R)^{-\frac{1}{p}} = 2^{\frac{1}{q}}(1+R)^{-\frac{1}{q}}$. Then we have

$$\|\chi_{E_j}\|_{\mathcal{M}^p_q} \sim (1+R)^{-jn/p}. \tag{2.16}$$

If we use E_j above, then we have

$$I_\alpha\chi_{E_j}(x) \ge c\,(R+1)^{-j\alpha}\chi_{E_j}(x). \tag{2.17}$$

Indeed, as the equality

$$I_\alpha\chi_{E_j}(x) = \int_{x-E_j}\frac{dy}{|y|^{n-\alpha}}, \quad x - E_j := \{x - y\,:\,y \in E_j\} \tag{2.18}$$

shows, $I_\alpha\chi_{E_j}$ is continuous. So we can take $c = \min\limits_{x\in[0,1]^n} I_\alpha\chi_{E_j}(x)$.

Also, from the definition of $\mathcal{M}^s_t$, we have

$$(R+1)^{-\frac{jnq}{pt}} = \|\chi_{E_j}\|_{L^t} \le \|\chi_{E_j}\|_{\mathcal{M}^s_t} \tag{2.19}$$

from the defintion of the Morrey norm. Consequently, it follows that

$$(R+1)^{-\frac{jnq}{pt}} \le C\,(R+1)^{j\alpha}\|\chi_{E_j}\|_{\mathcal{M}^p_q} = C\,(R+1)^{j\left(\alpha-\frac{n}{p}\right)} = C\,(R+1)^{-\frac{jn}{s}}.$$

Since $j \in \mathbb{N}$ is arbitrary, it follows that $t \leq \dfrac{qs}{p}$. $\square$

§ 3. Boundedness of the operators on Morrey spaces

Now we present a typical argument proving the Morrey-boundedness. We accept

$$\|Mf\|_q \leq C\,\|f\|_q,\ 1 < q \leq \infty, \tag{3.1}$$

where M denotes the Hardy-Littlewood maximal operator given by

$$Mf(x) = \sup_{x \in Q;cube} \frac{1}{|Q|} \int_Q |f(y)|\,dy. \tag{3.2}$$

Here and below given a cube Q, we define

$$\ell(Q) = |Q|^{1/n}. \tag{3.3}$$

Theorem 3.1. *Let $1 < q \leq p < \infty$. Then there exists $C_{p,q}$ such that*

$$\|Mf\|_{\mathcal{M}^p_q} \leq C_{p,q}\|f\|_{\mathcal{M}^p_q} \tag{3.4}$$

for all $f \in \mathcal{M}^p_q$.

Proof. For the proof we shall show, from the definition, that

$$|Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q Mf(y)^q\,dy \right)^{\frac{1}{q}} \leq C_{p,q}\|f\|_{\mathcal{M}^p_q}, Q \in \mathcal{Q}. \tag{3.5}$$

Once this is achieve, we have only to consider the supremum over all Q.

Write $f = f_1 + f_2$, where $f_1 = f$ on $5Q$ and $f_2 = f$ outside $5Q$. The estimate of (3.5) can be decomposed into

$$|Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q Mf_1(y)^q\,dy \right)^{\frac{1}{q}} \leq C_{p,q}\|f\|_{\mathcal{M}^p_q}, \tag{3.6}$$

$$|Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q Mf_2(y)^q\,dy \right)^{\frac{1}{q}} \leq C_{p,q}\|f\|_{\mathcal{M}^p_q} \tag{3.7}$$

by virtue of the triangle inequality.

As we have seen in (3.1), M is L^q-bounded. Thus (3.6) can be shown easily;

$$\begin{aligned}
|Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q Mf_1(y)^q\,dy \right)^{\frac{1}{q}} &\leq |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\mathbb{R}^n} Mf_1(y)^q\,dy \right)^{\frac{1}{q}} \\
&\leq C_{p,q}|Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{5Q} |f(y)|^q\,dy \right)^{\frac{1}{q}} \\
&\leq C_{p,q}|5Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{5Q} |f(y)|^q\,dy \right)^{\frac{1}{q}} \\
&\leq C_{p,q}\|f\|_{\mathcal{M}^p_q}.
\end{aligned}$$

Thus, we obtain (3.6).

To prove (3.7) we have to keep in mind the following fundamental geometric observation.

If $R \in \mathcal{Q}$ is a cube that meets both Q and $\mathbb{R}^n \setminus 5Q$, then $\ell(R) \geq 2\ell(Q)$ and $2R \supset Q$.

This geometric observation yields

$$Mf_2(y) \leq 2^n \sup_{Q \subset R \in \mathcal{Q}} \frac{1}{|R|} \int_R |f(y)|\,dy. \tag{3.8}$$

If we insert this inequality to (3.7), then we obtain

$$|Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q Mf_2(y)^q\,dy \right)^{\frac{1}{q}} \leq 2^n |Q|^{\frac{1}{p}} \sup_{Q \subset R \in \mathcal{Q}} \frac{1}{|R|} \int_R |f(y)|\,dy.$$

Taking into account $\mathcal{M}^p_q \subset \mathcal{M}^p_1$, we see that

$$|Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q Mf_2(y)^q\,dy \right)^{\frac{1}{q}} \leq 2^n \sup_{R \in \mathcal{Q}} |R|^{\frac{1}{p}-1} \int_R |f(y)|\,dy = 2^n \|f\|_{\mathcal{M}^p_1} \leq 2^n \|f\|_{\mathcal{M}^p_q}. \tag{3.9}$$

Estimate (3.7) is therefore proved. With (3.6) and (3.7) proved, we obtain (3.4). $\square$

We now turn to the boundedness of the fractional integral operators. The following lemma is named the comparison lemma by Volberg.

Lemma 3.2 (Comparison lemma)**.** *Let* $0 < \alpha < n$.

$$\int_0^\infty \chi_{B(x,l)}(y) \frac{dl}{l^{n+1-\alpha}} = \frac{|x-y|^{-n+\alpha}}{n-\alpha}. \tag{3.10}$$

Proof. The proof is simple. Indeed, we have

$$\int_0^\infty \chi_{B(x,l)}(y) \frac{dl}{l^{n+1-\alpha}} = \int_0^\infty \chi_{(|x-y|,\infty)}(l) \frac{dl}{l^{n+1-\alpha}} \tag{3.11}$$

$$= \int_{|x-y|}^\infty \frac{dl}{l^{n+1-\alpha}} \tag{3.12}$$

$$= \frac{|x-y|^{-n+\alpha}}{n-\alpha}, \tag{3.13}$$

which is the desired result. $\square$

We use a notation; if we are given a cube Q and a locally integrable function f, then we write

$$m_Q(f) = \frac{1}{|Q|} \int_Q f(x)\,dx,$$

the average of f over a cube Q.

Morrey spaces, the BMO space and Hölder spaces lie in a line. More precisely we have the following.

Theorem 3.3 (I_α and the modified fractional maximal operators $\tilde{I}_\alpha$). *Suppose that* $0<\alpha<n$. *We define*

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \tag{3.14}$$

$$\tilde{I}_\alpha f(x) := \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{\chi_{{Q_0}^c}(y)}{|x_0-y|^{n-\alpha}} \right) f(y)\,dy, \tag{3.15}$$

where Q_0 *is a fixed cube centered at* x_0.

1. (Subcritical case) *Let* $p<\dfrac{n}{\alpha}$. *Assume that the parameters* s,t *satisfy*

$$1<q\le p<\infty,\ 1<t\le s<\infty,\ \frac{1}{s}=\frac{1}{p}-\frac{\alpha}{n},\ \frac{t}{s}=\frac{q}{p}. \tag{3.16}$$

Then

$$\|I_\alpha f\|_{\mathcal{M}^s_t} \le C_{p,q,\alpha}\|f\|_{\mathcal{M}^p_q} \tag{3.17}$$

for every (positive) $f\in\mathcal{M}^p_q$.

2. (Critical case) *Assume that* $1\le q\le p=\dfrac{n}{\alpha}$. *Then*

$$\|\tilde{I}_\alpha f\|_* \le C_{p,q,\alpha}\|f\|_{\mathcal{M}^p_q} \tag{3.18}$$

for every $f\in\mathcal{M}^p_q$, *where* $\|\cdot\|_*$ *denotes the* BMO *norm given by*

$$\|g\|_* = \sup_{Q\in\mathcal{Q}} m_Q(|g-m_Q(g)|). \tag{3.19}$$

3. (Supercritical case) *Assume that* $1\le q\le p<\infty$ *and that* $0<\alpha-\dfrac{n}{p}<1$. *Then*

$$\|\tilde{I}_\alpha f\|_{\mathrm{Lip}\left(\alpha-\frac{n}{p}\right)} \le C_{p,q,\alpha}\|f\|_{\mathcal{M}^p_q} \tag{3.20}$$

for every $f\in\mathcal{M}^p_q$, *where we denoted*

$$\|g\|_{\mathrm{Lip}(\theta)} = \sup\left\{ \frac{|g(x)-g(y)|}{|x-y|^\theta} \,:\, x,y\in\mathbb{R}^n,\ x\ne y \right\},\quad 0<\theta<1 \tag{3.21}$$

for a function g.

Before we come to the proof of this theorem, let us remark that the integral kernel of $\tilde{I}_\alpha$ is better than that of I_α. With the better kernel, we can apply $\tilde{I}_\alpha$ for $\mathcal{M}_q^{n/\alpha}$ functions.

Here we content ourselves with proving (2), other results being proved similarly.

Proof. We have to prove

$$m_Q(|\tilde{I}_\alpha f - m_Q(\tilde{I}_\alpha f)|) \le c \, \|f\|_{\mathcal{M}_q^{n/\alpha}} \tag{3.22}$$

for all cubes Q. For the proof we may assume $q < \dfrac{n}{\alpha}$ because we always have

$$\|f\|_{\mathcal{M}_q^{n/\alpha}} \le \|f\|_{\mathcal{M}_{n/\alpha}^{n/\alpha}} = \|f\|_{L^{n/\alpha}}. \tag{3.23}$$

We decompose f according to $2Q$ as usual. That is, we split $f = f_1 + f_2$ with $f_1 = \chi_{2Q} \cdot f$ and $f_2 = f - f_1$. By virtue of the triangle inequality our present task is partitioned into proving

$$m_Q(|\tilde{I}_\alpha f_1 - m_Q(\tilde{I}_\alpha f_1)|) + m_Q(|\tilde{I}_\alpha f_2 - m_Q(\tilde{I}_\alpha f_2)|) \le c \, \|f\|_{\mathcal{M}_q^{n/\alpha}}. \tag{3.24}$$

Then the estimate for f_1 is simple. Indeed, to estimate f_1, we define an auxiliary index $w \in (q, \infty)$ by $\dfrac{1}{w} = \dfrac{1}{q} - \dfrac{\alpha}{n}$. By the triangle inequality and the Hölder inequality, we have

$$m_Q(|\tilde{I}_\alpha f_1 - m_Q(\tilde{I}_\alpha f_1)|) \le 2 m_Q(|\tilde{I}_\alpha f_1|) \le 2 m_Q^{(w)}(|\tilde{I}_\alpha f_1|), \tag{3.25}$$

where we wrote

$$m_Q^{(w)}(F) := \left(\frac{1}{|Q|} \int_Q F(x)^w \, dx \right)^{\frac{1}{w}} \tag{3.26}$$

for positive measurable functions F. By using the L^q-L^w boundedness of the fractional integral operator we obtain

$$\begin{aligned} m_Q(|\tilde{I}_\alpha f_1 - m_Q(\tilde{I}_\alpha f_1)|) &\le c \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} |\tilde{I}_\alpha f_1(x)|^w \, dx \right)^{1/w} \\ &\le c \, m_{2Q}(|f|^q)^{1/w} \\ &\le c \, \|f\|_{\mathcal{M}_q^{n/\alpha}}. \end{aligned}$$

For the proof of the second inequality, we write the left-side out in full.

$$\begin{aligned} & m_Q(|\tilde{I}_\alpha f_2 - m_Q(\tilde{I}_\alpha f_2)|) \\ &= \frac{1}{|Q|^2} \int_Q \left| \iint_{Q \times (\mathbb{R}^n \setminus 2Q)} \left(\frac{f(z)}{|x-z|^{n-\alpha}} - \frac{f(z)}{|y-z|^{n-\alpha}} \right) dy \, dz \right| dx. \end{aligned}$$

First, we bound the right-hand side with the triangle inequality and arrange it. Then the right-hand side is majorized by

$$\frac{1}{|Q|^2}\iiint_{Q\times Q\times(\mathbb{R}^n\setminus 2Q)}|f(z)|\cdot\left|\frac{1}{|x-z|^{n-\alpha}}-\frac{1}{|y-z|^{n-\alpha}}\right|dx\,dy\,dz. \tag{3.27}$$

Denote by $c(Q)$ the center of the cube Q. By virtue of the mean value theorem, we have

$$\left|\frac{1}{|x-z|^{n-\alpha}}-\frac{1}{|y-z|^{n-\alpha}}\right|\le c\,\frac{|x-y|}{|z-c(Q)|^{n-\alpha+1}}\le c\,\frac{\ell(Q)}{|z-c(Q)|^{n-\alpha+1}}. \tag{3.28}$$

Thus, inserting this inequality gives us

$$m_Q(|\tilde{I}_\alpha f_2-m_Q(\tilde{I}_\alpha f_2)|)\le c\,\ell(Q)\int_{\mathbb{R}^n\setminus 2Q}\frac{|f(z)|}{|z-c(Q)|^{n-\alpha+1}}\,dz. \tag{3.29}$$

By the comparison lemma, the integral of the right-hand side is bounded by

$$\int_{2\ell(Q)}^{\infty}\left(\int_{B(c(Q),\ell)}|f(z)|\,dz\right)\frac{d\ell}{\ell^{n-\alpha+2}}\le c\,\|f\|_{\mathcal{M}_1^{n/\alpha}}\cdot\int_{\ell(Q)}^{\infty}\frac{d\ell}{\ell^2}=c\frac{\|f\|_{\mathcal{M}_1^{n/\alpha}}}{\ell(Q)}\le c\frac{\|f\|_{\mathcal{M}_q^{n/\alpha}}}{\ell(Q)}. \tag{3.30}$$

Thus, the estimate of the second inequality is complete and the proof is concluded. □

§ 4. Proof of Theorem 1.1

In [10] the proof was rather lengthy. But our proof is a little shorter than that by Olsen.

We need the following observation below. This lemma dates back to [18] for example.

Lemma 4.1. *For a nonnegative function h in $L^\infty(Q_0)$ we let $\gamma_0=m_{Q_0}(h)$ and $c=2^{n+1}$. For $k=1,2,\dots$ let*

$$D_k:=\bigcup\{Q\,:\,Q\in\mathcal{D}_1(Q_0):\,m_Q(h)>\gamma_0c^k\}\subset\mathbb{R}^n.$$

(0) *Define*

$$\mathcal{Z}_k=\left\{Q\,:\,Q\in\mathcal{D}_1(Q_0):\,m_Q(h)>\gamma_0c^k\right\}. \tag{4.1}$$

Considering the maximal cubes in $\mathcal{Z}_k$ with respect to inclusion, we can write

$$D_k=\bigcup_j Q_{k,j},$$

where the cubes $\{Q_{k,j}\}\subset\mathcal{D}_1(Q_0)$ are nonoverlapping.

(1) *By virtue of the maximality of* $Q_{k,j}$ *one has that*

$$\gamma_0 c^k < m_{Q_{k,j}}(h) \le 2^n \gamma_0 c^k.$$

(2) *Let*

$$E_0 = Q_0 \setminus D_1 \text{ and } E_{k,j} = Q_{k,j} \setminus D_{k+1}.$$

Then $\{E_0\} \cup \{E_{k,j}\}$ *is a disjoint family of sets which decomposes* Q_0 *and satisfies*

$$|Q_0| \le 2|E_0| \text{ and } |Q_{k,j}| \le 2|E_{k,j}|. \tag{4.2}$$

(3) *Also, we set*

$$\mathcal{D}_0 := \{Q \in \mathcal{D}_1(Q_0) : m_Q(h) \le \gamma_0 c\} \subset \mathcal{D}$$
$$\mathcal{D}_{k,j} := \left\{Q \in \mathcal{D}_1(Q_0) : Q \subset Q_{k,j}, \gamma_0 c^k < m_Q(h) \le \gamma_0 c^{k+1}\right\} \subset \mathcal{D}.$$

Then $\mathcal{D}_1(Q_0)$ *is partitioned as follows*:

$$\mathcal{D}_1(Q_0) = \mathcal{D}_0 \cup \bigcup_{k,j} \mathcal{D}_{k,j}. \tag{4.3}$$

Proof. We choose $Q_{k,j}$ as is indicated in (0).

(1) The left inequality is a consequence of the fact that we chose $Q_{k,j}$ from $\mathcal{Z}_k$. If we consider the dyadic parent R of $Q_{k,j}$, then we have

$$m_R(h) \le \gamma_0 c^k. \tag{4.4}$$

Otherwise, instead of $Q_{k,j}$ R would have been chosen as an element of $\mathcal{Z}_k$. Since

$$m_{Q_{j,k}}(h) \le 2^n m_R(h), \tag{4.5}$$

we obtain the right inequality.

(2) A geometric observation shows that two dyadic cube never intersect unless one is not included in the other. Let j and k be freezed. We write

$$D_{k+1} = \bigcup_{j^*} Q_{k+1,j^*}, \quad Q_{k+1,j^*} \in \mathcal{Z}_{k+1}. \tag{4.6}$$

¿From the observation above, we have

$$D_{k+1} \cap Q_{k,j} = \bigcup \{Q_{k+1,j^*} \ : Q_{k+1,j^*} \subset Q_{k,j}\} \tag{4.7}$$

because $Q_{k+1,j^*} \supsetneq Q_{k,j}$ never happens thanks to the maximality of $Q_{k,j}$. Note that

$$|Q_{k+1,j^*}|\gamma_0 c^{k+1} \leq \int_{Q_{k+1,j^*}} h(x)\,dx \tag{4.8}$$

since $Q_{k+1,j^*} \in \mathcal{Z}_{k+1}$. If we add this estimate for all j^* such that $Q_{k+1,j^*} \subset Q_{k,j}$, we obtain

$$|D_{k+1} \cap Q_{k,j}|\gamma_0 c^{k+1} \leq \int_{Q_{k,j}} h(x)\,dx \leq |Q_{k,j}|2^n\gamma_0. \tag{4.9}$$

So, if we consider the complement set of D_{k+1}, then we obtain the desired result.

(3) By maximality if $R \subset Q_0$ is a dyadic cube such that $m_R(h) > \gamma_0$, then R is contained in some $Q_{k,j}$. So this assertion follows.

□

We begin by discretizing the operator $I_\alpha f$ following the idea of C. Pérez (see [11]):

$$I_\alpha f(x) = \sum_{\nu\in\mathbb{Z}} \int_{2^{-\nu-1}<|x-y|\leq 2^{-\nu}} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \leq C\sum_{\nu\in\mathbb{Z}} 2^{\nu(n-\alpha)} \int_{B(x,2^{-\nu})} f(y)\,dy.$$

Denote by $\mathcal{D}_\nu$ the set of all dyadic cubes of volume $2^{-\nu n}$;

$$\mathcal{D}_\nu = \{2^{-\nu}m + 2^{-\nu}[0,1)^n \,:\, m \in \mathbb{Z}^n\}. \tag{4.10}$$

Then we have

$$\begin{aligned} I_\alpha f(x) &\leq C\sum_{\nu\in\mathbb{Z}}\sum_{x\in Q\in\mathcal{D}_\nu} \frac{\ell(Q)^\alpha}{|Q|}\int_{3Q} f(y)\,dy \\ &= C\sum_{Q\in\mathcal{D}} \frac{\ell(Q)^\alpha}{|Q|}\int_{3Q} f(y)\,dy\cdot\chi_Q(x) \\ &= C\sum_{Q\in\mathcal{D}} \ell(Q)^\alpha m_{3Q}(f)\cdot\chi_Q(x). \end{aligned}$$

It suffices, from the definition of the Morrey norm, to show that

$$\left(\int_{Q_0} (g(x)I_\alpha f(x))^r\,dx\right)^{1/r} \leq C\|g\|_{\mathcal{M}^{q_0}_q}\cdot\|M_\alpha f\|_{\mathcal{M}^{r_0}_r}\cdot|Q_0|^{1/r-1/r_0},$$

for a given dyadic cube Q_0, where M_α denotes the fractional maximal operator given by

$$M_\alpha f(x) = \sup_{x\in Q\in\mathcal{Q}} |Q|^{\alpha/n}\left(\frac{1}{|Q|}\int_Q |f(y)|\,dy\right). \tag{4.11}$$

Indeed, a geometric observation shows

$$|Q|^{\alpha/n-1}\chi_Q(y) \le c|x-y|^{-n+\alpha}, \quad x \in Q. \tag{4.12}$$

So the pointwise estimate $M_\alpha f(x) \le c I_\alpha f(x)$ follows. If we invoke the Adams theorem, then we obtain the desired result. Furthermore, by a simple limiting argument, we can assume that $g \in L^\infty$. Hereafter, we write

$$\begin{cases} \mathcal{D}_1(Q_0) = \{Q \in \mathcal{D} :\, Q \subset Q_0\}, \\ \mathcal{D}_2(Q_0) = \{Q \in \mathcal{D} :\, Q \supsetneq Q_0\}. \end{cases}$$

Let us define for $i = 1, 2$

$$F_i(x) = \sum_{Q \in \mathcal{D}_i(Q_0)} \ell(Q)^\alpha m_{3Q}(f)\chi_Q(x)$$

and we shall estimate

$$\left(\int_{Q_0} (g(x)F_i(x))^r \, dx\right)^{1/r}.$$

First, we establish

$$\left(\int_{Q_0} (g(x)F_1(x))^r \, dx\right)^{1/r} \le C\|g\|_{\mathcal{M}^{q_0}_q} \left(\int_{Q_0} M_{\alpha-n/q_0}f(x)^r \, dx\right)^{1/r}, \tag{4.13}$$

by the duality argument. Take a nonnegative function $w \in L^{r'}$, $1/r+1/r' = 1$, satisfying that w is supported on $\overline{Q_0}$, that $\|w\|_{L^{r'}(Q_0)} = 1$ and that

$$\left(\int_{Q_0} (g(x)F_1(x))^r \, dx\right)^{1/r} = \int_{Q_0} g(x)F_1(x)w(x)\, dx.$$

Letting $h = g \cdot w$, we shall apply Lemma 4.1 to estimate this quantity. It follows that

$$\begin{aligned} \int_{Q_0} g(x)F_1(x)w(x)\, dx &= \sum_{Q \in \mathcal{D}_1(Q_0)} \ell(Q)^\alpha m_{3Q}(f) \int_Q g(x)w(x)\, dx \\ &= \sum_{Q \in \mathcal{D}_0} \ell(Q)^\alpha m_{3Q}(f) \int_Q g(x)w(x)\, dx \\ &\quad + \sum_{j=1}^\infty \sum_k \sum_{Q \in \mathcal{D}_{j,k}} \ell(Q)^\alpha m_{3Q}(f) \int_Q g(x)w(x)\, dx. \end{aligned}$$

First, we evaluate

$$\sum_{Q \in \mathcal{D}_{k,j}} \ell(Q)^\alpha m_{3Q}(f) \int_Q g(x)w(x)\, dx. \tag{4.14}$$

In view of the definition of $\mathcal{D}_{k,j}$, we have

$$\text{R.H.S. of (4.14)} = \sum_{l=0}^{\infty} \sum_{\substack{Q \in \mathcal{D}_{k,j} \\ \ell(Q)=2^{-l}\ell(Q_{k,j})}} \ell(Q)^{\alpha} m_{3Q}(f) \int_Q g(x)w(x)\,dx$$

$$= 3^n \sum_{l=0}^{\infty} \sum_{\substack{Q \in \mathcal{D}_{k,j} \\ \ell(Q)=2^{-l}\ell(Q_{k,j})}} \ell(Q)^{\alpha} m_{Q}(gw) \int_{3Q} f(x)\,dx$$

Let $Q \in \mathcal{D}_{k,j}$. Now that we have

$$\gamma_0 c^k \le m_Q(gw) \le \gamma_0 c^{k+1},\ \gamma_0 c^k \le m_{Q_{k,j}}(gw) \le 2^n \gamma_0 c^k,\ Q \subset Q_{k,j}, \tag{4.15}$$

we obtain $m_Q(gw) \sim m_{Q_{k,j}}(gw)$. Hence it follows that

$$\text{R.H.S. of (4.14)} \le C_{p,q,\alpha} m_{Q_{k,j}}(gw) \sum_{l=0}^{\infty} \sum_{\substack{Q \in \mathcal{D}_{k,j} \\ \ell(Q)=2^{-l}\ell(Q_{k,j})}} \ell(Q)^{\alpha} \int_{3Q} f(x)\,dx$$

$$\le C_{p,q,\alpha} \sum_{l=0}^{\infty} 2^{-l\alpha} \ell(Q_{k,j})^{\alpha} m_{Q_{k,j}}(gw) \int_{3Q_{k,j}} f(x)\,dx$$

$$\le C_{p,q,\alpha} \ell(Q_{k,j})^{\alpha} m_{Q_{k,j}}(gw) \int_{3Q_{k,j}} f(x)\,dx.$$

In summary, we have obtained

$$\sum_{Q \in \mathcal{D}_{k,j}} \ell(Q)^{\alpha} m_{3Q}(f) \int_Q g(x)w(x)\,dx \le C_{p,q,\alpha} \ell(Q_{k,j})^{\alpha} m_{3Q_{k,j}}(f)\, m_{Q_{k,j}}(g\,w)\,|E_{k,j}|. \tag{4.16}$$

Using Hölder's inequality, we have

$$m_{Q_{k,j}}(g\,w) \le m^{(q)}_{Q_{k,j}}(g) m^{(q')}_{Q_{k,j}}(w), \tag{4.17}$$

and

$$m^{(q)}_{Q_{k,j}}(g) \le \|g\|_{\mathcal{M}^{q_0}_q} \ell(Q_{k,j})^{-\frac{n}{q_0}} \tag{4.18}$$

We denote

$$M^{(q')}w(x) = \sup_{x \in Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \int_Q w(x)^{q'}\,dx \right)^{1/q'}. \tag{4.19}$$

Estimates (4.17) and (4.18) yield

$$(4.16) \le C\|g\|_{\mathcal{M}^{q_0}_q} \ell(Q_{k,j})^{\alpha - \frac{n}{q_0}} m_{3Q_{k,j}}(f) m^{(q')}_{Q_{k,j}}(w) |E_{k,j}|$$

$$\le C\|g\|_{\mathcal{M}^{q_0}_q} \ell(Q_{k,j})^{\alpha - \frac{n}{q_0}} m_{3Q_{k,j}}(f) |E_{k,j}| \inf_{x \in E_{k,j}} M^{(q')}w(x)$$

$$\le C\|g\|_{\mathcal{M}^{q_0}_q} \int_{E_{k,j}} M_{\alpha - n/q_0} f(x)\, M^{(q')}w(x)\,dx.$$

Similarly, we have

$$\sum_{Q \in \mathcal{D}_0} \ell(Q)^\alpha m_{3Q}(f) \int_Q g(x)w(x)\,dx \le C\|g\|_{\mathcal{M}^{q_0}_q} \int_{E_0} M_{\alpha-n/q_0}f(x)\, M^{(q')}w(x)\,dx.$$

Summing up all factors we obtain

$$(4.14) \le C\|g\|_{\mathcal{M}^{q_0}_{q_0}} \int_{Q_0} M_{\alpha-n/q_0}f(x)\, M^{({q_0}')}w(x)\,dx.$$

Another application of Hölder's inequality gives us that

$$(4.14) \le C\|g\|_{\mathcal{M}^{q_0}_q} \left(\int_{Q_0} M_{\alpha-n/q_0}f(x)^r\,dx\right)^{1/r} \left(\int_{Q_0} M^{(q_0')}w(x)^{r'}\,dx\right)^{1/r'}.$$

The fact $r' > q'$ and the $L^{r'/q'}$-boundedness of maximal operator M yield

$$\begin{aligned}(4.14) &\le C\|g\|_{\mathcal{M}^{q_0}_q} \left(\int_{Q_0} M_{\alpha-n/q_0}f(x)^r\,dx\right)^{1/r} \left(\int_{Q_0} w(x)^{r'}\,dx\right)^{1/r'}\\ &= C\|g\|_{\mathcal{M}^{q_0}_q} \left(\int_{Q_0} M_{\alpha-n/q_0}f(x)^r\,dx\right)^{1/r}.\end{aligned}$$

This is our desired inequality.

The case $i = 2$ A cruder estimate suffices in this case. By a property of the dyadic cubes, for all $x \in Q_0$ we have

$$F_2(x) = \sum_{Q \in \mathcal{D}_2(Q_0)} \ell(Q)^\alpha m_{3Q}(f) \le \sum_{Q \in \mathcal{D}_2(Q_0)} \ell(Q)^{\alpha-\frac{n}{p_0}} \|f\|_{\mathcal{M}^{p_0}_p}. \tag{4.20}$$

In view of the definition of $\mathcal{D}_2(Q_0)$, we have

$$F_2(x) \le \mu \ell(Q_0)^{\alpha-\frac{n}{p_0}} \|f\|_{\mathcal{M}^{p_0}_p} \quad \left(\mu = \sum_{j=0}^\infty 2^{j(\alpha-n/p_0)}\right) \tag{4.21}$$

Thus, for all $x \in Q_0$ we obtain

$$F_2(x) \le C\|f\|_{\mathcal{M}^{p_0}_p} \ell(Q_0)^{\alpha-\frac{n}{p_0}} \tag{4.22}$$

and

$$\left(\int_{Q_0} (g(x)F_2(x))^r\,dx\right)^{1/r} \le C\, m^{(q)}_{Q_0}(g) \|f\|_{\mathcal{M}^{p_0}_p} \ell(Q_0)^{\alpha-\frac{n}{p_0}+\frac{n}{r}}. \tag{4.23}$$

This is our desired inequality.

Remark. In the course of the proof we have proved

$$I_\alpha f(x) \le C \sum_{Q \in \mathcal{D}} \ell(Q)^\alpha m_{3Q}(f) \chi_Q(x). \tag{4.24}$$

Here is a converse inequality:

$$C^{-1} \sum_{Q \in \mathcal{D}} \ell(Q)^\alpha m_{3Q}(f) \chi_Q \le \sum_{Q \in \mathcal{D}} \ell(Q)^\alpha \left(\inf_Q Mf \right) \chi_Q \le C\, I_\alpha(Mf). \tag{4.25}$$

§ 5. Counterexample

Proposition 5.1. *Let $1 < r \le r_0 < \infty$ and $r < 1/\alpha$. Then, for any $c > 0$ we can find positive measurable functions f and g such that*

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r,r_0}} > c \|g\|_{\mathcal{M}^{r,1/\alpha}} \|f\|_{\mathcal{M}^{r,r_0}}.$$

Proof. The proof of this proposition is kind of lengthy and we use the predual of Morrey spaces investigated originally by Zorko [19]. Here we content ourselves with remarking that we chose $g = f = \chi_{E_j}$ with $j \in \mathbb{N}$, where E_j denotes the fractal set of this note (See Example 2.1) and $R > 1$ is appropriately chosen. □

§ 6. Applications

§ 6.1. Application to the variation problem

Here we apply Theorem 1.1 to a variation problem.

Denote by $X_q^{q_0}$ the closure of C_c^∞ with respect to $\mathcal{M}_q^{q_0}$.

Theorem 6.1. *Let $n \ge 3$ and $V \in \mathcal{M}_p^{n/2}$ with $1 < p \le n/2$. Let $W \in L^\infty$. Define a functional $\mathcal{E}_V$ and $\mathcal{E}_{V+W}$ by*

$$\mathcal{E}_V(\varphi) := \int_{\mathbb{R}^n} |\nabla \varphi(x)|^2 + V(x) |\varphi(x)|^2 \, dx \quad (\varphi \in H^1)$$
$$\mathcal{E}_{V+W}(\varphi) := \int_{\mathbb{R}^n} |\nabla \varphi(x)|^2 + (V(x) + W(x)) |\varphi(x)|^2 \, dx \quad (\varphi \in H^1).$$

(a) *There exists a constant $\alpha > 0$ such that*

$$\|f \cdot \sqrt{|V|}\|_2 \le \alpha \sqrt{\|V\|_{\mathcal{M}_p^{n/2}}} \|\nabla f\|_2 \tag{6.1}$$

for all $f \in C_c^\infty$.

(b) *If* $\|\min(0,V)\|_{\mathcal{M}_p^{n/2}} < \alpha^{-2}$ *and* $2 < p \le 2n$, *then* $H = -\Delta + V$ *is a positive operator on* $L^2(\mathbb{R}^n)$.

(c) *Suppose that* $\|\min(0,V)\|_{\mathcal{M}_p^{n/2}} < \dfrac{1}{4\alpha^2}$. *If* $\{\varphi_j\}_{j\in\mathbb{N}}$ *is an* $L^2(\mathbb{R}^n)$*-sequence such that* $\|\varphi_j\|_2 \le 1$ *for each* $j \in \mathbb{N}$ *and that* $\sup\limits_{j\in\mathbb{N}} \mathcal{E}_{V+W}(\varphi_j) < \infty$, *then* $\{\varphi_j\}_{j\in\mathbb{N}}$ *forms a bounded sequence in* $H^1(\mathbb{R}^n)$.

(d) *Suppose in addition that* $W \in L^\infty$. *Then we have*

$$\lim_{j\to\infty} \mathcal{E}_{V+W}(\varphi_j) \ge \mathcal{E}_{V+W}(\varphi) \tag{6.2}$$

if $\lim\limits_{j\to\infty} \varphi_j = \varphi$ *in the weak topology of* H^1.

(e) *Let* $V \in X_p^{n/2}$ *and* $1 < p \le n/2$. *Suppose that* $W \in L^\infty$. *Assume in addition that* $\|\min(0,V)\|_{\mathcal{M}_p^{n/2}} < \dfrac{1}{4\alpha^2}$ *and that*

$$E_0 = \inf\{\mathcal{E}_{V+W}(\varphi) \,:\, \varphi \in H^1,\ \|\varphi\|_2 = 1\} < 0 \tag{6.3}$$

then there exists $\varphi_0 \in L^2$ *such that*

$$\mathcal{E}_{V+W}(\varphi_0) = \inf\{\mathcal{E}_{V+W}(\varphi) \,:\, \varphi \in H^1,\ \|\varphi\|_2 = 1\},\ \|\varphi_0\|_2 = 1. \tag{6.4}$$

Assertions (a) and (b) are easy consequences of Theorem 1.1. Here we prove (c), (d) and (e).

Proof of (c). By (a)

$$\|\,\varphi_j\sqrt{|\min(V,0)|}\,\|_2 \le \frac{1}{2}\|\nabla\varphi_j\|_2 \tag{6.5}$$

for each $j \in \mathbb{N}$. Thus, it follows that

$$T_{\varphi_j} = \int_{\mathbb{R}^n} |\nabla\varphi_j|^2 \ge -4\int_{\mathbb{R}^n} |\varphi_j(x)|^2 \min(V(x),0)\,dx. \tag{6.6}$$

This pointwise estimate yields

$$\frac{3}{4}T_{\varphi_j} = T_{\varphi_j} - \frac{1}{4}T_{\varphi_j} \le T_{\varphi_j} + \int_{\mathbb{R}^n} |\varphi_j(x)|^2 \min(V(x),0)\,dx \le \mathcal{E}_V(\varphi_j). \tag{6.7}$$

Since

$$\sup_j \mathcal{E}_V(\varphi_j) \le \|W\|_\infty + \sup_j \mathcal{E}_{V+W}(\varphi_j) < \infty, \tag{6.8}$$

it follows that $\{\varphi_j\}_{j\in\mathbb{N}}$ forms a bounded sequence in H^1. $\square$

Proof of (d). Let us choose $\{V_k\}_{k\in\mathbb{N}} \subset C_c^\infty$ that approximates V in the $\mathcal{M}_p^{n/2}$ topology. Since by our theorem (see Theorem 1.1) we have

$$|\mathcal{E}_{V+W}(\varphi_j) - \mathcal{E}_{V_k+W}(\varphi_j)| \le C\,\|V - V_k\|_{\mathcal{M}_p^{n/2}}\|\varphi_j\|_{H^1} \tag{6.9}$$

and $\{\varphi_j\}_{j\in\mathbb{N}}$ forms a bounded family in H^1, we can assume that $V \in C_c^\infty$. Once we assume that $V \in C_c^\infty$, then we can use the compact embedding $H^1 \hookrightarrow L^{2+\varepsilon}$ to conclude the proof of (d). □

Proof of (e). Let us choose a sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subset L^2$ so that

$$E_0 = \lim_{j\to\infty} \mathcal{E}_{V+W}(\varphi_j),\ \|\varphi_j\|_2 = 1. \tag{6.10}$$

By virtue of the weak compactness of H^1 and (c), we can assume that $\varphi_j \to \varphi$ in the weak topology of H^1.

Since $E_0 = \mathcal{E}_V(\varphi) < 0$ by virtue of (d), it follows that $\varphi \ne 0$. Now that φ attains E_0, its L^2-norm must be 1. Therefore, we obtain the desired result. □

As an example to which we can apply our Theorem 1.1, we have the following function V_N. Recall that we have defined E_N as a union of cubes of equal size in Proposition 2.1. Here the auxiliary parameter R is chosen for the function space $\mathcal{M}_p^{n/2}$, that is, we have chosen R so that

$$\left(\frac{2^n}{(1+R)^n}\right)^{1/p} = \left(\frac{1}{(1+R)^n}\right)^{2/n}. \tag{6.11}$$

Write $E_N = \bigcup_{j=1}^{2^{nN}} Q(z_{N,j}, r_{N,j})$, where

$$Q(z,r) := \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \,:\, |x_1 - z_1|, |x_2 - z_2|, \cdots, |x_n - z_n| \le r\}.$$

We define

$$V_N(x) := -\kappa \sum_{j=1}^{2^{nN}} \chi_{Q(z_{N,j}, r_{N,j})}(x)|x - z_{N,j}|^{-2/n}. \tag{6.12}$$

Then there exists $\kappa_0 > 0$ with the following property; if $0 < \kappa < \kappa_0$ and $M \in \mathbb{R}$ satisfies

$$E_0 = \inf\{\mathcal{E}_{V_N+M}(\varphi) \,:\, \varphi \in H^1,\ \|\varphi\|_2 = 1\} < 0 \tag{6.13}$$

then for all $N \in \mathbb{N}$, there exists $\varphi \in H^1$ such that

$$\mathcal{E}_{V_N+W}(\varphi_0) = \inf\{\mathcal{E}_{V_N+W}(\varphi) \,:\, \varphi \in H^1,\ \|\varphi\|_2 = 1\},\ \|\varphi_0\|_2 = 1. \tag{6.14}$$

§ 6.2. Sobolev-Hardy inequality

Let $0 \le s \le 2$. Then we have

$$\|u\|_{L^{\frac{2n-2s}{n-2}}(|x|^{-s}\,dx)} \le C\,\|\nabla u\|_{L^2}. \tag{6.15}$$

Using Theorem 1.1, we can extend this theorem to some extent.

Here is our result.

Theorem 6.2. *Let $1 < p \le p_0 < n$, $1 < r \le r_0 < \infty$ and $0 < s < n$. Assume that*

$$\frac{1}{r_0} = \frac{s}{nr} + \frac{1}{p_0} - \frac{1}{n},\ \frac{r}{r_0} = \frac{p}{p_0},\ r \ge s. \tag{6.16}$$

Then we have

$$\left\|u|x|^{-\frac{s}{r}}\right\|_{\mathcal{M}^{r_0}_r} \le C\,\|\nabla u\|_{\mathcal{M}^{p_0}_p}.$$

Proof. This is just a rephrasement of Theorem 1.1 obtained by setting $q_0 = \dfrac{nr}{s}$ and $g(x) = |x|^{-s/r}$. Observe that $g \in \mathcal{M}^{q_0}_{q_0-\varepsilon}$ for all $0 < \varepsilon \ll 1$. □

§ 6.3. An extension of Olsen's result

Theorem 6.3. *Assume that*

$$0 < \alpha < n,\ 1 < \tilde{q} \le \frac{n}{2},\ 1 < q \le p < \infty,\ 1 < t \le s < \infty$$

and that

$$\tilde{q} > q,\ \frac{1}{s} = \frac{1}{p} - \frac{2}{n},\ \frac{t}{s} = \frac{q}{p}.$$

Denote by M_V the multiplication operator generated by the potential V. Then we have

$$\|VI_2W\|_{\mathcal{M}^p_q} \le C_0\,\|V\|_{\mathcal{M}^{n/2}_{\tilde{q}}}\|W\|_{\mathcal{M}^p_q} \tag{6.17}$$

and that

$$\|(I_2M_V)^nI_2W\|_{\mathcal{M}^s_t} \le C_1{C_0}^n\,{\|V\|_{\mathcal{M}^{n/2}_{\tilde{q}}}}^n\|W\|_{\mathcal{M}^p_q} \tag{6.18}$$

for some $C_0, C_1 > 1$.

In particular, if $V \in \mathcal{M}^{n/2}_{\tilde{q}} \cap \mathcal{M}^p_q$ and the norm is sufficiently small, then the formal solution (of $\Delta v + Vv = v$)

$$v = 1 + \sum_{j=1}^{\infty}(-1)^j(I_2M_V)^{j-1}(I_2V)$$

satisfies

$$\|v-1\|_{\mathcal{M}^s_t} \le C. \tag{6.19}$$

Proof. Inequality (6.17) is just a repetition of our results. As for (6.18), we have

$$\|(I_2 M_V)^n I_2 W\|_{\mathcal{M}^s_t} \le C_1 \, \|(M_V I_2)^n W\|_{\mathcal{M}^p_q} \le C_1 {C_0}^n \|V\|_{\mathcal{M}^{n/2}_{\tilde{q}}}{}^n \|W\|_{\mathcal{M}^p_q}$$

using the first estimate n-times. Finally, the last assertion can be obtained directly from (6.18). □

Remark. Olsen postulated

$$0 < \alpha < n, \; 1 < \tilde{q} \le \frac{n}{2}, \; 1 < t \le s < \infty, \; t < \frac{2}{n}\tilde{q}s \tag{6.20}$$

additionally on the parameters [10]. With this condition and the condition that all the functions are supported in a bounded set, he obtained $\|v - 1\|_{\mathcal{M}^s_t} \le C$. However, the last condition $t < \frac{2}{n}\tilde{q}s$ turned out to be superflous. We can replace this condition with $t < \frac{1}{p}\tilde{q}s$ with p given by $\frac{1}{s} = \frac{1}{p} - \frac{2}{n}$, keeping in mind that $p < \frac{n}{2}$.

§ 7. Full statement of our main results

Here we formulate our main theorem as the full statement.

§ 7.1. A passage from I_α to T_ρ and from $\mathcal{M}^p_q$ to generalized Morrey spaces

Here we describe what we actually obtained in our first paper [12]. Let $\rho : [0, \infty) \to [0, \infty]$ be a suitable function. We define the generalized fractional integral operator T_ρ and the generalized fractional maximal operator M_ρ by

$$T_\rho f(x) := \int_{\mathbb{R}^n} f(y) \frac{\rho(|x-y|)}{|x-y|^n} \, dy,$$
$$M_\rho f(x) := \sup_{x \in Q \in \mathcal{Q}} \rho(\ell(Q)) m_Q(|f|).$$

If $\rho(t) \equiv t^{n\alpha}$, $0 < \alpha < 1$, then $T_\rho = I_\alpha$ and $M_\rho = M_\alpha$. The Morrey norm $\|f\|_{p,\rho}$ is given by

$$\|f\|_{p,\rho} := \sup_{Q \in \mathcal{Q}} \rho(\ell(Q)) \left(\frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{1/p}. \tag{7.1}$$

In general, by the Dini condition we mean that

$$\int_0^1 \frac{\rho(s)}{s} \, ds < \infty, \tag{7.2}$$

while the doubling condition (with a doubling constant $C_0 > 0$) is that

$$\frac{1}{C_0} \le \frac{\rho(s)}{\rho(t)} \le C_0, \text{ if } \frac{1}{2} \le \frac{s}{t} \le 2. \tag{7.3}$$

A simple consequence that can be deduced from the doubling condition is

$$\frac{\log 2}{C_0}\rho(t) \le \int_{t/2}^{t} \frac{\rho(s)}{s}\,ds \le \log 2 \cdot C_0 \rho(t) \text{ for all } t > 0. \tag{7.4}$$

In the sequel, we always assume that ρ satisfies (7.2) and (7.3), and, then denote the set of all such functions by $\mathcal{G}_0$. We will write, when $\rho \in \mathcal{G}_0$,

$$\tilde{\rho}(t) := \int_0^t \frac{\rho(s)}{s}\,ds.$$

Let $\mathcal{G}_1$ be the set of all functions $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t)$ is nondecreasing but that $\phi(t)t^{-n}$ is nonincreasing. We notice that the condition $\phi \in \mathcal{G}_1$ is stronger than the doubling condition (7.3). More quantitatively, if we assume that $\phi \in \mathcal{G}_1$, then ϕ satisfies the doubling condition with the doubling constant 2^n.

Theorem 7.1. *Let $1 < p < \infty$, $q > r$, $0 \le b \le 1$, $a > 1$ and $(a + b - 1)r = ap$. Suppose that ρ satisfies the Dini condition, (7.3) and that $\tilde{\rho}(t)^{\max(ap,bq)}t^{-n}$ is nonincreasing. The condition $\tilde{\rho}(t)^{\max(ap,bq)}t^{-n}$ is nonincreasing implies that $\tilde{\rho}(t)^{ap}t^{-n}$ and $\tilde{\rho}(t)^{bq}t^{-n}$ are nonincreasing, since*

$$\tilde{\rho}(t)^{\min(ap,bq)}t^{-n} = \tilde{\rho}(t)^{\min(ap,bq)-\max(ap,bq)} \cdot \tilde{\rho}(t)^{\max(ap,bq)}t^{-n}.$$

Then

$$\|g \cdot T_\rho f\|_{r,\tilde{\rho}^{a+b-1}} \le C\|g\|_{q,\tilde{\rho}^b}\|f\|_{p,\tilde{\rho}^a},$$

where the constant C is independent of f and g.

Theorem 7.1 generalizes of [10, Theorem 2] and [17, Theorem 1]. Theorem 7.1 is not longer true when $q = r$ (see Proposition 5.1).

Letting $b = 0$ and $g \equiv 1$ in Theorem 7.1, we have the following:

Corollary 7.2. *Let $1 < p < \infty$, $a > 1$ and $(a - 1)r = ap$. Then*

$$\|T_\rho f\|_{r,\tilde{\rho}^{a-1}} \le C\|f\|_{p,\tilde{\rho}^a}.$$

Corollary 7.2 generalizes [1, Theorem 1.3].

Theorem 7.3. *Let $1 \le p < \infty$, $\begin{cases} p \le q \text{ if } p = 1, \\ p < q \text{ if } p > 1, \end{cases}$ $0 \le b \le 1$ and $b < a$. Suppose that ρ satisfies the Dini condition, (7.3) and that $\tilde{\rho}(t)^{\max(ap,bq)}t^{-n}$ is nonincreasing. Then*

$$\|g \cdot T_\rho f\|_{p,\tilde{\rho}^a} \le C\|g\|_{q,\tilde{\rho}^b}\|M_{\tilde{\rho}^{1-b}} f\|_{p,\tilde{\rho}^a},$$

where the constant C is independent of f and g.

Corollary 7.4. *Let $1 \le p < \infty$ and $a > 0$. Then*

$$\|T_\rho f\|_{p,\tilde{\rho}^a} \le C\|M_{\tilde{\rho}} f\|_{p,\tilde{\rho}^a}.$$

Corollary 7.4 generalizes [2, Theorem 4.2].

Letting $b = 1$ in Theorem 7.3, we have the following too:

Corollary 7.5. *Let $1 \le p < \infty$, $\begin{cases} p \le q \text{ if } p = 1, \\ p < q \text{ if } p > 1, \end{cases}$ and $a > 1$. Then*

$$\|g \cdot T_\rho f\|_{p,\tilde{\rho}^a} \le C\|g\|_{q,\tilde{\rho}}\|Mf\|_{p,\tilde{\rho}^a}.$$

These results are somehow generalized. We generalized them in [13] as follows; we content ourselves with stating them.

Theorem 7.6. *Let $1 \le p < \infty$, $\begin{cases} p \le q \text{ if } p = 1, \\ p < q \text{ if } p > 1. \end{cases}$ Suppose that $\phi(t)$ and $\eta(t)$ are nondecreasing but that $\phi(t)^p t^{-n}$ and $\eta(t)^q t^{-n}$ are nonincreasing. Assume also that*

$$\int_t^\infty \frac{\rho(s)\eta(s)}{s\,\tilde{\rho}(s)\phi(s)}\,ds \le C\frac{\eta(t)}{\phi(t)} \text{ for all } t > 0. \tag{7.5}$$

Then

$$\|g \cdot T_\rho f\|_{p,\phi} \le C\|g\|_{q,\eta}\|M_{\tilde{\rho}/\eta} f\|_{p,\phi},$$

where the constant C is independent of f and g.

Theorem 7.7. *Let $1 < p \le r < q < \infty$. Suppose that $\phi(t)$ and $\eta(t)$ are nondecreasing but that $\phi(t)^p t^{-n}$ and $\eta(t)^q t^{-n}$ are nonincreasing. Suppose also that*

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\,\phi(s)}\,ds \le C\frac{\eta(t)}{\phi(t)^{p/r}} \text{ for all } t > 0. \tag{7.6}$$

Then

$$\|g \cdot T_\rho f\|_{r,\phi^{p/r}} \le C\|g\|_{q,\eta}\|f\|_{p,\phi},$$

where the constant C is independent of f and g.

Theorem 7.8. *Let $0 < p < \infty$. Suppose that ρ, η and ϕ are nondecreasing and that $\eta(t)^p t^{-n}$ and $\phi(t)^p t^{-n}$ are nonincreasing. Then*

$$\|g \cdot M_\rho f\|_{p,\phi} \le C\|g\|_{p,\eta}\|M_{\rho/\eta} f\|_{p,\phi},$$

where the constant C is independent of f and g.

§ 7.2. Extension of Theorem 1.1 to Orlicz-Morrey spaces

To describe Orlicz-Morrey spaces, we recall some definitions and notation. Here we follow [15].

A function $\Phi : [0, \infty) \to [0, \infty]$ is said to be a Young function if it is left-continuous, convex and increasing, and if $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. We say that Φ is a normalized Young function when Φ is a Young function and $\Phi(1) = 1$. It is easy to see that t^p, $1 \le p < \infty$, is a normalized Young function.

A Young function Φ is said to satisfy the Δ_2-condition, denoted $\Phi \in \Delta_2$, if for some $K > 1$

$$\Phi(2t) \le K\Phi(t) \text{ for all } t > 0.$$

Meanwhile, a Young function Φ is said to satisfy the ∇_2-condition, denoted $\Phi \in \nabla_2$, if for some $K > 1$

$$\Phi(t) \le \frac{1}{2K}\Phi(Kt) \text{ for all } t > 0.$$

The function $\Phi(t) \equiv t$ satisfies the Δ_2-condition but fails the ∇_2-condition. If $1 < p < \infty$, then $\Phi(t) \equiv t^p$ satisfies both conditions. The complementary function $\bar{\Phi}$ of a Young function Φ is defined by

$$\bar{\Phi}(t) := \sup\{ts - \Phi(s) : s \in [0, \infty)\}.$$

Then $\bar{\Phi}$ is also a Young function and $\bar{\bar{\Phi}} = \Phi$. Notice that $\Phi \in \nabla_2$ if and only if $\bar{\Phi} \in \Delta_2$. For the other properties of Young functions and the examples, see [8, p196].

Given a Young function Φ, define the Orlicz space $\mathcal{L}^\Phi(\mathbb{R}^n) = \mathcal{L}^\Phi$ by the Luxemberg norm

$$\|f\|_{\mathcal{L}^\Phi} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

When $\Phi(t) \equiv t^p$, $1 \le p < \infty$, $\|f\|_{\mathcal{L}^\Phi} = \|f\|_{L^p}$. We need the following basic two facts.

Generalized Hölder's inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \le C\|f\|_{\mathcal{L}^\Phi}\|g\|_{\mathcal{L}^{\bar{\Phi}}};$$

The dual equation:

$$\|f\|_{\mathcal{L}^\Phi} \approx \sup\left\{\|fg\|_{L^1} : \|g\|_{\mathcal{L}^{\bar{\Phi}}} \le 1\right\}.$$

Given a Young function Φ, define the mean Luxemburg norm of f on a cube $Q \in \mathcal{Q}$ by

$$\|f\|_{\Phi,\, Q} := \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

When $\Phi(t) \equiv t^p$, $1 \le p < \infty$,

$$\|f\|_{\Phi,\,Q} = \left(\frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{1/p},$$

that is, the mean Luxemburg norm coincides with the (normalized) L^p norm. It should be noticed that

$$\|f\|_{\Phi,\,Q} = \|\tau_{\ell(Q)}[f\chi_Q]\|_{\mathcal{L}^\Phi}, \tag{7.7}$$

where τ_δ, $\delta > 0$, is the dilation operator $\tau_\delta f(x) = f(\delta x)$. It follows from this relation and generalized Hölder's inequality that for any cube $Q \in \mathcal{Q}$

$$m_Q(|fg|) \le C\|f\|_{\Phi,\,Q}\|g\|_{\bar{\Phi},\,Q}. \tag{7.8}$$

The Orlicz maximal operator, for any Young function Ψ, is defined by

$$M^\Psi f(x) := \sup_{x \in Q \in \mathcal{Q}} \|f\|_{\Psi,\,Q}.$$

Now let us introduce Orlicz-Morrey spaces.

Definition 7.9. Let $\phi \in \mathcal{G}_1$ and let Φ be a Young function. The Orlicz-Morrey space $\mathcal{L}^{\Phi,\,\phi}(\mathbb{R}^n) = \mathcal{L}^{\Phi,\,\phi}$ consists of all locally integrable functions f on $\mathbb{R}^n$ for which the norm

$$\|f\|_{\mathcal{L}^{\Phi,\,\phi}} := \sup_{Q \in \mathcal{Q}} \phi(\ell(Q))\|f\|_{\Phi,\,Q}$$

is finite. In particular, in order that the characteristic function of the unit cubes belongs to $\mathcal{L}^{\Phi,\,\phi}$, it is always assumed that

$$\sup_{t>1} \frac{\phi(t)}{\Phi^{-1}(t^n)} < \infty.$$

Example We define

$$\|f\|_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}} = \|f\|_{\Phi,\,\phi}, \tag{7.9}$$

when $\phi(t) = t^{n/p}$ and $\Phi(t) = t\log(3+t)$.

We have shown in [15] that

$$\|Mf\|_{\mathcal{M}^p_1} \sim \|f\|_{\mathcal{M}^p_{\mathrm{L}\log\mathrm{L}}}. \tag{7.10}$$

If $\Phi(t) \equiv t^p$ and $\phi(t) \equiv t^{n/p_0}$, $1 \le p \le p_0 < \infty$, then $\mathcal{L}^{\Phi,\,\phi} = \mathcal{M}^{p,\,p_0}$. When $\Phi(t) \equiv t^p$, $1 \le p < \infty$, we will denote $\mathcal{L}^{\Phi,\,\phi}$ by $\mathcal{M}^{p,\,\phi}$. In this case we will call it the (generalized) Morrey space. We consider $\mathcal{M}^{p,\,\phi}$ even for $0 < p < 1$. We define an auxiliary space too.

Definition 7.10. Let $\phi \in \mathcal{G}_1$ and let Φ be a Young function. The space

$$\tilde{\mathcal{L}}^{\Phi,\,\phi}(\mathbb{R}^n) = \tilde{\mathcal{L}}^{\Phi,\,\phi}$$

consists of all locally integrable functions g on $\mathbb{R}^n$ for which the norm

$$\|g\|_{\tilde{\mathcal{L}}^{\Phi,\,\phi}} := \sup\left\{\|M_\phi[gw\chi_Q]\|_{\bar{\Phi},\,Q} \,:\, Q \in \mathcal{Q},\, \|w\|_{\bar{\Phi},\,Q} \le 1\right\}$$

is finite.

Related to the space $\tilde{\mathcal{L}}^{\Phi,\,\phi}$, we need the following notion too.

Definition 7.11. Let Φ and Ψ be Young functions. One says that "M^Ψ is locally bounded in the norm determined by Φ", when it satisfies

$$\|M^\Psi[g\chi_Q]\|_{\Phi,\,Q} \le C\|g\|_{\Phi,\,Q} \text{ for all cubes } Q \in \mathcal{Q}.$$

We now state our first results, which extend those in [12, 13] to Orlicz-Morrey spaces.

Theorem 7.12. *Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$ and $\Phi \in \nabla_2$. Suppose that the condition*

$$\int_t^\infty \frac{\rho(s)}{s\phi(s)}\,ds \le C\frac{\tilde{\rho}(t)}{\phi(t)} \text{ for all } t > 0. \tag{7.11}$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi,\,\phi}} \le C\|g\|_{\tilde{\mathcal{L}}^{\Phi,\,\tilde{\rho}}}\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

Theorem 7.13. *Let Ψ be a Young function. With the same condition posed in Theorem 7.12, if, in addition, $M^{\bar{\Psi}}$ is locally bounded in the norm determined by $\bar{\Phi}$, then we have*

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Phi,\,\phi}} \le C\|g\|_{\mathcal{L}^{\Psi,\,\tilde{\rho}}}\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

Theorems 7.12 and 7.13 are the trace inequalities of the generalized fractional integral operators for Orlicz-Morrey spaces.

Theorem 7.14. *Let $\rho \in \mathcal{G}_0$, $\phi, \psi \in \mathcal{G}_1$, $\Phi \in \nabla_2$ and $0 < a \le 1$. Set*

$$\eta(t) \equiv \phi(t)^a, \quad \Psi(t) \equiv \Phi(t^{1/a}).$$

Suppose that the condition

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)}\,ds \le C\frac{\psi(t)}{\eta(t)} \text{ for all } t > 0. \tag{7.12}$$

Then

$$\|g \cdot T_\rho f\|_{\mathcal{L}^{\Psi,\,\eta}} \le C\|g\|_{\tilde{\mathcal{L}}^{\Psi,\,\psi}}\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

Theorem 7.14 is a general form of Theorem 7.12 (letting $a \equiv 1$) and is the Olsen inequality of the generalized fractional integral operators for Orlicz-Morrey spaces.

Letting $g(x) \equiv 1$ and $\psi(t) \equiv 1$ in Theorem 7.14, we can recover the boundedness property of T_ρ.

Corollary 7.15. *Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$, $\Phi \in \nabla_2$ and $0 < a \le 1$. Set*

$$\eta(t) \equiv \phi(t)^a, \quad \Psi(t) \equiv \Phi(t^{1/a}).$$

Suppose that $\bar{\Psi} \in \nabla_2$ and that the condition

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)}\,ds \le \frac{C}{\eta(t)} \text{ for all } t > 0.$$

Then

$$\|T_\rho f\|_{\mathcal{L}^{\Psi,\,\eta}} \le C\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

However, in the next theorem we reproved this corollary directly without the assumption $\bar{\Psi} \in \nabla_2$.

Theorem 7.16. *Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$, $\Phi \in \nabla_2$ and $0 < a \le 1$. Set*

$$\eta(t) \equiv \phi(t)^a, \quad \Psi(t) \equiv \Phi(t^{1/a}).$$

Suppose that the condition

$$\frac{\tilde{\rho}(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{s\phi(s)}\,ds \le \frac{C}{\eta(t)} \text{ for all } t > 0.$$

Then

$$\|T_\rho f\|_{\mathcal{L}^{\Psi,\,\eta}} \le C\|f\|_{\mathcal{L}^{\Phi,\,\phi}}.$$

Corollary 7.15 generalizes [13, Corollary 1.7]. In [7] Nakai studied the boundness of the generalized fractional integral operator T_ρ on Orlicz spaces. Since, we cannot recover Orlicz spaces as a special case of our Orlicz-Morrey spaces, we dare not compare Corollary 7.15 with [7, Theorem 3.1].

§ 8. Appendix-Boundedness of the fractional maximal operator

Lemma 8.1. *Let $p > 1$. Suppose that $\phi(t)$ is nondecreasing and $\phi(t)^p t^{-n}$ is nonincreasing. Then*

$$\|Mf\|_{p,\phi} \le C\|f\|_{p,\phi}.$$

Proof. Fix a cube Q_0. Let $f_1 = \chi_{3Q_0} f$ and $f_2 = f - f_1$. Then the subadditivity of M yields

$$Mf(x) \le Mf_1(x) + Mf_2(x).$$

It follows from the definition of M that for all $x \in Q_0$

$$Mf_2(x) = \sup_{x \in Q \in \mathcal{Q} \colon \ell(Q) \ge \ell(Q_0)} \frac{1}{|Q|} \int_Q |f(y)| \, dy.$$

Suppose that $x \in Q_0$, $x \in Q \in \mathcal{Q}$ and $\ell(Q) \ge \ell(Q_0)$. Then

$$\phi(\ell(Q_0)) m_Q(|f|) \le \phi(\ell(Q)) m_Q^{(p)}(|f|) \le \|f\|_{p,\phi},$$

where we have used Hölder's inequality and the fact that ϕ is nondecreasing.

This gives us that

$$\phi(\ell(Q_0)) Mf_2(x) \le \|f\|_{p,\phi} \text{ for all } x \in Q_0,$$

and that

$$\begin{aligned} \phi(\ell(Q_0)) m_{Q_0}^{(p)}(Mf)^p &\le \phi(\ell(Q_0)) m_{Q_0}^{(p)}(Mf_1)^p + \phi(\ell(Q_0)) m_{Q_0}^{(p)}(Mf_2)^p \\ &\le C\phi(\ell(3Q_0)) m_{3Q_0}^{(p)}(f) + \|f\|_{p,\phi} \le C\|f\|_{p,\phi}, \end{aligned}$$

where we have used L^p boundedness of maximal operator M. This implies our desired inequality. □

Lemma 8.2. *Let $1 < p \le q < \infty$. Suppose that $\phi(t)$ is nondecreasing and $\phi(t)^p t^{-n}$ is nonincreasing. Then*

$$\|M_{\phi^{1-p/q}} f\|_{q,\phi^{p/q}} \le C\|f\|_{p,\phi}.$$

It is worth noting that the surjectivity of ϕ was superflous.

Proof. Let $x \in \mathbb{R}^n$ be a fixed point. For every cube $Q \ni x$ we see that

$$\begin{aligned} \phi(\ell(Q))^{1-p/q} m_Q(|f|) &\le \min(\phi(\ell(Q))^{1-p/q} Mf(x)\,,\, \phi(\ell(Q))^{-p/q} \|f\|_{p,\phi}) \\ &\le \sup_{t \ge 0} \min(t^{1-p/q} Mf(x)\,,\, t^{-p/q} \|f\|_{p,\phi}) \\ &= \|f\|_{p,\phi}^{1-p/q} Mf(x)^{p/q}. \end{aligned}$$

This implies

$$M_{\phi^{1-p/q}} f(x)^q \le \|f\|_{p,\phi}^{q-p} Mf(x)^p.$$

It follows from Lemma 8.1 that for every cube Q_0

$$m_{Q_0}^{(q)}(M_{\phi^{1-p/q}} f) \le \|f\|_{p,\phi}^{1-p/q} m_{Q_0}^{(p)}(Mf)^{p/q} \le C\|f\|_{p,\phi} \phi(\ell(Q_0))^{-p/q}.$$

The desired inequality then follows. □

§ 9. Acknowledgement

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