

A METHOD OF CLASSIFYING EXPANSIVE SINGULARITIES

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Introduction

To study singularities is in a sense to study the classification of germs of varieties. It is therefore important to give a method of classification. The purpose of this paper is to show the classification of a class of germs of varieties, which will be called expansive singularities in this paper, is included in that of Lie algebras of formal vector fields. As a matter of course, the classification of the latter does not seem easy. However, note that such a Lie algebra is given by an inverse limit of finite dimensional Lie algebras of polynomial vector fields truncated at the order k , $k \geq 0$. Therefore such Lie algebras can be understood by step by step method in the order k .

Let $\mathbb{C}^n$ be the Cartesian product of n copies of complex numbers $\mathbb{C}$ with natural coordinate system $(x_1, \dots, x_n)$. By $\mathcal{O}$, we mean the ring of all convergent power series in $x_1, \dots, x_n$ centered at the origin 0 . Let V be a germ of variety in $\mathbb{C}^n$ at 0 , and $\mathcal{J}(V)$ the ideal of V in $\mathcal{O}$ (cf. [2] pp86-7 for the definitions). Two germs V, V' are called bi-holomorphically equivalent if there is a germ of holomorphic diffeomorphism φ such that $\varphi(0) = 0$ and $\varphi(V) = V'$.

Let $\mathcal{X}$ be the Lie algebra of all germs of holomorphic vector fields at 0 , and $\mathcal{X}(V)$ the subalgebra defined by

$$\mathcal{X}(V) = \{u \in \mathcal{X} ; u \mathcal{J}(V) \subset \mathcal{J}(V)\}.$$

$\mathcal{X}(V)$ is then an $\mathcal{O}$ -module. If there are $v_1, \dots, v_s$, linearly independent at 0, then Corollary 3,4 of [9] shows that V is bi-holomorphically equivalent to the direct product $\mathbb{C}^s \times W$, where $W \subset \mathbb{C}^{n-s}$. Thus, for the structure of singularities we have only to consider the germ W . Taking this fact into account, we may restrict our concern to the varieties such that all $u \in \mathcal{X}(V)$ vanishes at 0, which we assume throughout this paper, i.e. $\mathcal{X}(V)(0) = \{0\}$.

$u \in \mathcal{X}(V)$ ($u(0) = 0$) is called a semi-simple expansive vector field, if after a suitable bi-holomorphic change of variables at 0, u can be written in the form

$$(1) \quad u = \sum_{i=1}^n \hat{\mu}_i y_i \partial / \partial y_i,$$

where $\hat{\mu}_1, \dots, \hat{\mu}_n$ lie in the same open half-plane in $\mathbb{C}$ about the origin. (See also §2.A for a justification of this definition.) The origin 0 is called to be an expansive singularity, if $\mathcal{X}(V)$ contains a semi-simple expansive vector field. If V is given by the locus of zeros of a weighted homogeneous polynomial, then V has an expansive singularity at 0. The advantage of existence of such a vector field u is that one can extend through $\exp tu$ a germ V to a subvariety $\tilde{V}$ in $\mathbb{C}^n$. In this paper we restrict our concern to the germs of varieties with expansive singularities at the origin.

For such $\mathcal{X}(V)$, we set $\mathcal{X}_k(V) = \{u \in \mathcal{X}(V) ; j^k u = 0\}$, where $j^k u$ is the k -th jet at 0. Since $\mathcal{X}(V) = \mathcal{X}_0(V)$, $\mathcal{X}_k(V)$ is a finite codimensional ideal of $\mathcal{X}(V)$ such that $[\mathcal{X}_k(V), \mathcal{X}_\ell(V)] \subset \mathcal{X}_{k+\ell}(V)$ and $\bigcap \mathcal{X}_k(V) = \{0\}$. We denote by $\mathcal{G}(V)$ the inverse limit of $\{\mathcal{X}(V) / \mathcal{X}_k(V)\}_{k \geq 0}$ with the inverse limit topology. Since

$\mathcal{X}(V) / \mathcal{X}_k(V)$ is finite dimensional, $\mathcal{G}(V)$ is a Frechet space such that the Lie bracket product $[\cdot, \cdot] : \mathcal{G}(V) \times \mathcal{G}(V) \mapsto \mathcal{G}(V)$ is

continuous. Namely, $\mathcal{G}(V)$ is a Frechet-Lie algebra. It is obvious that $\mathcal{G}(V)$ is a Lie algebra of formal vector fields, where a formal vector field u is a vector field $u = \sum_{i=1}^n u_i \partial / \partial x_i$ such that each u_i is a formal power series in $x_1, \dots, x_n$ without constant terms. The statement to be proved in this paper is as follows :

Theorem I Let V, V' be germs of varieties with expansive singularities at the origins of $\mathbb{C}^n, \mathbb{C}^{n'}$ respectively. Notations and assumptions being as above, V and V' are bi-holomorphically equivalent, if and only if $\mathcal{G}(V)$ and $\mathcal{G}(V')$ are isomorphic as topological Lie algebras.

By the above result, we see especially that any isomorphism Φ of $\mathcal{G}(V)$ onto $\mathcal{G}(V')$ preserves orders, that is, $\Phi \mathcal{G}_k(V) = \mathcal{G}_k(V')$ for every k . Hence, to classify $\mathcal{G}(V)$ is to classify the inverse system $\{\mathcal{X}(V) / \mathcal{X}_k(V)\}_{k \geq 0}$. Note that $\mathcal{X}(V) / \mathcal{X}_k(V)$ is an extension of $\mathcal{X}(V) / \mathcal{X}_{k-1}(V)$ with an abelian kernel $\mathcal{X}_{k-1}(V) / \mathcal{X}_k(V)$. Such extensions can be classified by representations and second cohomologies (cf. [6]).

The proof of the above theorem is divided into several steps as follows :

Step 1. We define the concept of Cartan subalgebras and prove the conjugacy of Cartan subalgebras.

Step 2. Using the assumption that V (resp. V') has an expansive singularity at 0 , we prove that there is a Cartan subalgebra $\mathfrak{h}$ of $\mathcal{G}(V)$ such that $\mathfrak{h} \subset \mathcal{X}(V)$ (resp. $\mathfrak{h}' \subset \mathcal{X}(V')$). By a suitable bi-holomorphic change of variables, every element of $\mathfrak{h}$ (resp. $\mathfrak{h}'$) can be changed simultaneously into a normal form, which is a polynomial vector field. Moreover, every "eigenvector with respect to $\text{ad}(\mathfrak{h})$ " is a polynomial vector field.

Step 3. Now, suppose there is an isomorphism Φ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$. Then, by definition $\Phi(\xi)$ is a Cartan subalgebra of $\mathfrak{g}(V')$. Hence by Steps 1, 2 we may assume that $\Phi(\xi) \subset \mathfrak{X}(V')$. Thus, considering the eigenspace decomposition of $\mathfrak{g}(V)$, $\mathfrak{g}(V')$ with respect to $\text{ad}(\xi)$ $\text{ad}(\xi')$ respectively, we see that Φ induces an isomorphism of $\mathfrak{p}$ onto $\mathfrak{p}'$, where $\mathfrak{p}$ (resp. $\mathfrak{p}'$) is the totality of $u \in \mathfrak{g}(V)$ (resp. $\mathfrak{g}(V')$) which can be expressed as a polynomial vector field with respect to the local coordinate system which normalizes ξ (resp. ξ').

Step 4. From isomorphism $\Phi : \mathfrak{p} \rightarrow \mathfrak{p}'$, we conclude by the same procedure as in [5] that there is a bi-holomorphic diffeomorphism $\mathcal{Y}$ of $\mathbb{C}^n$ onto $\mathbb{C}^{n'}$ such that $\mathcal{Y}(0) = 0$ and $d\mathcal{Y}\mathfrak{p} = \mathfrak{p}'$. The main idea of making such $\mathcal{Y}$ is roughly in the fact that every maximal subalgebra of $\mathfrak{p}$ corresponds to a point. However, since $\mathfrak{p}(0) = \{0\}$, the situation is much more difficult than that of [1]. Existence of expansive vector field plays an important role at this step as well as in the above steps.

Step 5. Recapturing V from the Lie algebra $\mathfrak{p}$, we can conclude $\mathfrak{g}(V) = V'$.

The theorem is proved by this way. Note that the converse is trivial.

§1 Conjugacy of Cartan subalgebras

We denote a formal power series f in a form $f = \sum_{|\alpha| \geq 0} a_\alpha x^\alpha$, where $a_\alpha \in \mathbb{C}$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$. We denote by $\mathcal{F}$ the Lie algebra of all formal vector fields and $\mathcal{F}_k$ the subalgebra

$$\{u \in \mathcal{F} ; u = \sum_{i=1}^n \sum_{|\alpha| \geq k} a_{i,\alpha} x^\alpha \partial / \partial x_i\}$$

$\mathcal{F}$ is then regarded as the inverse limit of the system $\{\mathcal{F} / \mathcal{F}_k ; p_k\}$, where $p_k : \mathcal{F} / \mathcal{F}_{k+1} \rightarrow \mathcal{F} / \mathcal{F}_k$ is the natural projection. We denote by $\tilde{p}_k$ the projection of $\mathcal{F}$ onto $\mathcal{F} / \mathcal{F}_k$. p_k and $\tilde{p}_k$ are sometimes called forgetful mappings. Since $\mathcal{F} / \mathcal{F}_k$ is a finite dimensional vector space over $\mathbb{C}$, $\mathcal{F}$ is a Frechet space, and the Lie bracket product is continuous.

Let $\mathcal{G}$ be a closed Lie subalgebra of $\mathcal{F}$, and $\mathcal{G}_k = \mathcal{G} \cap \mathcal{F}_k$. The closedness of $\mathcal{G}$ implies that $\mathcal{G}$ is the inverse limit of the system $\{\mathcal{G} / \mathcal{G}_k ; p_k\}_{k \geq 0}$. In this paper, we restrict our concern to a closed subalgebra $\mathcal{G}$ of $\mathcal{F}_0$. For any subalgebra $\mathcal{S}$ of $\mathcal{G}$, we denote by $\pi(\mathcal{S})$ the normalizer of $\mathcal{S}$, i.e. $\pi(\mathcal{S}) = \{u \in \mathcal{G} ; [u, \mathcal{S}] \subset \mathcal{S}\}$, and by $\mathcal{G}^{(0)}(\mathcal{S})$ the 0-eigenspace of $\text{ad}(\mathcal{S})$, i.e. $\mathcal{G}^{(0)}(\mathcal{S})$ is the totality of $v \in \mathcal{G}$ satisfying that there are non-negative integers $m_k, k \geq 0$, (depending on v) such that $\text{ad}(S)^{m_k} v \in \mathcal{G}_k$ for all $S \in \mathcal{S}$ and for all $k \geq 0$, where $\text{ad}(u)v = [u, v]$. If $\mathcal{S}$ is nilpotent, then $\mathcal{G}^{(0)}(\mathcal{S}) \supset \pi(\mathcal{S})$. Therefore, if $\mathcal{G}^{(0)}(\mathcal{S}) = \mathcal{S}$, then $\pi(\mathcal{S}) = \mathcal{S}$. The converse is also true if $\dim \mathcal{G}^{(0)}(\mathcal{S}) < \infty$ (cf. [6]).

A subalgebra $\mathcal{H}$ of $\mathcal{G}$ is called a Cartan subalgebra of $\mathcal{G}$, if the following conditions are satisfied :

(H, 1) $\mathcal{H}$ is a closed subalgebra of $\mathcal{G}$ such that $\tilde{p}_k \mathcal{H}$ is a nilpotent

subalgebra of $\mathfrak{g}/\mathfrak{g}_k$ for every $k \geq 0$.

$$(\mathfrak{g}, 2) \quad \mathfrak{g} = \mathfrak{g}^{(0)}(\mathfrak{g}).$$

Note that if $\dim \mathfrak{g} < \infty$ above $\mathfrak{g}$ is a usual Cartan subalgebra. The statement to be proved in this chapter is as follows :

Proposition A Let $\mathfrak{g}$ be a closed subalgebra of $\mathfrak{F}_0$. Then, there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. For Cartan subalgebras $\mathfrak{h}, \hat{\mathfrak{h}}$ of $\mathfrak{g}$, there is an inner automorphism A of $\mathfrak{g}$ such that $A\mathfrak{h} = \hat{\mathfrak{h}}$.

1.A. Automorphisms of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a closed Lie subalgebra of $\mathfrak{F}_0$, and $\mathfrak{g}_k = \mathfrak{g} \cap \mathfrak{F}_k$. For every $u \in \mathfrak{g}$ the adjoint action $\text{ad}(u)$ leaves each $\mathfrak{g}_k$ invariant, hence $\text{ad}(u)$ induces a linear mapping $a_k(u)$ of $\mathfrak{g}/\mathfrak{g}_k$ into itself. $\text{ad}(u)$ is then regarded as the inverse limit of the system $\{a_k(u)\}_{k \geq 0}$. Define a linear mapping $e^{t \cdot \text{ad}(u)} : \mathfrak{g} \rightarrow \mathfrak{g}$ by the inverse limit of $\{e^{t \cdot a_k(u)}\}_{k \geq 0}$. Since $\text{ad}(u)$ is a derivation of $\mathfrak{g}$, $e^{t \cdot \text{ad}(u)}$ is a one parameter family of automorphisms of $\mathfrak{g}$. The group $\mathcal{O}(\mathfrak{g})$ generated by $\{e^{\text{ad}(u)}; u \in \mathfrak{g}\}$ is called the group of inner automorphisms of $\mathfrak{g}$. The purpose of this section is to investigate the structure of $\mathcal{O}(\mathfrak{g})$.

Let $\hat{\mathcal{O}}$ be the ring of all formal power series $\sum_{|\alpha| \geq 0} a_\alpha x^\alpha$ and $\hat{\mathcal{O}}_k$ the ideal given by $\hat{\mathcal{O}}_k = \{\sum_{|\alpha| \geq k+1} a_\alpha x^\alpha\}$. $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ is then a finite dimensional algebra over $\mathbb{C}$. We denote by $\tilde{\pi}_k, \pi_k$ the projections $\hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}/\hat{\mathcal{O}}_k, \hat{\mathcal{O}}/\hat{\mathcal{O}}_{k+1} \rightarrow \hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ respectively. Every $u \in \mathfrak{F}_0$ acts naturally on $\hat{\mathcal{O}}$ as a derivation such that $u\hat{\mathcal{O}}_k \subset \hat{\mathcal{O}}_k$ for every k . Conversely, $u \in \mathfrak{F}_0$ can be characterised by the above property. Every $u \in \mathfrak{F}_0$ induces, therefore, a derivation $u^{(k)}$ of the algebra $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ and $u^{(k)}$ is canonically identified with $\tilde{p}_k u$. Conversely,

for every derivation δ of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ such that $\delta \hat{\mathcal{O}}_0/\hat{\mathcal{O}}_k \subset \hat{\mathcal{O}}_0/\hat{\mathcal{O}}_k$ there is an element $u \in \mathcal{F}_0$ such that $\delta = \tilde{p}_k u$.

Since a derivation $u : \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}$ can be regarded as an inverse limit of derivations $\{\tilde{p}_k u : \hat{\mathcal{O}}/\hat{\mathcal{O}}_k \rightarrow \hat{\mathcal{O}}/\hat{\mathcal{O}}_k\}$, we define an automorphism $\exp u$ of $\hat{\mathcal{O}}$ by an inverse limit of $\{e^{\tilde{p}_k u}\}$. We denote by G' the group generated by $\{\exp u ; u \in \mathcal{F}\}$.

Define an automorphism $\text{Ad}(\exp u)$ of $\mathcal{F}$ by

$$(2) \quad (\text{Ad}(\exp u)v)f = (\exp u)v(\exp -u)f, \quad f \in \hat{\mathcal{O}}.$$

Since $(d/dt)_{t=0}(\exp tu)f = uf$, we see easily that

$$(3) \quad \frac{d}{dt} \text{Ad}(\exp tu)v = [u, \text{Ad}(\exp tu)v].$$

On the other hand, $e^{t \cdot \text{ad}(u)}$ satisfies the same differential equation.

Thus, by uniqueness, we obtain

$$(4) \quad \text{Ad}(\exp u) = e^{\text{ad}(u)}.$$

Especially, if $\mathcal{G}$ is a closed Lie subalgebra of $\mathcal{F}_0$, then

$\text{Ad}(\exp u)\mathcal{G} = \mathcal{G}$ for every $u \in \mathcal{G}$. Since

$$e^{\text{ad}(u)}e^{\text{ad}(v)} = \text{Ad}(\exp u \cdot \exp v),$$

we obtain that $\mathcal{O}(\mathcal{G}) = \{\text{Ad}(g) ; g \in G'\}$.

Let $G^{(k)}$ be the group generated by $\{e^{\tilde{p}_k u} ; u \in \mathcal{G}\}$. Since $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ is finite dimensional, $G^{(k)}$ is a Lie group with Lie algebra $\mathcal{G}/\mathcal{G}_k$. For every integer l such that $l \leq k$, the group $G^{(k)}$ leaves $\mathcal{G}_l/\mathcal{G}_k$ invariant. Hence $\{G^{(k)}\}_{k \geq 0}$ forms an inverse system. We denote by G the inverse limit. Obviously, G' is a subgroup of G . However, note that if a sequence $(u_0, u_1, \dots, u_n, \dots)$ satisfies $u_l \in \mathcal{G}_l$ for every $l \geq 0$, then $\exp u_0 \cdot \exp u_1 \cdots \exp u_n \cdots$ is an element of G . Since $G^{(k)}$ is a Lie group, G is a topological group under the inverse limit topology. The purpose of the remainder of this section is to show $G = G'$ and that G is a Frechet-Lie group with

Lie algebra $\mathfrak{g}$.

Let $G_1^{(k)}$, $k \geq 1$, be the group generated by $\{e^{\tilde{p}_k u} ; u \in \mathfrak{g}_1\}$, and G_1 the inverse limit of $\{G_1^{(k)}\}_{k \geq 1}$.

1.1 Lemma exp is a bijective mapping of $\mathfrak{g}_1$ onto G_1 .

Proof. Let $\exp_k$ be the exponential mapping of $\mathfrak{g}_1/\mathfrak{g}_k$ into $G_1^{(k)}$, i.e. $\exp_k u = e^{\tilde{p}_k u}$. Since $\exp : \mathfrak{g}_1 \rightarrow G_1$ is defined by the inverse limit of $\{\exp_k\}$, we have only to show that $\exp_k : \mathfrak{g}_1/\mathfrak{g}_k \rightarrow G_1^{(k)}$ is bijective. Since $\mathfrak{g}_1/\mathfrak{g}_k = \tilde{p}_1 \mathfrak{g}_1$ is a nilpotent Lie algebra, we see that $\exp_k$ is regular and surjective (cf. [3] p 229). However, the derivation $\tilde{p}_k u : \hat{\partial}/\hat{\partial}_k \mapsto \hat{\partial}/\hat{\partial}_k$ is expressed by a triangular matrix with zeros in the diagonal. Therefore, one can define $\log(1 + N)$ by $\sum_{n=1}^{\infty} (-1)^{n-1} N^n/n$, which gives the inverse of $\exp_k$. Thus $\exp_k$ is bijective.

1.2 Corollary $G' = G$.

Proof. We have only to show $G' \supset G$. Since $G^{(1)} = G/G_1$ is generated by $\{\tilde{p}_1 u ; u \in \mathfrak{g}\}$, every $g \in G$ can be written in the form $g = \exp u_1 \cdot \exp u_2 \cdots \exp u_m \cdot h$, where $u_1, \dots, u_m \in \mathfrak{g}$ and $h \in G_1$. Thus, the above lemma shows $G \subset G'$.

We next prove that G is a Frechet-Lie group. Although such a structure of G has no direct relevance to our present purpose, there is an advantage of making analogies easy from the theory of finite dimensional Lie groups.

Let $\varsigma : \tilde{p}_1 \mathfrak{g} \rightarrow \mathfrak{g}$ be a linear mapping such that $\tilde{p}_1 \varsigma \tilde{u} = \tilde{u}$ for $\tilde{u} \in \tilde{p}_1 \mathfrak{g}$. It is not hard to see that $\xi(u) = \exp \varsigma \tilde{p}_1 u \cdot \exp(u - \varsigma \tilde{p}_1 u)$ gives a homeomorphism of an open neighborhood U of 0 of $\mathfrak{g}$ onto an open neighborhood $\tilde{U}$ of the identity e of G . Since G is a topological group, there is an open neighborhood V of 0 of $\mathfrak{g}$ such that

$\xi(v)^{-1} = \xi(v)$, $\xi(v)^2 \subset \xi(u)$. We set $\eta(u, v) = \xi^{-1}(\xi(u)\xi(v))$ and $i(u) = \xi^{-1}(\xi(u)^{-1})$ for $u, v \in V$. We have next to prove the differentiability of η and i . However, the differentiability is defined by inverse limits of differentiable mappings, hence that of η and i are trivial in our case. Thus, we get the following:

1.3 Lemma G is a Frechet-Lie group with Lie algebra $\mathfrak{g}$.

1.B. Simultaneous normalization and eigenspace decomposition

For any $u \in \mathcal{F}_0$, the linear mapping $u^{(k)} : \hat{\mathcal{O}}/\hat{\mathcal{O}}_k \rightarrow \hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ splits uniquely into a sum of semi-simple part $u_s^{(k)}$ and nilpotent part $u_N^{(k)}$ such that $[u_s^{(k)}, u_N^{(k)}] = 0$. Using eigenspace decomposition of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$, we see that $u_s^{(k)}$ is also a derivation of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ hence so is $u_N^{(k)}$. For $u^{(k+1)}$, we have that $[p_k u_s^{(k+1)}, p_k u_N^{(k+1)}] = 0$, $p_k u_N^{(k+1)}$ is nilpotent, and that $p_k u_s^{(k+1)}$ is semi-simple by considering eigenspace decomposition of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_{k+1}$. Therefore, $p_k u_s^{(k+1)} = u_s^{(k)}$ and $p_k u_N^{(k+1)} = u_N^{(k)}$. Hence, taking inverse limit, we get formal vector fields u_s, u_N which will be called the semi-simple part and the nilpotent part of u respectively. A formal vector field is called to be semi-simple if it has no nilpotent part.

Let $\mathfrak{J}^k$ be a nilpotent subalgebra of $\mathcal{F}_0/\mathcal{F}_k$ for an arbitrarily fixed k . Set $\mathfrak{J}_s^k = \{u_s^{(k)} ; u^{(k)} \in \mathfrak{J}^k\}$, and denote by p_k^l the forgetful projection of $\mathcal{F}_0/\mathcal{F}_k$ onto $\mathcal{F}_0/\mathcal{F}_l$ that is, $p_k^l = p_l p_{l+1} \cdots p_{k-1}$. Since $p_k^1 \mathfrak{J}^k$ is a nilpotent subalgebra of $\mathcal{F}_0/\mathcal{F}_1$, there is a basis $(f_1^{(1)}, \dots, f_n^{(1)})$ of $\hat{\mathcal{O}}_0/\hat{\mathcal{O}}_1$ such that every $u^{(1)} \in p_k^1 \mathfrak{J}^k$ is represented by an upper triangular matrix. Let $(\mu_1(u^{(1)}), \dots, \mu_n(u^{(1)}))$ be the diagonal part. μ_j is then a linear mapping of $p_k^1 \mathfrak{J}^k$ into $\mathbb{C}$ for every j , which one may regard as a

linear mapping of $\mathfrak{S}^k$ into $\mathbb{C}$. Since $u_s^{(1)}$ is the semi-simple part of $u^{(1)}$, it must satisfy

$$(5) \quad u_s^{(1)} f_j^{(1)} = \mu_j(u^{(1)}) f_j^{(1)}.$$

By a simple linear algebra, we see that there are $f_1^{(k)}, \dots, f_n^{(k)} \in \hat{\mathcal{O}}_0 / \hat{\mathcal{O}}_k$ such that

$$(6) \quad u_s^{(k)} f_j^{(k)} = \mu_j(u^{(k)}) f_j^{(k)}, \quad \pi_k^\ell f_j^{(k)} = f_j^{(\ell)} \quad (1 \leq j \leq n)$$

for every $u^{(k)} \in \mathfrak{S}^k$, where π_k^ℓ is the forgetful projection of $\hat{\mathcal{O}}_0 / \hat{\mathcal{O}}_k$ onto $\hat{\mathcal{O}}_0 / \hat{\mathcal{O}}_\ell$, that is, $\pi_k^\ell = \pi_\ell \cdot \pi_{\ell+1} \cdots \pi_{k-1}$.

Since $f_j^{(k)} \in \hat{\mathcal{O}}_0 / \hat{\mathcal{O}}_k$, $f_j^{(k)}$ is expressed in the form

$$(7) \quad f_j^{(k)} = \sum_{0 < |\alpha| \leq k} a_{j,\alpha} x^\alpha.$$

Set $y_j = \sum_{0 < |\alpha| \leq k} a_{j,\alpha} x^\alpha$. Since $f_1^{(1)}, \dots, f_n^{(1)}$ are linearly independent, these give a formal change of variables and every $u_s^{(k)}$ can be written in the form

$$(8) \quad u_s^{(k)} = \sum_{i=1}^n \mu_i(u^{(k)}) y_i \partial / \partial y_i.$$

Since $[\mathfrak{S}_s^k, \mathfrak{S}^k] = 0$, because $\mathfrak{S}^k$ is nilpotent, every $u^{(k)} \in \mathfrak{S}^k$ should be written in the form

$$(9) \quad u^{(k)} = \sum_{i=1}^n \sum_{\substack{\langle \alpha, \mu \rangle = \mu_i \\ 0 < |\alpha| \leq k}} a_{i,\alpha} y^\alpha \partial / \partial y_i$$

where $\langle \alpha, \mu \rangle = \alpha_1 \mu_1 + \dots + \alpha_n \mu_n$. It should be noted that the semi-simple part $u_s^{(k)}$ of $u^{(k)}$ has been changed into a linear diagonal vector field such as (8).

Let $\mathfrak{S}^{k+1}$ be another nilpotent subalgebra of $\mathcal{F}_0 / \mathcal{F}_{k+1}$ such that $p_k \mathfrak{S}^{k+1} \subset \mathfrak{S}^k$, and let $\mathfrak{S}_s^{k+1} = \{u_s^{(k+1)} ; u^{(k+1)} \in \mathfrak{S}^{k+1}\}$. Since $p_{k+1}^1 \mathfrak{S}^{k+1} \subset p_k^1 \mathfrak{S}^k$, the equality (5) holds also for every $u^{(1)} \in p_{k+1}^1 \mathfrak{S}^{k+1}$ and the equality (6) does for every $p_k \mathfrak{S}^{k+1}$. By a simple linear algebra, we see that there are $f_1^{(k+1)}, \dots, f_n^{(k+1)} \in \hat{\mathcal{O}}_0 / \hat{\mathcal{O}}_{k+1}$ such that

$$(10) \quad u_s^{(k+1)} f_j^{(k+1)} = \mu_j(u^{(k+1)}) f_j^{(k+1)}, \quad \pi_k f_j^{(k+1)} = f_j^{(k)}.$$

Note that $f_j^{(k+1)} = f_j^{(k)} + \sum_{|\alpha|=k+1} a_{j,\alpha} x^\alpha$. Hence by putting

$$(11) \quad y_j = \sum_{0 < |\alpha| \leq k+1} a_{j,\alpha} x^\alpha$$

instead of (7), we get the same equations as (8) and (9) with respect to $\mathcal{J}^k$. Moreover we have

$$(12) \quad u_s^{(k+1)} = \sum_{i=1}^n \mu_i(u^{(k+1)}) y_i \partial / \partial y_i,$$

$$(13) \quad u^{(k+1)} = \sum_{i=1}^n \sum_{\substack{\langle \alpha, \mu \rangle = \mu_i \\ 0 < |\alpha| \leq k+1}} a_{i,\alpha} y^\alpha \partial / \partial y_i$$

for every $u^{(k+1)} \in \mathcal{J}^{k+1}$. Especially we obtain the following :

1.4 Lemma Notations and assumptions being as above, the forgetful projection $p_k : \mathcal{J}_s^{k+1} \mapsto \mathcal{J}_s^k$ is injective.

Let $\{\mathcal{J}^k\}_{k \geq 1}$ be a series of nilpotent subalgebras $\mathcal{J}^k$ of $\mathcal{F}_0/\mathcal{F}_k$ such that $p_k \mathcal{J}^{k+1} \subset \mathcal{J}^k$ for every $k \geq 1$. We denote by $\mathcal{J}$ the inverse limit, and set $\mathcal{J}_s = \{u_s ; u \in \mathcal{J}\}$. Note that $\dim \mathcal{J}_s^k \leq n$ for every $k \geq 1$. Thus, there is an integer k_0 such that $p_k : \mathcal{J}_s^{k+1} \mapsto \mathcal{J}_s^k$ is bijective for every $k \geq k_0$. By a method of inverse limit, we see that there is a formal change of variables

$$(14) \quad y_j = f_j(x_1, \dots, x_n) \quad 1 \leq j \leq n, \quad f_j \in \hat{\mathcal{O}}_0$$

such that (8) and (9) hold for every $u^{(k)} \in \mathcal{J}^k$ ($k \geq 1$), and

$$(15) \quad u_s = \sum_{i=1}^n \mu_i(u) y_i \partial / \partial y_i,$$

$$(16) \quad u = \sum_{i=1}^n \sum_{\langle \alpha, \mu \rangle = \mu_i} a_{i,\alpha} y^\alpha \partial / \partial y_i$$

for every $u \in \mathcal{J}$.

Now, let $\mathcal{G}$ be a closed subalgebra of $\mathcal{F}_0$, and suppose the above $\mathcal{J}^k$'s are subalgebras of $\mathcal{G}/\mathcal{G}_k$ respectively. Hence, the inverse limit $\mathcal{J}$ is a closed subalgebra of $\mathcal{G}$. We next consider the eigen-space decomposition of $\mathcal{G}$ with respect to $\text{ad}(\mathcal{J})$. Since

$\text{ad}(u) : \mathcal{F}_0 \mapsto \mathcal{F}_0$ leaves $\mathcal{G}$ invariant for every $u \in \mathcal{J}$, and $[\text{ad}(u), \text{ad}(u_s)] = 0$, we see that $\text{ad}(u_s) : \mathcal{F}_0 \mapsto \mathcal{F}_0$ is the semi-simple part of $\text{ad}(u)$ and hence $\text{ad}(u_s)\mathcal{G} \subset \mathcal{G}$. Therefore, we have only to consider the eigenspace decomposition with respect to $\text{ad}(\mathcal{J}_s)$.

For a linear mapping λ of $\check{p}_1\mathcal{J}_s$ into $\mathbb{C}$, i.e. $\lambda \in (\check{p}_1\mathcal{J}_s)^*$, we denote by $\mathcal{F}_\lambda$ the subspace

$$\{u \in \mathcal{F}_0 ; u = \sum_{i=1}^n \sum_{\langle \alpha, \mu \rangle, \mu_i = \lambda} a_{i,\alpha} y^\alpha \partial / \partial y_i\}.$$

Note that $\mathcal{F}_\lambda = \{0\}$ for almost all $\lambda \in (\check{p}_1\mathcal{J}_s)^*$ except countably many λ 's. By $\Pi(\mathcal{J})$ we denote the set of all $\lambda \in (\check{p}_1\mathcal{J}_s)^*$ such that $\mathcal{F}_\lambda \neq \{0\}$. If $\check{p}_1\mathcal{J}_s = \{0\}$, then we set $\Pi(\mathcal{J}) = 0$, because all μ_j 's are zeros.

1.5 Lemma If $\check{p}_1\mathcal{J}_s = 0$, then $\mathcal{G}^{(0)}(\mathcal{J}) = \mathcal{G}$.

Proof. By (16), every $u \in \mathcal{J}$ can be written in the form $u = u_1 + u_2$ such that

$$u_1 = \sum_{i=1}^{n-1} \sum_{j=i+1}^n a_j^i y_j \partial / \partial y_i, \quad u_2 = \sum_{i=1}^n \sum_{|\alpha| \geq 2} a_{i,\alpha} y^\alpha \partial / \partial y_i.$$

The reason for the shape of u_1 is that the linear part of u is an upper triangular matrix. Therefore, for every $k \geq 1$, there is an integer m_k such that $\text{ad}(u)^{m_k} \mathcal{F}_0 \subset \mathcal{F}_k$ for every $u \in \mathcal{J}$. This means $\mathcal{G} = \mathcal{G}^{(0)}(\mathcal{J})$ by definition.

Now, we set $\mathcal{G}^{(\lambda)}(\mathcal{J}) = \mathcal{G} \cap \mathcal{F}_\lambda$ for every $\lambda \in \Pi(\mathcal{J})$.

1.6 Lemma Every $u \in \mathcal{G}$ can be rearranged in the form

$$u = \sum_{\lambda \in \Pi(\mathcal{J})} u_\lambda, \quad u_\lambda \in \mathcal{F}_\lambda$$

Moreover, every u_λ is contained in $\mathcal{G}^{(\lambda)}(\mathcal{J})$.

Proof. Since the first assertion is trivial, we have only to show the second one. Since $\Pi(\mathcal{J})$ is a countable set, there is $v_0 \in \mathcal{J}_s$ such that $\lambda(v_0^{(1)}) \neq \lambda'(v_0^{(1)})$ for any $\lambda, \lambda' \in \Pi(\mathcal{J})$ such that $\lambda \neq \lambda'$. For every k , let $u^{(k)}$ be the truncation of $u \in \mathcal{G}$ at the

order k . $u^{(k)}$ is canonically identified with $\tilde{p}_k u$. $u^{(k)}$ can be rearranged in the form $u^{(k)} = \sum_{\lambda \in \Pi(\mathfrak{g})} u_\lambda^{(k)}$, where each $u_\lambda^{(k)}$ is the truncation of u_λ at the order k . Since $\mathfrak{g}/\mathfrak{g}_k$ is finite dimensional, only finite number of $u_\lambda^{(k)}$'s do not vanish. Apply $\text{ad}(v_o^{(k)})^\ell$ to $u^{(k)}$. Since $\text{ad}(\mathfrak{g}_s)\mathfrak{g} \subset \mathfrak{g}$, we have

$$\text{ad}(v_o^{(k)})^\ell u^{(k)} = \sum_{\lambda \in \Pi(\mathfrak{g})} (v_o)^\ell u_\lambda^{(k)} \in \mathfrak{g}/\mathfrak{g}_k$$

Hence, considering Vandermonde's matrix, we get $u_\lambda^{(k)} \in \mathfrak{g}/\mathfrak{g}_k$. Thus, taking inverse limit, we get $u_\lambda \in \mathfrak{g}$, hence the desired result.

1.7 Corollary $\tilde{p}_k \mathfrak{g}^{(o)}(\mathfrak{g})$ is the zero-eigenspace of $\text{ad}(\tilde{p}_k \mathfrak{g})$:

$$\mathfrak{g}/\mathfrak{g}_k \mapsto \mathfrak{g}/\mathfrak{g}_k.$$

Proof. It is trivial that $\tilde{p}_k \mathfrak{g}^{(o)}(\mathfrak{g})$ is contained in the zero-eigenspace of $\text{ad}(\tilde{p}_k \mathfrak{g})$, for $[\mathfrak{g}_s, \mathfrak{g}^{(o)}(\mathfrak{g})] = \{0\}$. Thus, we have only to show the converse. The zero-eigenspace of $\text{ad}(\tilde{p}_k \mathfrak{g})$ is equal to that of $\text{ad}(\tilde{p}_k \mathfrak{g}_s)$, that is, the space of all $v^{(k)} \in \mathfrak{g}/\mathfrak{g}_k$ such that $[\tilde{p}_k \mathfrak{g}_s, v^{(k)}] = \{0\}$. Thus, $v^{(k)}$ should be written in the form (9). Let $v \in \mathfrak{g}$ be an element such that $\tilde{p}_k v = v^{(k)}$, and let $v = \sum_{\lambda \in \Pi(\mathfrak{g})} v_\lambda$ be the decomposition in accordance with the above lemma. Then it is clear that $\tilde{p}_k v_o = v^{(k)}$. Since $v_o \in \mathfrak{g}^{(o)}(\mathfrak{g})$, we get the desired result.

1.C Existence and conjugacy of Cartan subalgebras

Let $\mathfrak{g}$ be a closed subalgebra of $\mathfrak{F}_o$. If $\mathfrak{g}/\mathfrak{g}_1 = \{0\}$, then $\mathfrak{g}/\mathfrak{g}_k$ is nilpotent for every $k \geq 1$, for $[\mathfrak{g}_k, \mathfrak{g}_\ell] \subset \mathfrak{g}_{k+\ell}$. Therefore, by 1.5 Lemma, we see that $\mathfrak{g}$ itself is the only Cartan subalgebra of $\mathfrak{g}$. Thus, the conjugacy is trivial in this case.

Now, suppose $\mathfrak{g}/\mathfrak{g}_1 \neq \{0\}$, and let $\mathfrak{h}^1$ be a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_1$.

1.8 Lemma Let $\mathfrak{g}^1, \dots, \mathfrak{g}^k$ be a series of Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_1, \dots, \mathfrak{g}/\mathfrak{g}_k$ respectively such that $p_{l-1} \mathfrak{g}^l = \mathfrak{g}^{l-1}$ for $2 \leq l \leq k$. Then, there is a Cartan subalgebra $\mathfrak{g}^{k+1}$ of $\mathfrak{g}/\mathfrak{g}_{k+1}$ such that $p_k \mathfrak{g}^{k+1} = \mathfrak{g}^k$.

Proof. Let $\mathfrak{g}'$ be a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_{k+1}$. We prove at first that $p_k \mathfrak{g}'$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$. Since $\mathfrak{g}'$ is nilpotent, so is $p_k \mathfrak{g}'$. Let $\mathfrak{g}'_s = \{u_s^{(k+1)}; u_s^{(k+1)} \in \mathfrak{g}'\}$, and let $v^{(k)}$ be an element of the zero-eigenspace of $p_k \mathfrak{g}'_s$. Then, $[v^{(k)}, p_k \mathfrak{g}'_s] = \{0\}$ and hence $v^{(k)}$ can be written in the form (9). Let $v^{(k+1)}$ be an element of $\mathfrak{g}/\mathfrak{g}_{k+1}$ such that $p_k v^{(k+1)} = v^{(k)}$. Using the eigenspace decomposition of $\mathfrak{g}/\mathfrak{g}_{k+1}$ with respect to $\text{ad}(\mathfrak{g}'_s)$, we see that $v^{(k+1)} = \sum_{\lambda \in \Pi(\mathfrak{g}'_s)} v_\lambda^{(k+1)}$. Note that this decomposition is given by only rearranging of the terms of $v^{(k+1)}$ (cf. 1.6 Lemma). Hence it is clear that $p_k v_o^{(k+1)} = v^{(k)}$. $v_o^{(k+1)}$ is an element of the zero-eigenspace of $\mathfrak{g}'_s$. However, since $\mathfrak{g}'$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_{k+1}$ we get $v_o^{(k+1)} \in \mathfrak{g}'$. Thus, $v^{(k)} \in p_k \mathfrak{g}'$. Hence $p_k \mathfrak{g}'$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$.

By the well-known conjugacy of Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_k$, there is an inner automorphism A such that $A(p_k \mathfrak{g}') = \mathfrak{g}^k$. Since there is a natural projection of $G^{(k+1)}$ onto $G^{(k)}$ (cf. 1.A), there is an inner automorphism A' of $\mathfrak{g}/\mathfrak{g}_{k+1}$ which induces naturally A . Thus, by setting $A' \mathfrak{g}' = \mathfrak{g}^{k+1}$, $\mathfrak{g}^{k+1}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_{k+1}$ such that $p_k \mathfrak{g}^{k+1} = \mathfrak{g}^k$.

By the above lemma, we have a series $\{\mathfrak{g}^k\}_{k \geq 1}$ of Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_k$ such that $p_k \mathfrak{g}^{k+1} = \mathfrak{g}^k$. Let $\mathfrak{g}$ be the inverse limit of $\mathfrak{g}^k$.

1.9 Lemma Notations and assumptions being as above, $\mathfrak{g}$ is a Cartan

subalgebra of $\mathfrak{g}$.

Proof. Since $\tilde{p}_k \mathfrak{g} = \mathfrak{g}^k$, $\tilde{p}_k \mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{g}/\mathfrak{g}_k$ for every $k \geq 1$. By 1.7 Corollary, $\tilde{p}_k \mathfrak{g}^{(0)}(\mathfrak{g})$ is the zero-eigenspace of $\text{ad}(\tilde{p}_k \mathfrak{g})$. Since $\tilde{p}_k \mathfrak{g} = \mathfrak{g}^k$ is a Cartan subalgebra, we have $\tilde{p}_k \mathfrak{g}^{(0)}(\mathfrak{g}) = \mathfrak{g}^k$ and hence $\mathfrak{g}^{(0)}(\mathfrak{g}) = \mathfrak{g}$. Thus, $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$.

We next consider the converse of the above lemma.

1.10 Lemma Let $\mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$. Then, $\tilde{p}_k \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$ for every $k \geq 1$.

Proof. By 1.7 Corollary, the zero-eigenspace of $\text{ad}(\tilde{p}_k \mathfrak{g})$ is equal to $\tilde{p}_k \mathfrak{g}^{(0)}(\mathfrak{g})$. Since $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$, we see $\tilde{p}_k \mathfrak{g}^{(0)}(\mathfrak{g}) = \tilde{p}_k \mathfrak{g}$. Thus, $\tilde{p}_k \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$.

As in 1.A, we denote by $G^{(k)}$ the Lie group generated by $\{e^{\tilde{p}_k u} ; u \in \mathfrak{g}\}$. Let $\pi_k : G^{(k+1)} \rightarrow G^{(k)}$ be the natural projection. We shall next prove the conjugacy of Cartan subalgebras, which completes the proof of Proposition A. Let $\mathfrak{g}, \hat{\mathfrak{g}}$ be Cartan subalgebras of $\mathfrak{g}$. By the argument in the first part of this section, we may assume $\mathfrak{g}/\mathfrak{g}_1 \neq \{0\}$. Since $\tilde{p}_1 \mathfrak{g}, \tilde{p}_1 \hat{\mathfrak{g}}$ are Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_1$, there is $g_1 \in G^{(1)}$ such that $\text{Ad}(g_1)(\tilde{p}_1 \mathfrak{g}) = \tilde{p}_1 \hat{\mathfrak{g}}$. Therefore, one may assume without loss of generality that $\tilde{p}_1 \mathfrak{g} = \tilde{p}_1 \hat{\mathfrak{g}}$. Let $G_\ell^{(k)}$ be the Lie group generated by $\{e^{\tilde{p}_\ell u} ; u \in \mathfrak{g}_\ell\}$ for any $\ell, \ell \leq k$.

1.11 Lemma Let $\mathfrak{g}, \hat{\mathfrak{g}}$ be Cartan subalgebras of $\mathfrak{g}$ such that $\tilde{p}_k \mathfrak{g} = \tilde{p}_k \hat{\mathfrak{g}}$. Then, there is $g_{k+1} \in G_k^{(k+1)}$ such that $\text{Ad}(g_{k+1})(\tilde{p}_{k+1} \mathfrak{g}) = \tilde{p}_{k+1} \hat{\mathfrak{g}}$.

Proof. Since $\tilde{p}_k \mathfrak{g} = \tilde{p}_k \hat{\mathfrak{g}}$, $\tilde{p}_{k+1} \mathfrak{g}$ and $\tilde{p}_{k+1} \hat{\mathfrak{g}}$ are Cartan subalgebras of $\tilde{p}_k^{-1} \tilde{p}_k \mathfrak{g} = \tilde{p}_k^{-1} \tilde{p}_k \hat{\mathfrak{g}}$. Let

$$p_k^{-1} \tilde{p}_k \mathfrak{g} = \tilde{p}_{k+1} \mathfrak{g} \oplus \sum_{\lambda \neq 0} \mathfrak{g}'_{\lambda}, \quad p_k^{-1} p_k = \tilde{p}_{k+1} \hat{\mathfrak{g}} \oplus \sum_{\lambda \neq 0} \mathfrak{g}''_{\lambda}$$

be the eigenspace decompositions with respect to $\text{ad}(\tilde{p}_{k+1} \mathfrak{g})$ and $\text{ad}(\tilde{p}_{k+1} \hat{\mathfrak{g}})$ respectively. Since $p_k \tilde{p}_{k+1} \mathfrak{g} = p_k \tilde{p}_{k+1} \hat{\mathfrak{g}} = p_k \mathfrak{g}$, we see that $\sum \mathfrak{g}'_{\lambda} \subset \mathfrak{g}_k / \mathfrak{g}_{k+1}$ and $\sum \mathfrak{g}''_{\lambda} \subset \mathfrak{g}_k / \mathfrak{g}_{k+1}$. It is well-known (cf. [6] pp59-66) that there are $v_1, \dots, v_m \in \sum_{\lambda \neq 0} \mathfrak{g}'_{\lambda}$, $w_1, \dots, w_\ell \in \sum_{\lambda \neq 0} \mathfrak{g}''_{\lambda}$ such that

$$\text{Ad}(\exp v_1) \cdots \text{Ad}(\exp v_m) \text{Ad}(\exp w_1) \cdots \text{Ad}(\exp w_\ell) \tilde{p}_{k+1} \mathfrak{g} = \tilde{p}_{k+1} \hat{\mathfrak{g}}$$

Since $\exp v_i, \exp w_j \in G_k^{(k+1)}$, we see that there is $g_{k+1} \in G_k^{(k+1)}$ such that $\text{Ad}(g_{k+1})(\tilde{p}_{k+1} \mathfrak{g}) = \tilde{p}_{k+1} \hat{\mathfrak{g}}$.

Let G_k be the subgroup of G generated by $\{e^u; u \in \mathfrak{g}_k\}$. For Cartan subalgebras $\mathfrak{g}, \hat{\mathfrak{g}}$ of $\mathfrak{g}$, the above lemma shows that there are elements $g_1, g_2, \dots, g_k, \dots$ such that $g_k \in G_k$ and

$$\text{Ad}(g_1) \text{Ad}(g_2) \cdots \text{Ad}(g_k) \hat{\mathfrak{g}} = \mathfrak{g} \mod \mathfrak{g}_{k+1}.$$

Note that $g_1 g_2 \cdots g_k \cdots \in G$, hence putting $g = g_1 g_2 \cdots g_k \cdots$, we see $\text{Ad}(g) \mathfrak{g} = \hat{\mathfrak{g}}$. This shows the conjugacy of Cartan subalgebras.

Proposition A is thereby proved.

§2 Cartan subalgebras at expansive singularities

2.A Semi-simple expansive vector fields

In this section, notations are as in the introduction. A germ of holomorphic vector field $u \in \mathcal{X}(V)$ is called expansive, if the eigenvalues of the linear term of u at 0 lie in the same open half plane in $\mathbb{C}$ about the origin. u is called to be semi-simple expansive if u is expansive and semi-simple as a formal vector field. The purpose of this section is to show the following :

2.1 Lemma Let $u \in \mathcal{X}(V)$ be a semi-simple expansive vector field. Then, there is a germ $y_j = f_j(x_1, \dots, x_n)$, $1 \leq j \leq n$, of biholomorphic change of variables such that u can be written in the form

$$u = \sum_{i=1}^n \hat{\mu}_i y_i \partial / \partial y_i$$

Proof. By a suitable change of variables $y_j = \sum_{0 < |k| \leq k} a_{j,k} x^k$ such as in (7), we have that u can be written in the form

$$u = \sum_{i=1}^n \hat{\mu}_i y_i \partial / \partial y_i + w, \quad w \in \mathcal{X}_k(V)$$

for sufficiently large k . For the proof that u is linearizable, it is enough to show that there are holomorphic functions $f_1, \dots, f_n$ in $y_1, \dots, y_n$ such that $u f_j = \hat{\mu}_j f_j$ ($1 \leq j \leq n$) and $f_j = y_j + \text{higher order terms}$. Set $f_j = y_j + g_j$ and consider the equation $u(y_j + g_j) = \hat{\mu}_j(y_j + g_j)$. Then we get

$$(17) \quad (u - \hat{\mu}_j) g_j = -w y_j.$$

Since k is sufficiently large, we have

$$(18) \quad \lim_{t \rightarrow \infty} e^{-t(u - \hat{\mu}_j)} w y_j = 0$$

and

$$(19) \quad - \int_0^\infty e^{-t(u - \hat{\mu}_j)} w y_j dt$$

exists as a germ of holomorphic functions (cf. [5]). Set $g_j =$

$-\int_0^\infty e^{-t(u - \hat{\mu}_j)} w y_j dt$. Then,

$$(u - \hat{\mu}_j)g_j = \int_0^\infty \frac{d}{dt} e^{-t(u - \hat{\mu}_j)} w y_j dt = [e^{-t(u - \hat{\mu}_j)} w y_j]_0^\infty = -w y_j.$$

2.B Lie algebras containing semi-simple expansive vector fields.

Let $\mathfrak{g}$ be a closed subalgebra of $\mathfrak{f}_0$ such that $\mathfrak{g}$ contains a semi-simple expansive vector field X .

2.2 Lemma Let X be a semi-simple expansive vector field in $\mathfrak{g}$. Then, there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing X .

Proof. By the same proof as in the above lemma, we see that X can be linearizable by a suitable formal change of variables, and hence we may assume that X can be written in the form $X = \sum_{i=1}^n \hat{\mu}_i y_i \partial / \partial y_i$, $\operatorname{Re} \hat{\mu}_i > 0$. Let $\mathfrak{g}^{(0)}(X) = \{u \in \mathfrak{g} ; [X, u] = 0\}$. Since every $u \in \mathfrak{g}^{(0)}(X)$ can be written in the form

$$(20) \quad u = \sum_{i=1}^n \sum_{\langle \alpha, \hat{\mu} \rangle = \hat{\mu}_i} a_{i, \alpha} y^\alpha \partial / \partial y_i,$$

we see that $\mathfrak{g}^{(0)}(X)$ is a finite dimensional Lie subalgebra of $\mathfrak{g}$. Since $\operatorname{ad}(X) : \mathfrak{g}^{(0)}(X) \rightarrow \mathfrak{g}^{(0)}(X)$ is of diagonal type, there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{(0)}(X)$ containing X . We shall show that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. For that purpose we have only to show $\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathfrak{h}$. Since $X \in \mathfrak{h}$, we see $\mathfrak{g}^{(0)}(\mathfrak{h}) \subset \mathfrak{g}^{(0)}(X)$ and hence $\mathfrak{g}^{(0)}(\mathfrak{h})$ is the zero-eigenspace of $\operatorname{ad}(\mathfrak{h})$ in $\mathfrak{g}^{(0)}(X)$. However since $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}^{(0)}(X)$, we have $\mathfrak{h} = \mathfrak{g}^{(0)}(\mathfrak{h})$.

2.3 Corollary If $\mathfrak{g}$ has a semi-simple expansive vector field, then every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is finite dimensional and $\mathfrak{g}^{(\lambda)}(\mathfrak{h})$ is finite dimensional for every $\lambda \in \pi(\mathfrak{h})$.

Proof. By the above lemma, there is a finite dimensional Cartan subalgebra of $\mathfrak{g}$. However by Proposition A it implies that all Cartan subalgebras are finite dimensional and every Cartan subalgebra contains a semi-simple expansive vector field. Note that

$$\mathfrak{g}_\lambda = \{u \in \mathfrak{g}_0 ; u = \sum_{i=1}^n \sum_{\langle \mu, \mu \rangle - \mu_i = \lambda} a_{i,\alpha} y_i^\alpha \partial / \partial y_i\}$$

Since $\mathfrak{g}$ contains an expansive vector field, we see that $\dim \mathfrak{g}_\lambda < \infty$ and hence $\dim \mathfrak{g}^{(\lambda)}(\mathfrak{g}) < \infty$.

2.4 Corollary Notations being as in the introduction, if $\mathfrak{X}(V)$ contains a semi-simple expansive vector field X , then there is a Cartan subalgebra $\mathfrak{g}$ of $\mathfrak{g}(V)$ such that $\mathfrak{g} \subset \mathfrak{X}(V)$. Moreover, for that $\mathfrak{g}$, $\mathfrak{g}^{(\lambda)}(\mathfrak{g})$ is contained in $\mathfrak{X}(V)$ for every $\lambda \in \pi(\mathfrak{g})$.

Proof. Since $X \in \mathfrak{X}(V)$, 2.1 Lemma shows that X can be written in the form $X = \sum_{i=1}^n \hat{\mu}_i y_i^\alpha \partial / \partial y_i$ by a suitable biholomorphic change of variables. Therefore, every $u \in \mathfrak{g}^{(\lambda)}(\mathfrak{g})$ is contained in $\mathfrak{X}(V)$, because u is a polynomial vector field in $y_1, \dots, y_n$.

2.C Isomorphisms of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$.

Let V, V' be germs of varieties in $\mathbb{C}^n, \mathbb{C}^{n'}$ respectively.

Suppose there is a bicontinuous isomorphism Φ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$.

2.5 Lemma Let $\mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}(V)$. Then, so is $\Phi(\mathfrak{g})$ of $\mathfrak{g}(V')$.

Proof. Set $\mathfrak{g}' = \Phi(\mathfrak{g})$. Since $\Phi : \mathfrak{g}(V) \rightarrow \mathfrak{g}(V')$ is continuous, for every k' there is an integer $k = k(k')$ such that $\Phi(\mathfrak{g}_k(V)) \subset \mathfrak{g}_{k'}(V')$. Thus, $\tilde{\mathfrak{p}}_k, \mathfrak{g}'$ is a nilpotent subalgebra of $\mathfrak{g}(V')/\mathfrak{g}_{k'}(V')$ and $\mathfrak{g}^{(0)}(\mathfrak{g}') \supset \Phi(\mathfrak{g}^{(0)}(\mathfrak{g}))$. Thus, replacing Φ by Φ^{-1} , we get the desired result.

Now, suppose that V and V' have expansive singularities at the origins respectively. By 2.4 Corollary, $\mathcal{X}(V)$ and $\mathcal{X}(V')$ contain Cartan subalgebras of $\mathfrak{g}(V)$ and $\mathfrak{g}(V')$ respectively.

2.6 Corollary Assumptions being as above, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}(V)$ contained in $\mathcal{X}(V)$. Suppose there is a bicontinuous isomorphism Φ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$. Then, there is a bicontinuous isomorphism Ψ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$ such that $\Psi(\mathfrak{h}) \subset \mathcal{X}(V')$, that is, $\Psi(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}(V')$ contained in $\mathcal{X}(V')$.

Proof. By the above lemma, $\Phi(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}(V')$. By 2.4 Corollary, there is a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}(V')$ contained in $\mathcal{X}(V')$. By Proposition A, there is $g \in G$ such that $\text{Ad}(g)\Phi(\mathfrak{h}) = \mathfrak{h}'$. Note that $\text{Ad}(g) : \mathfrak{g}(V') \rightarrow \mathfrak{g}(V')$ is a bicontinuous isomorphism. Thus, $\Psi = \text{Ad}(g)\Phi$ is the desired one.

In the remainder of this section, we assume that there is a bicontinuous isomorphism $\Phi : \mathfrak{g}(V) \rightarrow \mathfrak{g}(V')$ such that $\Phi(\mathfrak{h}) = \mathfrak{h}'$ where $\mathfrak{h}, \mathfrak{h}'$ are Cartan subalgebras of $\mathfrak{g}(V), \mathfrak{g}(V')$ respectively such that $\mathfrak{h} \subset \mathcal{X}(V)$ and $\mathfrak{h}' \subset \mathcal{X}(V')$. By 2.3-4 Corollaries, there is a local coordinate system $(y_1, \dots, y_n)$, related biholomorphically to the original one such that every $\mathfrak{g}^{(\lambda)}(\mathfrak{h})$ is a finite dimensional space of polynomial vector fields in $y_1, \dots, y_n$. We choose such a local coordinate system $(z_1, \dots, z_n)$ for $\mathfrak{g}(V')$. Let $\mathcal{P}(V; y_1, \dots, y_n)$ (resp. $\mathcal{P}(V'; z_1, \dots, z_n)$) be the totality of $u \in \mathfrak{g}(V)$ (resp. $\mathfrak{g}(V')$) such that u can be expressed as a polynomial vector field in $y_1, \dots, y_n$ (resp. $z_1, \dots, z_n$), $\mathcal{P}(V; y_1, \dots, y_n)$ and $\mathcal{P}(V'; z_1, \dots, z_n)$ are Lie subalgebras of $\mathcal{X}(V), \mathcal{X}(V')$ respectively. Since $\mathfrak{g}^{(\lambda)}(\mathfrak{h}) \subset \mathcal{P}(V; y_1, \dots, y_n)$ for every $\lambda \in \Pi(\mathfrak{h})$, we get the

following :

2.7 Corollary Notations and assumptions being as above, the above isomorphism $\Phi : \mathcal{G}(V) \mapsto \mathcal{G}(V')$ induces an isomorphism of

$\mathcal{P}(V; y_1, \dots, y_n)$ onto $\mathcal{P}(V'; z_1, \dots, z_n)$.

Proof. Note that $\Phi(\mathcal{G}^{(\lambda)}(\xi)) = \mathcal{G}^{(\lambda)}(\xi')$, because $\mathcal{G}^{(\lambda)}(\xi)$ is an eigenspace of $\text{ad}(\xi)$. Every $u \in \mathcal{P}(V; y_1, \dots, y_n)$ can be written in the form $u = \sum_{\lambda \in \Pi(\xi)} u_\lambda$, but the summation in this case is a finite sum. Since $\Phi(u) = \sum_{\lambda \in \Pi(\xi)} \Phi(u_\lambda)$ and $\Phi(u_\lambda) \in \mathcal{G}^{(\lambda)}(\xi')$, we see that $\Phi(u) \in \mathcal{P}(V'; z_1, \dots, z_n)$. Replacing Φ by Φ^{-1} , we get the desired result.

Let $\mathbb{C}[y_1, \dots, y_n]$ be the ring of all polynomials in $y_1, \dots, y_n$. Then, since $\mathcal{G}(V)$ is an $\hat{\mathcal{O}}$ -module, $\mathcal{P}(V; y_1, \dots, y_n)$ is a $\mathbb{C}[y_1, \dots, y_n]$ -module.

3 Theorem of Pursell-Shanks' type

In this chapter, we consider two Lie algebras $\mathfrak{g}(V; y_1, \dots, y_n)$ and $\mathfrak{g}(V'; z_1, \dots, z_n)$ of polynomial vector fields such that they are $\mathbb{C}[y_1, \dots, y_n]$ and $\mathbb{C}[z_1, \dots, z_n]$ -module respectively and that there is an isomorphism Φ of $\mathfrak{g}(V; y_1, \dots, y_n)$ onto $\mathfrak{g}(V'; z_1, \dots, z_n)$. The goal is as follows :

Theorem II Notations and assumptions being as above, there is a bi-holomorphic mapping φ of $\mathbb{C}^n$ onto $\mathbb{C}^n$ such that $d\varphi \mathfrak{g}(V; y_1, \dots, y_n) = \mathfrak{g}(V'; z_1, \dots, z_n)$. Moreover, $\varphi(V) = V'$ as germs of varieties.

Note at first that Theorem II implies Theorem I in the introduction, for 2.6-7 Corollaries show that an isomorphism between $\mathfrak{g}(V)$ and $\mathfrak{g}(V')$ induces an isomorphism between $\mathfrak{g}(V; y_1, \dots, y_n)$ and $\mathfrak{g}(V'; z_1, \dots, z_n)$.

3.A Characterization of maximal subalgebras

Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{g}(V; y_1, \dots, y_n)$. We denote by $\mathfrak{g}^{(\infty)}$ the ideal consisting of all $u \in \mathfrak{g}$ such that $\text{ad}(v_1) \dots \text{ad}(v_k)u \in \mathfrak{g}$ for every $k \geq 0$ and any $v_1, \dots, v_k \in \mathfrak{g}(V; y_1, \dots, y_n)$. Let $V_{\mathfrak{g}}$ be the set of all points $q \in \mathbb{C}^n$ such that $\mathfrak{g}(V; y_1, \dots, y_n)$ does not span n -dimensional vector space at q , that is, $\dim \mathfrak{g}(V; y_1, \dots, y_n)(q) < n$. For a point $p \in \mathbb{C}^n$, let $\mathfrak{g}_p$ be the isotropy subalgebra of $\mathfrak{g}(V; y_1, \dots, y_n)$ at p , i.e. $\mathfrak{g}_p = \{u \in \mathfrak{g}(V; y_1, \dots, y_n) ; u(p) = 0\}$.

3.1 Lemma For a point $p \in \mathbb{C}^n - V_{\mathfrak{g}}$, $\mathfrak{g}_p$ is a maximal, finite co-dimensional subalgebra such that $\mathfrak{g}_p^{(\infty)} = \{0\}$.

Proof. Since $p \in \mathbb{C}^n - V_{\mathfrak{g}}$, there are $u_1, \dots, u_n \in \mathfrak{g}(V; y_1, \dots, y_n)$

such that $u_j(p) = \partial/\partial y_j|_p$ for $1 \leq j \leq n$. Consider

$$(\text{ad}(u_1)^{l_1} \cdots \text{ad}(u_n)^{l_n} v)(p) = 0$$

for any $l_1, \dots, l_n$, and we get easily that $\mathfrak{p}_p^{(\infty)} = \{0\}$.

We next prove the maximality of $\mathfrak{p}_p$. Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{p}(V; y_1, \dots, y_n)$ such that $\mathfrak{g} \supsetneq \mathfrak{p}_p$. There is then an element $v \in \mathfrak{g}$ such that $v(p) \neq 0$. By a suitable linear change of variables, we may assume that v is written in the form

$$(21) \quad v = g \partial/\partial y_1 + \sum_{j=2}^n h_j \partial/\partial y_j, \quad g(p) \neq 0, h_j(p) = 0.$$

Let $(p_1, \dots, p_n)$ be the coordinate of p . Then, $(y_1 - p_1)u_j \in \mathfrak{p}_p$ for $1 \leq j \leq n$. Therefore, $[v, (y_1 - p_1)u_j] = v(y_1)u_j + (y_1 - p_1)[v, u_j] \in \mathfrak{g}$. Since $v(y_1)(p) = g(p) \neq 0$, we have $\mathfrak{g}(p) = \mathfrak{p}(V; y_1, \dots, y_n)(p)$ and hence $\mathfrak{g} = \mathfrak{p}(V; y_1, \dots, y_n)$.

Let $\mathcal{W}_{\mathfrak{p}}$ be the set of all points q such that $\mathfrak{p}_q$ is a maximal subalgebra and $\mathfrak{p}_q^{(\infty)} = \{0\}$. By the above lemma, $\mathcal{W}_{\mathfrak{p}}$ contains $\mathbb{C}^n - V_{\mathfrak{p}}$. The goal of this section is as follows :

3.2 Proposition Let $\mathfrak{g}$ be a maximal, finite codimensional subalgebra of $\mathfrak{p}(V; y_1, \dots, y_n)$ such that $\mathfrak{g}^{(\infty)} = \{0\}$. Then, there is a unique point $p \in \mathcal{W}_{\mathfrak{p}}$ such that $\mathfrak{g} = \mathfrak{p}_p$.

Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{p}(V; y_1, \dots, y_n)$, and let $J = \{f \in \mathbb{C}[y_1, \dots, y_n] ; f \mathfrak{p}(V; y_1, \dots, y_n) \subset \mathfrak{g}\}$. Obviously, J is an ideal of $\mathbb{C}[y_1, \dots, y_n]$, for $\mathfrak{p}(V; y_1, \dots, y_n)$ is a $\mathbb{C}[y_1, \dots, y_n]$ -module.

3.3 Lemma Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{p}(V; y_1, \dots, y_n)$ such that $\mathbb{C}[y_1, \dots, y_n] \mathfrak{g} = \mathfrak{p}(V; y_1, \dots, y_n)$. Then $J \mathfrak{p}(V; y_1, \dots, y_n)$ is an ideal of $\mathfrak{p}(V; y_1, \dots, y_n)$ contained in $\mathfrak{g}$.

Proof. By definition $J\phi(V; y_1, \dots, y_n) \subset \mathcal{J}$. Since $(uf)v = [u, fv] - f[u, v]$, we have $\mathcal{J}J \subset J$, hence $(\mathbb{C}[y_1, \dots, y_n]\mathcal{J})J \subset J$. By the assumption, we get $\phi(V; y_1, \dots, y_n)J \subset J$. Therefore, $J\phi(V; y_1, \dots, y_n)$ is an ideal of $\phi(V; y_1, \dots, y_n)$.

By the above lemma, we see also that $J\phi(V; y_1, \dots, y_n) \subset \mathcal{J}^{(\infty)}$. The next lemma is due to Amamiya [1]. The proof is seen also in [5], however we repeat the proof for the sake of selfcontainedness.

3.4 Lemma Let $\mathcal{J}$ be a finite codimensional subalgebra of $\phi(V; y_1, \dots, y_n)$. Then, $J \neq \{0\}$.

Proof. Set $\mathcal{J}^{(1)} = \{u \in \mathcal{J} ; [u, \phi(V; y_1, \dots, y_n)] \subset \mathcal{J}\}$. Since $\text{codim } \mathcal{J} < \infty$ and $\text{ad}(u)$ for every $u \in \mathcal{J}$ induces a linear mapping of $\phi(V; y_1, \dots, y_n)/\mathcal{J}$ into itself, we see that $\text{codim } \mathcal{J}^{(1)} < \infty$ and hence in particular $\mathcal{J}^{(1)} \neq \{0\}$.

Let v be a non-trivial element in $\mathcal{J}^{(1)}$, and let f be a polynomial such that $vf \neq 0$. Consider a sequence $fv, f^2v, f^3v, \dots$. Since $\text{codim } \mathcal{J}^{(1)} < \infty$, there is a polynomial $P(t)$ in t such that $P(f)v \in \mathcal{J}^{(1)}$.

We next prove that if v and gv are contained in $\mathcal{J}^{(1)}$, then $(vg)^2 \in J$. For that purpose, let w be an arbitrary element of $\phi(V; y_1, \dots, y_n)$. Then, we have

$$[v, gw] = (vg)w + g[w, v] \in \mathcal{J}$$

$$[gv, w] = -(wg)v + g[w, v] \in \mathcal{J}$$

Hence

$$(22) \quad (vg)w + (wg)v \in \mathcal{J}$$

for every $w \in \phi(V; y_1, \dots, y_n)$. Replacing w by $(wg)v$, we have

$(vg)(wg)v \in \mathcal{J}$. Replacing w in (22) by $(vg)w$, we have also

$$(vg)^2w + (vg)(wg)v \in \mathcal{J}$$

Hence $(vg)^2w \in \mathcal{J}$. Thus, $(vg)^2 \in J$.

Set $g = P(f)$. Then, $v, gv \in \mathcal{G}^{(1)}$ and $vg \neq 0$ because of $vf \neq 0$. Thus, we get $J \neq \{0\}$.

3.5 Corollary Let $\mathcal{G}$ be a maximal finite codimensional subalgebra of $\mathcal{P}(V; y_1, \dots, y_n)$ such that $\mathcal{G}^{(\infty)} = \{0\}$. Then, $\mathcal{G}$ is a $\mathbb{C}[y_1, \dots, y_n]$ -module.

Proof. We have only to show that $\mathbb{C}[y_1, \dots, y_n]\mathcal{G} \subseteq \mathcal{P}(V; y_1, \dots, y_n)$, because if so, the maximality of $\mathcal{G}$ shows that $\mathbb{C}[y_1, \dots, y_n]\mathcal{G} = \mathcal{G}$. Thus, assume that $\mathbb{C}[y_1, \dots, y_n]\mathcal{G} = \mathcal{P}(V; y_1, \dots, y_n)$. Then by the above lemma, we get that $\mathcal{G}^{(\infty)} \supset J\mathcal{P}(V; y_1, \dots, y_n) \neq 0$, contradicting the assumption.

Now, we have only to consider a maximal finite codimensional subalgebra $\mathcal{G}$ of $\mathcal{P}(V; y_1, \dots, y_n)$ such that $\mathcal{G}^{(\infty)} = \{0\}$ and $\mathcal{G}$ is a $\mathbb{C}[y_1, \dots, y_n]$ -module. Let $M_p = \{f \in \mathbb{C}[y_1, \dots, y_n] ; f(p) = 0\}$.

3.6 Lemma For a $\mathbb{C}[y_1, \dots, y_n]$ -submodule $\mathcal{G}$ of $\mathcal{P}(V; y_1, \dots, y_n)$, if

$$\mathcal{G} + M_p \mathcal{P}(V; y_1, \dots, y_n) = \mathcal{P}(V; y_1, \dots, y_n)$$

for every $p \in \mathbb{C}^n$, then $\mathcal{G} = \mathcal{P}(V; y_1, \dots, y_n)$.

Proof. By Nakayama's lemma, we see that for each $p \in \mathbb{C}^n$, there is $f_p \in \mathbb{C}[y_1, \dots, y_n]$ such that $f_p(p) \neq 0$ and $f_p \mathcal{P}(V; y_1, \dots, y_n) = \mathcal{G}$. Since the ideal J generated by $\{f_p ; p \in \mathbb{C}^n\}$ has no common zero, we see that $J = \mathbb{C}[y_1, \dots, y_n]$ and hence there are $f_{p_1}, f_{p_2}, \dots, f_{p_\ell}$, $g_1, g_2, \dots, g_\ell \in \mathbb{C}[y_1, \dots, y_n]$ such that $1 = \sum_{j=1}^{\ell} g_j f_{p_j}$. Therefore,

$$\mathcal{P}(V; y_1, \dots, y_n) = \left(\sum_{j=1}^{\ell} g_j \right) \mathcal{G} \subset \mathcal{G}.$$

3.7 Corollary Let $\mathcal{G}$ be a maximal, finite codimensional subalgebra of $\mathcal{P}(V; y_1, \dots, y_n)$ such that $\mathcal{G}^{(\infty)} = \{0\}$. Then, there exists uniquely a point $p \in \mathcal{W}_{\mathcal{P}}$ such that $\mathcal{G} = \mathcal{P}_p$.

Proof. By 3.5 Corollary, $\mathcal{G}$ is a $\mathbb{C}[y_1, \dots, y_n]$ -module, and hence

there is a point $p \in \mathbb{C}^n$ such that $\mathcal{J} + M_p \mathcal{L}(V; y_1, \dots, y_n) \subsetneq \mathcal{L}(V; y_1, \dots, y_n)$. Thus, $\mathcal{J} \supset M_p \mathcal{L}(V; y_1, \dots, y_n)$ by the maximality of $\mathcal{J}$. It is easy to see that such a point is unique, because $M_p + M_q = \mathbb{C}[y_1, \dots, y_n]$ if $p \neq q$.

If $\mathcal{L}(V; y_1, \dots, y_n)(p) = \{0\}$, then $M_p \mathcal{L}(V; y_1, \dots, y_n)$ is an ideal of $\mathcal{L}(V; y_1, \dots, y_n)$, hence it must be contained in $\mathcal{J}^{(\infty)}$. Thus, by the assumption, it must be $\{0\}$, contradicting the assumption. Therefore we get $\mathcal{L}(V; y_1, \dots, y_n)(p) \neq \{0\}$. Now, there is $u \in \mathcal{L}(V; y_1, \dots, y_n)$ such that $u(p) \neq 0$ and $f \in \mathbb{C}[y_1, \dots, y_n]$ such that $f(p) = 0$ and $(uf)(p) \neq 0$. For every $v \in \mathcal{L}(V; y_1, \dots, y_n)$, fv is an element of $\mathcal{J}$. Therefore if u were contained in $\mathcal{J}$, then $[u, fv] \in \mathcal{J}$. Thus, $(uf)v \in \mathcal{J}$. It follows that $(uf)(p)v \in (uf - (uf)(p))v + \mathcal{J} \subset \mathcal{J}$. Since $(uf)(p) \neq 0$, we get $v \in \mathcal{J}$, hence $\mathcal{J} = \mathcal{L}(V; y_1, \dots, y_n)$, contradicting the assumption.

By the above argument, we see that $\mathcal{J} \subset \mathcal{L}_p$, and hence $\mathcal{J} = \mathcal{L}_p$ by the maximality of $\mathcal{J}$. Since $\mathcal{J}^{(\infty)} = \{0\}$, we see $p \in \mathcal{W}_{\mathcal{L}}$ by definition.

This completes the proof of 3.2 Proposition.

3.B A diffeomorphism induced from Φ .

Let $\mathcal{L}(V'; z_1, \dots, z_n)$ be another Lie algebra of polynomial vector fields on $\mathbb{C}^n$. Subsets $\mathcal{V}_{\mathcal{L}'}$, $\mathcal{W}_{\mathcal{L}'}$ are defined by the same way as in $\mathcal{L}(V; y_1, \dots, y_n)$. Suppose there is an isomorphism Φ of $\mathcal{L}(V; y_1, \dots, y_n)$ onto $\mathcal{L}(V'; z_1, \dots, z_n)$. For a point $p \in \mathcal{W}_{\mathcal{L}}$, $\mathcal{L}_p$ is a maximal finite codimensional subalgebra such that $\mathcal{L}_p^{(\infty)} = 0$. Then, $\Phi(\mathcal{L}_p)$ has the same property, hence there is a point $q(p) \in \mathcal{W}_{\mathcal{L}'}$ such that $\Phi(\mathcal{L}_p) = \mathcal{L}'_{q(p)}$, where $\mathcal{L}'_{q(p)}$ is defined by the same manner as in $\mathcal{L}(V; y_1, \dots, y_n)$. $\varphi : \mathcal{W}_{\mathcal{L}} \rightarrow \mathcal{W}_{\mathcal{L}'}$ is a bijective mapping. The goal of

this section is as follows :

3.8 Proposition Notations and assumptions being as above, assume further that $\mathcal{O}(V; y_1, \dots, y_n)$ (resp. $\mathcal{O}(V'; z_1, \dots, z_{n'})$) contains a vector field X (resp. X') such that $X = \sum_{j=1}^n \hat{\mu}_j y_j \partial / \partial y_j$ (resp. $X' = \sum_{j=1}^{n'} \hat{\mu}'_j z_j \partial / \partial z_j$). Then φ can be extended to a holomorphic diffeomorphism of $\mathbb{C}^n$ onto $\mathbb{C}^{n'}$ such that $\varphi(V_\rho) = V_{\rho'}$.

Note that the existence of X and X' are obtained by 2.1 Lemma.

Let Ψ_ρ be the totality of $\mathbb{C}$ -valued functions f on $\mathcal{W}_\rho$ such that $f u$ can be extended to an element of $\mathcal{O}(V; y_1, \dots, y_n)$ for every $u \in \mathcal{O}(V; y_1, \dots, y_n)$. Remark that the extension of $f u$ is unique, because $\mathcal{W}_\rho$ is dense in $\mathbb{C}^n$. Ψ_ρ is a ring and $\mathcal{O}(V; y_1, \dots, y_n)$ is an Ψ_ρ -module. For $\mathcal{O}(V'; z_1, \dots, z_{n'})$, we define $\Psi_{\rho'}$ by the same manner as above.

3.9 Lemma Notations and assumptions being as above, φ induces an isomorphism of $\Psi_{\rho'}$ onto Ψ_ρ .

Proof. Let $f \in \Psi_{\rho'}$ and p an arbitrary point in $\mathcal{W}_\rho$. By definition, $f \Phi(u)$ can be extended to an element of $\mathcal{O}(V'; z_1, \dots, z_{n'})$, which will be denoted by the same notation. $f \Phi(u) - f(\varphi(p)) \Phi(u) \in \mathcal{O}'_{\varphi(p)}$, hence $\Phi^{-1}(f \Phi(u) - f(\varphi(p)) \Phi(u)) \in \mathcal{O}_p$, that is, $\Phi^{-1}(f \Phi(u) - f(\varphi(p)) \Phi(u))(p) = 0$. Therefore, $\Phi^{-1}(f \Phi(u))(p) = f(\varphi(p)) u$, that is, $\Phi^{-1}(f \Phi(u)) = (\varphi^* f) u$. Since the left hand member is contained in $\mathcal{O}(V; y_1, \dots, y_n)$, we see $\varphi^* f \in \Psi_\rho$. It is easy to see that $\varphi^* : \Psi_{\rho'} \rightarrow \Psi_\rho$ is an isomorphism.

3.10 Lemma Under the same assumption as in the statement of 3.8

Proposition, we have $\Psi_\rho = \mathbb{C}[y_1, \dots, y_n]$. Hence φ is a bi-holomorphic diffeomorphism of $\mathbb{C}^n$ onto $\mathbb{C}^{n'}$.

Proof. Obviously $\Psi_\rho \supset \mathbb{C}[y_1, \dots, y_n]$. For any $f \in \Psi_\rho$, $f X$ is an element of $\mathcal{O}(V; y_1, \dots, y_n)$. Thus, $f y_1, \dots, f y_n \in \mathbb{C}[y_1, \dots, y_n]$. Hence

it is not hard to see $f \in \mathbb{C}[y_1, \dots, y_n]$.

3.11 Lemma $\mathcal{V}(\mathbb{C}^n - V_{\mathcal{P}}) = \mathbb{C}^{n'} - V_{\mathcal{P}'}$.

Proof. By the above lemma, we have $n = n'$. Let p be a point of $\mathbb{C}^n - V_{\mathcal{P}}$. Then $\text{codim } \mathcal{P}_p = n$, hence $\text{codim } \mathcal{P}'_{\varphi(p)} = n$, because $\mathcal{P}'_{\varphi(p)} = \varphi(\mathcal{P}_p)$. Therefore, we see $\mathcal{V}(\mathbb{C}^n - V_{\mathcal{P}}) = \mathbb{C}^{n'} - V_{\mathcal{P}'}$.

This completes the proof of 3.8 Proposition.

3.C Recapture of the germ.

Recall that V is a germ of variety with 0 as an expansive singularity. Hence there is $X = \sum_{i=1}^n \hat{\mu}_i y_i \partial / \partial y_i \in \mathfrak{X}(V)$ such that $\text{Re } \hat{\mu}_i > 0$ for $1 \leq i \leq n$. Since X is a linear vector field, $\exp tX$ is a bi-holomorphic diffeomorphism of $\mathbb{C}^n$ onto itself. Remark that $(\exp tX)V = V$ as germs of varieties, for $X\mathfrak{J}(V) \subset \mathfrak{J}(V)$ where $\mathfrak{J}(V)$ is the ideal of V in $\hat{\mathcal{O}}$. Let $\tilde{V} = \bigcup_{t \in \mathbb{R}} (\exp tX)V$. Though V is a germ of variety at 0 , the expansive property of X yields that $\tilde{V}$ is a closed subset of $\mathbb{C}^n$ such that $(\exp tX)\tilde{V} = \tilde{V}$. Obviously, $\tilde{V} = V$ as germs of varieties.

In this section, we shall prove that $V_{\mathcal{P}} = \tilde{V}$, hence $V_{\mathcal{P}} = V$ as germs of varieties. Let $\hat{\mathfrak{J}}(V)$ be the closure of $\mathfrak{J}(V)$ in $\hat{\mathcal{O}}$. Note that $\mathfrak{J}(V)$ is also the closure of $\mathfrak{X}(V)$ in $\mathfrak{F}_0$. Hence $\mathfrak{J}(V) \hat{\mathfrak{J}}(V) \subset \hat{\mathfrak{J}}(V)$. Recall that $\mathcal{P}(V; y_1, \dots, y_n)$ is given by using the eigenspace decomposition of $\mathfrak{J}(V)$ with respect to $\text{ad}(X)$, that is, every $u \in \mathfrak{J}(V)$ can be rearranged in the form $u = \sum u_\lambda$ as in 1.6 Lemma, and $\mathcal{P}(V; y_1, \dots, y_n)$ is generated by the u_λ 's. Similarly, we decompose $\hat{\mathfrak{J}}(V)$ into eigenspaces of X . Let f be an element of $\hat{\mathfrak{J}}(V)$. Then, f can be rearranged in the form

$$(23) \quad f = \sum_{\nu} f_{\nu}, \quad f_{\nu} = \sum_{\langle \alpha, \hat{\mu} \rangle = \nu} a_{\alpha} y^{\alpha}.$$

Then, f_{ν} is a polynomial such that $Xf_{\nu} = \nu f_{\nu}$. By the same proof

as in 1.6 Lemma, we see that $f_\nu \in \hat{\mathcal{J}}(V)$. We denote by $I_{\hat{\mathcal{P}}}$ the ideal of $\mathbb{C}[y_1, \dots, y_n]$ generated by all f_ν 's with $f \in \hat{\mathcal{J}}(V)$.

3.11 Lemma $I_{\hat{\mathcal{P}}} \subset \mathcal{J}(V)$.

Proof. Let $f \in \mathcal{J}(V)$. f can be rearranged in the form $f = \sum_{i=1}^{\infty} f_{\nu_i}$, $f_{\nu_i} = \sum_{\langle \hat{\mu}, \hat{\nu} \rangle = \nu_i} a_{\hat{\mu}} y^{\hat{\mu}}$. We may assume $0 < \nu_1 < \nu_2 < \dots < \nu_k < \dots$. First of all, we shall show $f_{\nu_i} \in \mathcal{J}(V)$. Note that $e^{\nu_i t} (\exp -tX) f = \sum_j e^{-(\nu_j - \nu_i)t} f_{\nu_j} \in \mathcal{J}(V)$ for $t > 0$. Suppose f is defined on a neighborhood N of 0 in $\mathbb{C}^n$. Then, $(\exp -tX)f$ is defined on $(\exp tX)N$. Note that $\bigcup_{t>0} (\exp tX)N = \mathbb{C}^n$ and $\bigcup_{t>0} (\exp tX)(N \cap V) = \tilde{V}$. Since $e^{\nu_i t} (\exp -tX)f = 0$ on $(\exp tX)(N \cap V)$, taking $\lim_{t \rightarrow \infty}$ we see that $f_{\nu_i} = 0$ on $\tilde{V}$. Since $\tilde{V} = V$ as germs of varieties, we have $f_{\nu_i} \in \mathcal{J}(V)$. Repeating the same procedure to $f - f_{\nu_1}$, we have $f_{\nu_2} \in \mathcal{J}(V)$, and so on. Hence $f_{\nu_j} \in \mathcal{J}(V)$.

Let $f \in \hat{\mathcal{J}}(V)$. Then, there is a sequence $\{f^{(m)}\}$ in $\mathcal{J}(V)$ such that $\lim f^{(m)} = f$ in the topology of formal power series. For any eigenvalue ν of $X : \hat{\mathcal{O}} \rightarrow \hat{\mathcal{O}}$, we see $f_\nu^{(m)} \in \mathcal{J}(V)$, and $\lim_{m \rightarrow \infty} f_\nu^{(m)} = f_\nu$ as polynomials, because the degrees of $f_\nu^{(m)}$, f_ν are bounded from above by a number related only to $\hat{\mu}_1, \dots, \hat{\mu}_n$ and ν . Since $f_\nu^{(m)}|_V \equiv 0$, we have $f_\nu|_V \equiv 0$, hence $f_\nu \in \mathcal{J}(V)$. Recall that the f_ν 's generate $I_{\hat{\mathcal{P}}}$. Thus, we see $I_{\hat{\mathcal{P}}} \subset \mathcal{J}(V)$.

3.12 Lemma Notations and assumptions being as above, a polynomial vector field u with $u(0) = 0$ is contained in $\hat{\mathcal{P}}(V; y_1, \dots, y_n)$ if and only if $uI_{\hat{\mathcal{P}}} \subset I_{\hat{\mathcal{P}}}$.

Proof. For $u \in \mathcal{G}(V)$, $f \in \hat{\mathcal{J}}(V)$, let $u = \sum_{\lambda} u_{\lambda}$, $f = \sum_{\nu} f_{\nu}$ be the decompositions of eigenvectors with respect to $\text{ad}(X)$, X respectively. Then, $u_{\lambda} \in \hat{\mathcal{P}}(V; y_1, \dots, y_n)$, $f_{\nu} \in I_{\hat{\mathcal{P}}}$. Since $Xu_{\lambda} f_{\nu} = [X, u_{\lambda}]f_{\nu} + u_{\lambda}Xf_{\nu} = (\lambda + \nu)u_{\lambda}f_{\nu}$, $u_{\lambda}f_{\nu}$ is also an eigenvector of X . Since $uf \in \hat{\mathcal{J}}(V)$, the $u_{\lambda}f_{\nu}$'s appear in the eigenspace decomposition of uf , and hence $u_{\lambda}f_{\nu} \in I_{\hat{\mathcal{P}}}$. Thus, we have $\hat{\mathcal{P}}(V; y_1, \dots, y_n)I_{\hat{\mathcal{P}}} \subset I_{\hat{\mathcal{P}}}$.

Conversely, if $uI_{\mathfrak{p}} \subset I_{\mathfrak{p}}$ for a polynomial vector field u with $u(0) = 0$. Then, $u\hat{\mathcal{J}}(V) \subset \hat{\mathcal{J}}(V)$ by taking the closure in the formal power series. Note that $u\mathcal{J}(V) \subset \mathcal{O} \cap \hat{\mathcal{J}}(V)$. We next prove that $\mathcal{J}(V) = \mathcal{O} \cap \hat{\mathcal{J}}(V)$. For that purpose, we have only to show $\mathcal{J}(V) \supset \mathcal{O} \cap \hat{\mathcal{J}}(V)$, because the converse is trivial. Let $f \in \mathcal{O} \cap \hat{\mathcal{J}}(V)$, and $f = \sum_{\nu} f_{\nu}$, the eigenvector decomposition of f with respect to X . Then, by 3.11 Lemma, we have $f_{\nu} \in I_{\mathfrak{p}} \subset \mathcal{J}(V)$. Thus, $f_{\nu} = 0$ on V , hence $f = 0$ on V . This means $f \in \mathcal{J}(V)$. Thus, $uI_{\mathfrak{p}} \subset I_{\mathfrak{p}}$ yields $u \in \mathfrak{X}(V) \subset \mathfrak{G}(V)$. However u is a polynomial vector field in $y_1, \dots, y_n$, hence $u \in \mathfrak{p}(V; y_1, \dots, y_n)$.

3.13 Lemma $V_{\mathfrak{p}} = V_{I_{\mathfrak{p}}}$: the locus of zeros of $I_{\mathfrak{p}}$.

Proof. Let p be a point in $\mathbb{C}^n - V_{\mathfrak{p}}$. By definition there are $u_1, \dots, u_n \in \mathfrak{p}(V; y_1, \dots, y_n)$ such that $u_1(p), \dots, u_n(p)$ are linearly independent. Assume for a while that $p \in V_{I_{\mathfrak{p}}}$. Since $u_i I_{\mathfrak{p}} \subset I_{\mathfrak{p}}$, we have

$$(u_1^{l_1} u_2^{l_2} \dots u_n^{l_n} f)(p) = 0$$

for every $f \in I_{\mathfrak{p}}$ and any $l_1, l_2, \dots, l_n$. Thus, $f = 0$, contradicting the fact $I_{\mathfrak{p}} \neq \{0\}$. Therefore, $V_{\mathfrak{p}} \supset V_{I_{\mathfrak{p}}}$.

Conversely, let $p \in \mathbb{C}^n - V_{I_{\mathfrak{p}}}$. There is then $g \in I_{\mathfrak{p}}$ such that $g(p) \neq 0$. By 3.12 Lemma, $g\partial/\partial y_1, \dots, g\partial/\partial y_n \in \mathfrak{p}(V; y_1, \dots, y_n)$, which are linearly independent at p . Hence $p \in \mathbb{C}^n - V_{\mathfrak{p}}$. Thus, $V_{I_{\mathfrak{p}}} \supset V_{\mathfrak{p}}$.

3.14 Lemma $V_{I_{\mathfrak{p}}} = V$ as germs of varieties.

Proof. By 3.11 Lemma, we have $\mathcal{O}I_{\mathfrak{p}} \subset \mathcal{J}(V)$, hence $V_{I_{\mathfrak{p}}} \supset V$. Assume for a while that $V_{I_{\mathfrak{p}}} \neq V$. Then there is $f \in \mathcal{J}(V)$ such that $f \neq 0$ on V . Let $f = \sum_{\nu} f_{\nu}$ be the eigenvector decomposition of f . Then $f_{\nu} \in I_{\mathfrak{p}}$. Therefore $f_{\nu} = 0$ on V , hence $f = 0$ on V contradicting the assumption. Thus, we get $V_{I_{\mathfrak{p}}} = V$ as germs of varieties, and hence $V_{I_{\mathfrak{p}}} = \tilde{V}$.

By the above result, we get that $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^{n'}$ maps $\tilde{V}$ onto $\tilde{V}'$ and $\varphi(V) = V'$ as germs. This implies that $\varphi^*J(V') = J(V)$ and hence $d\varphi X(V) = X(V')$. This completes the proof of Theorem I in the introduction.

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