

Numerical Cylindrical Algebraic Decomposition with Certification via Symbolic Reconstruction

穴井 宏和

HIROKAZU ANAI

(株) 富士通研究所/(独) 科学技術振興機構

FUJITSU LABORATORIES LTD / CREST, JST *

横山 和弘

KAZUHIRO YOKOYAMA

立教大学/(独) 科学技術振興機構

RIKKYO UNIVERSITY /CREST, JST †

Abstract

Recently real quantifier elimination (QE) has been widely used as an effective tool for solving real algebraic constraints (in particular parametric and non-convex cases) arising in many engineering and industrial problems. In fact there are several successful applications of quantifier elimination to practical industrial problems. It is, however, still strongly required to develop practically efficient realizations of quantifier elimination. In this paper we propose a new scheme for efficient cylindrical algebraic decomposition (CAD) algorithm, which is the most fundamental part of quantifier elimination, based on symbolic-numeric computation.

1 Introduction

Cylindrical algebraic decomposition (CAD) is a general-purpose symbolic method aiming for quantifier elimination (QE) which is a powerful tool to resolve non-convex and parametric optimization problems exactly. However, QE based on CAD is not practical on real computers, since CAD usually consists of many of purely symbolic computations and has bad computational complexity in nature.

Against the inherent computational complexity of QE based on a CAD algorithm, several researchers have focused on QE algorithms specialized to particular types of input formulas, see [Wei88, LW93, Wei97, Hon93, GV98]. This direction is quite promising in practice since number of important problems in engineering have been successfully reduced to the particular input formulas and resolved by using the specialized QE algorithms. For the concrete applications of the scheme, see [Wei96, SW97, DSW98, GV96, AH00, AYSH04].

However, there still remain many significant problems in engineering that can not be recast as such particular formulas. Therefore, it is strongly desired to develop an efficient algorithm to realize CAD. One

*anai@jp.fujitsu.com

†yokoyama@rkmath.rikkyo.ac.jp

effective way for efficient CAD construction is achieved by utilizing numerical computation and derived numerical information on algebraic numbers as far as possible without violating correctness of the results instead of symbolic treatment. So far there are several related works to introduce numerical computation into CAD construction [Hon93b, Rat02, AP03]: for example, Ratschan proposed a combination of CAD and interval arithmetic [Rat02], and Anai & Parrilo presented improved CAD by using the information of a numerical feasible point particularly for convex optimization [AP03].

In this paper we propose a new scheme for an efficient realization of CAD based on symbolic-numeric computation [AY04], where numerical computation is used particularly in handling algebraic numbers in the lifting phase of CAD construction and we apply “symbolic reconstruction” with a smaller number of symbolic computations only to the unreliable numerical results to validate them. Our symbolic reconstruction procedure in the lifting phase is based on a dynamic evaluation (DE) technique [Duv94], and combined with successive representations of algebraic extensions.

2 Numeric Computation with Symbolic Reconstruction

Computational difficulty of the lifting phase of CAD comes from

- **Computation over Algebraic Extension:** For exact computation, we have to construct towers of algebraic extension fields successively over the rational number field $\mathbb{Q}$. This requires heavy computations such as algebraic factorization, GCD of polynomials.
- **Combinatorial Explosion:** CAD decomposes the whole space $\mathbb{R}^n$ into numerous sub-algebraic sets. The computational flow of the decomposition can be illustrated by a computational tree. However, in QE, many such sub-algebraic sets are unnecessary, and many subtree should be discarded.

To resolve these difficulties we consider using floating points number in CAD construction. In contrast to symbolic construction of CAD, replacing every symbolic computation (except projection phase in CAD construction) with its approximate numerical computation, we can compute an numerical CAD (i.e. approximate CAD) quite efficiently due to avoidance of computation over algebraic extension.

2.1 Using numerical computation in CAD

Using numerical computation in handling algebraic numbers greatly improves the efficiency of CAD construction because we can avoid symbolic computations over algebraic extension fields and also prune unnecessary branches of a CAD tree by numerical values of algebraic numbers, while this causes uncertainty of the computed results depending on accuracy of numerical computation. Hence we propose to use numerical computation with machinery for validating numerically computed results by reconstructing them exactly with a smaller number of symbolic computations. Some possible computational flow of our scheme in the lifting phase is illustrated in Fig.1. The dotted arrows and solid arrows in Fig.1 stand for numerical computation and symbolic reconstruction, respectively.

Actually computations over algebraic extension fields are required in the lifting phase of CAD to compute respective isolating interval of algebraic numbers over successive extension fields. Then what we need to do is “sign determination” of many algebraic numbers, i.e., to determine exactly whether they are 0 or not and their sign if not 0. We can usually expect that sign determinations are mostly properly

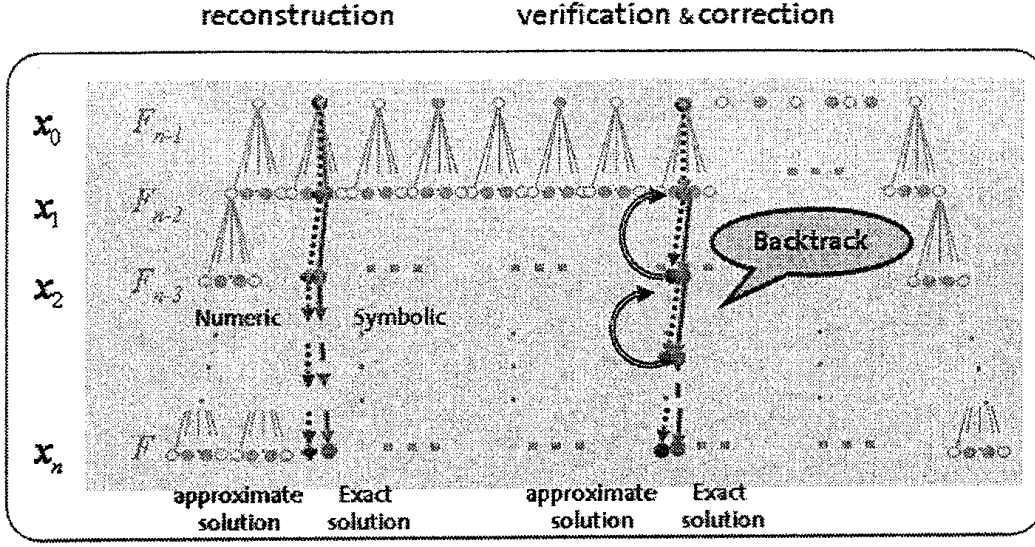


Figure 1: CAD construction – numerical computation with symbolic reconstruction

done by only numerical computation. We check again the sign exactly by symbolic computation only for unreliable numerical results (We call this symbolic reconstruction.) This is the ground why our method would improve the efficiency of CAD construction.

2.2 Efficient symbolic reconstruction by using a numerical CAD

Moreover we improve the reconstruction procedure by employing a dynamic evaluation technique integrated with successive representations of algebraic extensions as follows: Algebraic extension is expressed by a residue class ring $\mathbb{Q}[X]/\mathcal{M}$, where X is a set of variables and $\mathcal{M}$ is a maximal ideal in $\mathbb{Q}[X]$. Usually, computation of maximal ideals tends to be very hard. Instead of $\mathbb{Q}[X]/\mathcal{M}$, we utilize “lazy representation” $\mathbb{Q}[X]/\mathcal{I}$, where $\mathcal{I}$ is not necessary maximal but easily computable. $\mathbb{Q}[X]/\mathcal{I}$ may not be a domain, i.e., it could have zero-divisors. In the computation over $\mathbb{Q}[X]/\mathcal{I}$, if a given algebraic number does not correspond to a zero-divisor, then it is not equal to 0 and we check its sign by using a certain numerical method. If we met some zero-divisors $ab = 0$, then we can split the ideal $\mathcal{I}$ as follows:

$$\mathcal{I} = (\mathcal{I} + \langle a \rangle) \cap (\mathcal{I} + \langle b \rangle).$$

This decomposition is achieved by gcd computation for univariate case, i.e. simple algebraic extension, and Gröbner basis computation for multivariate case. Thus, by using lazy representation successively for towers of extensions, where defining polynomials for algebraic numbers are used as generators of the ideal $\mathcal{I}$, we do not require any algebraic factorization and a primitive element computation for a simple extension which are often difficult especially in the case of tall towers of extensions. Furthermore the most crucial point is that we can easily choose one necessary branch $\mathcal{I} + \langle a \rangle$ in the decomposition by virtue of numerical information of algebraic numbers and hence can prune unnecessary branches for the further computation of towers of extensions.

3 Concluding Remarks

We present a new scheme for realizing efficient CAD based on symbolic-numeric computation. In order to examine its effectiveness, implementation of our scheme into SyNRAC, which is a maple package for solving real algebraic constraints developed at FUJITSU LABORATORIES LTD [YA04], is ongoing. We remark that this work also provides one of promising directions for validated numerics for optimization problems.

Acknowledgements : This work has been partially supported by CREST, JST (Japan Science and Technology Agency).

References

- [AH00] H. Anai and S. Hara. Fixed-structure robust controller synthesis based on sign definite condition by a special quantifier elimination. In *Proceedings of American Control Conference 2000*, pages 1312–1316, 2000.
- [AP03] H. Anai and P. A. Parrilo. Convex quantifier elimination for semidefinite programming. In *Proceedings of the 6th International Workshop on Computer Algebra in Scientific Computing 2003*. To appear, 2003.
- [AY04] H. Anai and K. Yokoyama. Numerical cylindrical algebraic decomposition with certificated reconstruction. In *Proceedings of 11th GAMM - IMACS International Symposium on Scientific Computing, Computer Arithmetic, and Validated Numerics (Fukuoka, Japan)*, page 31, 2004.
- [AYSH04] H. Anai, H. Yanami, K. Sakabe, and S. Hara. Fixed-structure robust controller synthesis based on symbolic-numeric computation: design algorithms with a CACSD toolbox (invited paper). In *Proceedings of CCA/ISIC/CACSD 2004 (Taipei, Taiwan)*, pages 1540–1545, 2004.
- [DSW98] A. Dolzmann, T. Sturm, and V. Weispfenning. Real quantifier elimination in practice. In B. H. Matzatz, G.-M. Greuel, and G. Hiss, editors, *Algorithmic Algebra and Number Theory*, pages 221–247. Springer, Berlin, 1998.
- [Duv94] D. Duval. Algebraic numbers: an example of dynamic evaluation. *Journal of Symbolic Computation*, 18:429–445, 1994.
- [GV96] L. González-Vega. Applying quantifier elimination to the Birkhoff interpolation problem. *Journal of Symbolic Computation*, 1996. To appear.
- [GV98] L. González-Vega. A combinatorial algorithm solving some quantifier elimination problems. In B.F. Caviness and J.R. Johnson, editors, *Quantifier Elimination and Cylindrical Algebraic Decomposition*, Texts and Monographs in Symbolic Computation, pages 365–375. Springer, Wien, New York, 1998.
- [Hon93] H. Hong. Quantifier elimination for formulas constrained by quadratic equations. In Manuel Bronstein, editor, *Proceedings of the International Symposium on Symbolic and Algebraic Computation (ISSAC 93)*, pages 264–274, Kiev, Ukraine, July 1993. ACM, ACM Press.
- [Hon93b] H. Hong. Efficient method for analyzing topology of plane real algebraic curves. In *Proceedings of IMACS-SC 93*, Lille, France, 1993.

- [LW93] R. Loos and V. Weispfenning. Applying linear quantifier elimination. *The Computer Journal*, 36(5):450–462, 1993. Special issue on computational quantifier elimination.
- [Rat02] S. Ratschan. Approximate quantified constraint solving by cylindrical box decomposition. *Reliable Computing*, 8(1):21–42, 2002.
- [SW97] T. Sturm and V. Weispfenning. Rounding and blending of solids by a real elimination method. In Achim Sydow, editor, *Proceedings of the 15th IMACS World Congress on Scientific Computation, Modeling, and Applied Mathematics (IMACS 97)*, volume 2, pages 727–732, Berlin, August 1997. IMACS, Wissenschaft & Technik Verlag.
- [Wei88] V. Weispfenning. The complexity of linear problems in fields. *Journal of Symbolic Computation*, 5(1–2):3–27, February–April 1988.
- [Wei96] V. Weispfenning. Applying quantifier elimination to problems in simulation and optimization. Technical Report MIP-9607, FMI, Universität Passau, D-94030 Passau, Germany, April 1996. To appear in the *Journal of Symbolic Computation*.
- [Wei97] V. Weispfenning. Quantifier elimination for real algebra—the quadratic case and beyond. *Applicable Algebra in Engineering Communication and Computing*, 8(2):85–101, February 1997.
- [YA04] H. Yanami and H. Anai. Development of SyNRAC – formula description and new functions. In *Proceedings of International Workshop on Computer Algebra Systems and their Applications (CASA) 2004: ICCS 2004, LNCS 3039*, pages 286–294. Springer, 2004.