

Unbounded C^* -seminorms and $*$ -representations of $*$ -algebras

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1. INTRODUCTION

Unbounded C^* -seminorms on $*$ -algebras in the sense that they are C^* -seminorms defined on $*$ -subalgebras have appeared in many mathematical and physical subjects (for example, locally convex $*$ -algebras and the quantum field theory etc.). But this systematical study has not yet done sufficiently. The main purpose of this paper is to do a systematical study of unbounded C^* -seminorms and to apply it to a study of unbounded $*$ -representations.

The paper is organized as follows: In Section 2 we construct unbounded $*$ -representations of a $*$ -algebra from unbounded C^* -seminorms and investigate them. Let $\mathcal{A}$ be a $*$ -algebra. Let p be a C^* -seminorm defined on $\mathcal{A}$. Every $*$ -representation of the Hausdorff completion of $(\mathcal{A}, p)$ gives rise to a $*$ -representation of $\mathcal{A}$ into bounded Hilbert space operators. However, there are a number of situations in which natural C^* -seminorms are defined on $*$ -subalgebras of $\mathcal{A}$. Then they should lead to unbounded operator representations of $\mathcal{A}$. An unbounded m^* -(resp. C^* -)seminorm is a submultiplicative $*$ (resp. C^* -)seminorm p defined on a $*$ -subalgebra $\mathcal{D}(p)$ of $\mathcal{A}$. Then $\mathfrak{N}_p \equiv \{x \in \mathcal{D}(p); ax \in \mathcal{D}(p), \forall a \in \mathcal{A}\}$ is a left ideal of $\mathcal{A}$. It is shown that any $*$ -representation $\Pi_p: \mathcal{A}_p \longrightarrow \mathfrak{B}(\mathcal{H})$ of the Hausdorff completion $\mathcal{A}_p$ of $(\mathcal{D}(p), p)$ leads to an unbounded $*$ -representation π_p of $\mathcal{A}$ such that $\|\overline{\pi_p(x)}\| \leq p(x)$ for all $x \in \mathcal{D}(p)$. We denote by $\text{Rep}(\mathcal{A}, p)$ the set of all such $*$ -representations π_p of $\mathcal{A}$. In order to investigate representations in $\text{Rep}(\mathcal{A}, p)$ in details, we introduce the notions of nondegenerate, finite, uniformly semifinite, semifinite and weakly semifinite unbounded C^* -seminorms, and show that if p is (weakly) semifinite, then there exists a strongly nondegenerate $*$ -representation π_p in $\text{Rep}(\mathcal{A}, p)$ such

that $\|\overline{\pi_p(x)}\| = p(x)$ for all $x \in \mathcal{D}(p)$. Such a π_p is called well-behaved. In Section 3 we consider the converse direction of Section 2. We construct an unbounded C^* -seminorm r_π on $\mathcal{A}$ from a $*$ -representation π of $\mathcal{A}$ and a natural representation $\pi_{r_\pi}^N$ of $\mathcal{A}$ constructed from r_π which is the restriction of the closure $\tilde{\pi}$ of π . It is shown that π is strongly nondegenerate if and only if $\pi_{r_\pi}^N$ is a well-behaved $*$ -representation of $\mathcal{A}$. Further, it is shown that if p is a weakly semifinite unbounded C^* -seminorm on $\mathcal{A}$ and π_p is any well-behaved $*$ -representation, then r_{π_p} is a maximal extension of p .

2. REPRESENTATIONS INDUCED BY UNBOUNDED C^* -SEMINORMS

In this section we construct a family of $*$ -representations of a $*$ -algebra $\mathcal{A}$ induced by an unbounded C^* -seminorm on $\mathcal{A}$ and investigate the properties. We begin with the review of (unbounded) $*$ -representations of $\mathcal{A}$. Throughout this section let $\mathcal{A}$ be a $*$ -algebra. Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$ and let $\mathcal{L}^+(\mathcal{D})$ denote the set of all linear operators X in $\mathcal{H}$ with the domain $\mathcal{D}$ for which $X\mathcal{D} \subset \mathcal{D}$, $\mathcal{D}(X^*) \supset \mathcal{D}$ and $X^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^+(\mathcal{D})$ is a $*$ -algebra under the usual operations and the involution $X \rightarrow X^\dagger \equiv X^*|_{\mathcal{D}}$. A $*$ -subalgebra of the $*$ -algebra $\mathcal{L}^+(\mathcal{D})$ is said to be an O^* -algebra on $\mathcal{D}$ in $\mathcal{H}$. A $*$ -representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ with a domain $\mathcal{D}$ is a $*$ -homomorphism of $\mathcal{A}$ into $\mathcal{L}^+(\mathcal{D})$ and $\pi(1)=I$ if $\mathcal{A}$ has identity 1, and then we write $\mathcal{D}$ and $\mathcal{H}$ by $\mathcal{D}(\pi)$ and $\mathcal{H}_\pi$, respectively. Let π_1 and π_2 be $*$ -representations of $\mathcal{A}$. If $\mathcal{H}_{\pi_1}$ is a closed subspace of $\mathcal{H}_{\pi_2}$ and $\pi_1(x) \subset \pi_2(x)$ for each $x \in \mathcal{A}$, then π_2 is said to be an extension of π_1 and denoted by $\pi_1 \subset \pi_2$. In particular, if $\pi_1 \subset \pi_2$ and $\mathcal{H}_{\pi_1} = \mathcal{H}_{\pi_2}$, then π_2 is said to be an extension of π_1 as the same Hilbert space. Let π be a $*$ -representation of $\mathcal{A}$. If $\mathcal{D}(\pi)$ is complete with the graph topology t_π defined by the family of seminorms $\{\|\bullet\|_{\pi(x)} \equiv \|\bullet\| + \|\pi(x)\bullet\|; x \in \mathcal{A}\}$, then π is said to be closed. It is well known that π is closed if and only if $\mathcal{D}(\pi) = \bigcap_{x \in \mathcal{A}} \mathcal{D}(\overline{\pi(x)})$. The

closure $\tilde{\pi}$ of π is defined by

$$\mathcal{D}(\tilde{\pi}) = \bigcap_{x \in \mathcal{A}} \mathcal{D}(\overline{\pi(x)}) \text{ and } \tilde{\pi}(x)\xi = \overline{\pi(x)}\xi \text{ for } x \in \mathcal{A}, \xi \in \mathcal{D}(\tilde{\pi}).$$

Then $\tilde{\pi}$ is the smallest closed extension of π . The weak commutant $\pi(\mathcal{A})'_w$ of π is defined by

$$\pi(\mathcal{A})'_w = \left\{ C \in \mathfrak{B}(\mathcal{H}_\pi); C\pi(x)\xi = \pi(x^*)^* C\xi, \forall x \in \mathcal{A}, \forall \xi \in \mathcal{D}(\pi) \right\},$$

where $\mathfrak{B}(\mathcal{H}_\pi)$ is the set of all bounded linear operators on $\mathcal{H}_\pi$, and it is a weakly closed $*$ -invariant subspace of $\mathfrak{B}(\mathcal{H}_\pi)$, but it is not necessarily an algebra. It is known that $\pi(\mathcal{A})'_w \mathcal{D}(\pi) \subset \mathcal{D}(\pi)$ if and only if $\pi(\mathcal{A})'_w$ is a von Neumann algebra and $\overline{\pi(x)}$ is affiliated with the von Neumann algebra $\left(\pi(\mathcal{A})'_w \right)'$ for each $x \in \mathcal{A}$.

Definition 2.1. A mapping p of a subspace $\mathcal{D}(p)$ of $\mathcal{A}$ into $\mathbb{R}^+ = [0, \infty)$ is said to be an unbounded (semi) norm on $\mathcal{A}$ if it is a (semi) norm on $\mathcal{D}(p)$, and p is said to be an unbounded m^* -(resp. C^* -) (semi) norm on $\mathcal{A}$ if $\mathcal{D}(p)$ is a $*$ -subalgebra of $\mathcal{A}$ and p is a submultiplicative $*$ -(resp. C^* -) (semi) norm on $\mathcal{D}(p)$.

If a seminorm p on a $*$ -algebra $\mathcal{A}$ is a C^* -seminorm, that is, it satisfies the C^* -property $p(x^*x) = p(x)^2$, $\forall x \in \mathcal{A}$, then it is a m^* -seminorm on $\mathcal{A}$, that is, $p(x^*) = p(x)$ and $p(xy) \leq p(x)p(y)$ for $\forall x, y \in \mathcal{A}$.

Let p be an unbounded C^* -seminorm on $\mathcal{A}$. We put

$$N_p = \{x \in \mathcal{D}(p); p(x) = 0\} \text{ and } \mathfrak{N}_p \equiv \{x \in \mathcal{D}(p); ax \in \mathcal{D}(p), \forall a \in \mathcal{A}\}.$$

Then N_p is a $*$ -ideal of $\mathcal{D}(p)$ and $\mathfrak{N}_p$ is a left ideal of $\mathcal{A}$, and the quotient $*$ -algebra $\mathcal{D}(p)/N_p$ is a normed $*$ -algebra with the C^* -norm $\|x + N_p\|_p \equiv p(x)$ ($x \in \mathcal{D}(p)$). We denote by $\mathcal{A}_p$ the C^* -algebra obtained by the completion of $\mathcal{D}(p)/N_p$, and denote by $\text{Rep}(\mathcal{A}_p)$ the set of all $*$ -representations Π_p of the C^* -algebra $\mathcal{A}_p$ on Hilbert space $\mathcal{H}_{\Pi_p}$. Put

$$\text{FRep}(\mathcal{A}_p) = \left\{ \Pi_p \in \text{Rep}(\mathcal{A}_p); \Pi_p \text{ is faithful} \right\}$$

$$\text{FNRep}(\mathcal{A}_p) = \left\{ \Pi_p \in \text{Rep}(\mathcal{A}_p); \Pi_p \text{ is faithful and nondegenerate} \right\}.$$

It is well known that $\text{FNRep}(\mathcal{A}_p) \neq \emptyset$. For each $\Pi_p \in \text{Rep}(\mathcal{A}_p)$ we can define a bounded $*$ -representation π_p^0 of $\mathcal{D}(p)$ on the Hilbert space $\mathcal{H}_{\Pi_p}$ by

$$\pi_p^0(x) = \Pi_p(x + N_p), \quad x \in \mathcal{D}(p).$$

The natural question arises: Can we extend the bounded $*$ -representation π_p^0 of the

*-algebra $\mathcal{D}(p)$ to a (generally unbounded) *-representation of the *-algebra $\mathcal{A}$?

We show that this question has affirmative answer.

Proposition 2.2. Let p be an unbounded C^* -seminorm on $\mathcal{A}$. For any $\Pi_p \in \text{Rep}(\mathcal{A}_p)$, there exists a *-representation π_p of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\pi_p}$ such that $\|\overline{\pi_p(b)}\| \leq p(b)$ for each $b \in \mathcal{D}(p)$. In particular, if $\Pi_p \in \text{FRep}(\mathcal{A}_p)$, then $\|\overline{\pi_p(x)}\| = p(x)$ for each $x \in \mathfrak{N}_p$.

Proof. We put

$$\begin{aligned} \mathcal{D}(\pi_p) &= \text{linear span of } \left\{ \Pi_p(x + N_p)\xi; x \in \mathfrak{N}_p, \text{ and } \xi \in \mathcal{H}_{\Pi_p} \right\} \\ \pi_p(a) \left(\sum_k \Pi_p(x_k + N_p)\xi_k \right) &= \sum_k \Pi_p(ax_k + N_p)\xi_k \quad (\text{finite sums}) \\ &\text{for } a \in \mathcal{A}, \{x_k\} \subset \mathfrak{N}_p \text{ and } \{\xi_k\} \subset \mathcal{H}_{\Pi_p}. \end{aligned}$$

Since

$$\begin{aligned} (\Pi_p(ax + N_p)\xi | \Pi_p(y + N_p)\eta) &= (\xi | \Pi_p((ax + N_p)^*(y + N_p))\eta) \\ &= (\xi | \Pi_p(x^*a^*y + N_p)\eta) \\ &= (\xi | \Pi_p(x^* + N_p)\Pi_p(a^*y + N_p)\eta) \\ &= (\Pi_p(x + N_p)\xi | \Pi_p(a^*y + N_p)\eta) \end{aligned}$$

for each $a \in \mathcal{A}$, $x, y \in \mathfrak{N}_p$ and $\xi, \eta \in \mathcal{H}_{\Pi_p}$, it follows that $\pi_p(a)$ is a well-defined linear operator on $\mathcal{D}(\pi_p)$ for each $a \in \mathcal{A}$, so that it is easily shown that π_p is a *-representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_{\pi_p} = [\mathcal{D}(\pi_p)] = \overline{\mathcal{D}(\pi_p)}^{\|\cdot\|}$ (the closure of $\mathcal{D}(\pi_p)$ in $\mathcal{H}_{\Pi_p}$) with domain $\mathcal{D}(\pi_p)$. Take an arbitrary $b \in \mathcal{D}(p)$. By the definition of π_p we have $\pi_p(b) = \pi_p^\circ(b)|_{\mathcal{D}(\pi_p)}$, and hence

$$\|\overline{\pi_p(b)}\| \leq \|\Pi_p(b + N_p)\| \leq \|b + N_p\|_p = p(b).$$

Suppose $\Pi_p \in \text{FRep}(\mathcal{A}_p)$ and $x \in \mathfrak{N}_p$. It is sufficient to show that $\|\overline{\pi_p(x)}\| \geq p(x)$.

If $p(x)=0$, then it is obvious. Suppose $p(x) \neq 0$. We put $y = \frac{x}{p(x)} \in \mathfrak{N}_p$. For each

$\xi \in \mathcal{H}_{\Pi_p}$ with $\|\xi\| \leq 1$, we have

$$\|\Pi_p(y + N_p)\xi\| \leq \|\Pi_p(y + N_p)\| \|\xi\| = p(y)\|\xi\| \leq 1,$$

and so

$$\begin{aligned}
\|\overline{\pi_p(y)}\| &= \|\overline{\pi_p(y^*)}\| \geq \sup\{\|\pi_p(y^*)\Pi_p(y + N_p)\xi\|; \xi \in \mathcal{H}_{\Pi_p} \text{ s.t. } \|\xi\| \leq 1\} \\
&= \sup\{\|\Pi_p(y^*y + N_p)\xi\|; \xi \in \mathcal{H}_{\Pi_p} \text{ s.t. } \|\xi\| \leq 1\} \\
&= \|\Pi_p(y^*y + N_p)\| \\
&= p(y^*y) = p(y)^2 = 1.
\end{aligned}$$

Hence, we have $\|\overline{\pi_p(x)}\| \geq p(x)$. This completes the proof.

We have the following diagram:

$$\begin{array}{ccccc}
\mathcal{D}(p) & \xrightarrow{\quad} & \mathcal{D}(p)/N_p & \xrightarrow{\quad} & \mathcal{A}_p \text{ (} C^* \text{-algebra)} \\
& \searrow \pi_p^0 & & \text{completion} & \downarrow \Pi_p \\
& & & & \Pi_p(\mathcal{A}_p) \text{ (} C^* \text{-algebra on } \mathcal{H}_{\Pi_p} \text{)} \\
& & & & \downarrow \\
\mathcal{A} & \xrightarrow{\pi_p} & \pi_p(\mathcal{A}) \text{ (} O^* \text{-algebra in } \mathcal{H}_{\pi_p} \subset \mathcal{H}_{\Pi_p} \text{)}.
\end{array}$$

Remark: The $*$ -representation π_p of $\mathcal{A}$ defined above by an unbounded C^* -seminorm p on $\mathcal{A}$ and an element Π_p of $\text{Rep}(\mathcal{A}_p)$ is non-zero if and only if $\mathcal{A} \cap N_p \not\subset N_p$. In what follows, we discuss several situations keeping this in mind.

Let p be an unbounded C^* -seminorm on $\mathcal{A}$. We denote by $\text{Rep}(\mathcal{A}, p)$, $\text{FRep}(\mathcal{A}, p)$ and $\text{FNRep}(\mathcal{A}, p)$ the sets of all $*$ -representations of $\mathcal{A}$ constructed as above by $(\mathcal{A}, p)$, that is,

$$\begin{aligned}
\text{Rep}(\mathcal{A}, p) &= \{\pi_p; \Pi_p \in \text{Rep}(\mathcal{A}_p)\}, \\
\text{FRep}(\mathcal{A}, p) &= \{\pi_p; \Pi_p \in \text{FRep}(\mathcal{A}_p)\}, \\
\text{FNRep}(\mathcal{A}, p) &= \{\pi_p; \Pi_p \in \text{FNRep}(\mathcal{A}_p)\}.
\end{aligned}$$

Definition 2.3. An unbounded m^* -seminorm q on $\mathcal{A}$ is said to be nondegenerate if $\mathcal{D}(q)^2$ is total in $\mathcal{D}(q)$ with respect to the seminorm q . An unbounded m^* -seminorm q on $\mathcal{A}$ is said to be finite if $\mathcal{D}(q) = \mathfrak{N}_q$; and q is said to be uniformly semifinite if there exists a net $\{u_\alpha\}$ in $\mathfrak{N}_q$ such that $u_\alpha^* = u_\alpha$ and $q(u_\alpha) \leq 1$ for each α and $\lim_\alpha q(xu_\alpha - x) = 0$ for each $x \in \mathcal{D}(q)$; and q is said to be semifinite if $\mathfrak{N}_q$ is dense in $\mathcal{D}(q)$ with respect to the seminorm q . An unbounded C^* -seminorm p on $\mathcal{A}$ is said to be weakly semifinite if $\text{FRep}^{WB}(\mathcal{A}, p) \equiv \{\pi_p \in \text{FRep}(\mathcal{A}, p); \mathcal{H}_{\pi_p} = \mathcal{H}_{\pi_p}\} \neq \emptyset$. An element π_p of $\text{Rep}^{WB}(\mathcal{A}, p)$ is said to be a well-behaved $*$ -representation of $\mathcal{A}$ in $\text{Rep}(\mathcal{A}, p)$.

Definition 2.4. A $*$ -representation π of $\mathcal{A}$ is said to be nondegenerate if $[\pi(\mathcal{A})\mathcal{D}(\pi)] = \mathcal{H}_\pi$; and π is said to be strongly nondegenerate if there exists a left ideal $\mathcal{I}$ of $\mathcal{A}$ contained in the bounded part $\mathcal{A}_b^\pi \equiv \{x \in \mathcal{A}; \overline{\pi(x)} \in \mathfrak{B}(\mathcal{H}_\pi)\}$ of π such that $[\overline{\pi(\mathcal{I})}\mathcal{H}_\pi] = \mathcal{H}_\pi$.

Proposition 2.5. Let p be an unbounded C^* -seminorm on $\mathcal{A}$. Then the following statements hold:

(1) $\text{Rep}^{WB}(\mathcal{A}, p) \subset \text{FNRep}(\mathcal{A}, p)$ and every $\pi_p \in \text{Rep}^{WB}(\mathcal{A}, p)$ satisfies the following conditions (i), (ii) and (iii):

(i) $[\overline{\pi_p(\mathfrak{N}_p)}\mathcal{H}_{\pi_p}] = \mathcal{H}_{\pi_p}$, and π_p is strongly nondegenerate.

(ii) $\|\pi_p(x)\| = p(x)$, $\forall x \in \mathcal{D}(p)$.

(iii) $\pi_p(\mathcal{A})'_w = \overline{\pi_p(\mathcal{D}(p))}'$ and $\pi_p(\mathcal{A})'_w \mathcal{D}(\pi_p) \subset \mathcal{D}(\pi_p)$.

Conversely suppose $\pi_p \in \text{FRep}(\mathcal{A}, p)$ satisfies conditions (i) and (ii) above. Then there exists an element π_p^{WB} of $\text{Rep}^{WB}(\mathcal{A}, p)$ which is a representation of π_p .

(2) Suppose p is semifinite. Then $\text{Rep}^{WB}(\mathcal{A}, p) = \text{FNRep}(\mathcal{A}, p)$ and $\mathfrak{N}_p^2$ is total in $\mathcal{D}(p)$ with respect to p , and so p is nondegenerate.

(3) Suppose p is uniformly semifinite. Then

$$\mathcal{A}_b^{\pi_p} = \mathcal{A}_b^p \equiv \{a \in \mathcal{A}; \exists k_a > 0 \text{ s.t. } p(ax) \leq k_a p(x), \forall x \in \mathfrak{N}_p\},$$

$$\|\pi_p(b)\| = \sup\{p(bx); x \in \mathfrak{N}_p \text{ and } p(x) \leq 1\}, \quad \forall b \in \mathcal{A}_b^p$$

for each $\pi_p \in \text{FRep}(\mathcal{A}, p)$.

(4) p is finite if and only if $\mathcal{D}(p)$ is a left ideal of $\mathcal{A}$.

3. UNBOUNDED C^* -SEMINORMS DEFINED BY $*$ -REPRESENTATIONS

In Section 2 we constructed a family $\text{Rep}(\mathcal{A}, p)$ (resp. $\text{Rep}^{\text{WB}}(\mathcal{A}, p)$) of $*$ -representation of $\mathcal{A}$ from an (resp. weakly semifinite) unbounded C^* -seminorm p on $\mathcal{A}$. Conversely we shall construct an unbounded C^* -seminorm r_π on $\mathcal{A}$ from a $*$ -representation π of $\mathcal{A}$ and the natural representation $\pi_{r_\pi}^N$ of $\mathcal{A}$ constructed from r_π , and investigate the relation π and $\pi_{r_\pi}^N$. Let π be a $*$ -representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\pi$. We put

$$\mathcal{A}_b^\pi = \{x \in \mathcal{A}; \overline{\pi(x)} \in \mathfrak{B}(\mathcal{H}_\pi)\} \text{ and } \pi_b(x) = \overline{\pi(x)}, \quad x \in \mathcal{A}_b^\pi.$$

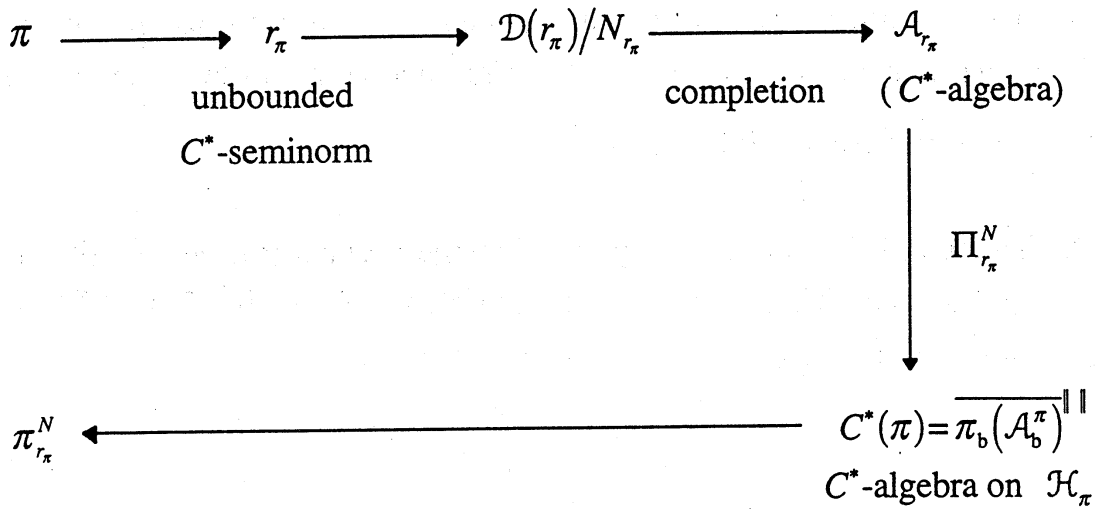
Then $\mathcal{A}_b^\pi$ is a $*$ -subalgebra of $\mathcal{A}$ and π_b is a bounded $*$ -representation of $\mathcal{A}_b^\pi$ on $\mathcal{H}_\pi$. We denote by $C^*(\pi)$ the C^* -algebra generated by $\pi_b(\mathcal{A}_b^\pi)$. We now define an unbounded C^* -seminorm r_π on $\mathcal{A}$ as follows;

$$\mathcal{D}(r_\pi) = \mathcal{A}_b^\pi \text{ and } r_\pi(x) = \|\pi_b(x)\|, \quad x \in \mathcal{D}(r_\pi).$$

Then we put

$$\Pi(x + N_{r_\pi}) = \pi_b(x), \quad x \in \mathcal{A}_b^\pi.$$

Since $\|\Pi(x + N_{r_\pi})\| = r_\pi(x) = \|x + N_{r_\pi}\|_{r_\pi}$ for each $x \in \mathcal{A}_b^\pi$, it follows that Π can be extended to a faithful $*$ -representation $\Pi_{r_\pi}^N$ of $\mathcal{A}_{r_\pi}$ on the Hilbert space $\mathcal{H}_\pi$. The $*$ -representation $\pi_{r_\pi}^N$ of $\mathcal{A}$ defined by $\Pi_{r_\pi}^N$ as above is called the natural representation of $\mathcal{A}$ induced by π . Since $\mathcal{H}_{\Pi_{r_\pi}^N} = \mathcal{H}_\pi$, it follows that $\mathcal{H}_{\pi_{r_\pi}^N}$ is a closed subspace of $\mathcal{H}_\pi$. We simply note the above method of the construction of $\pi_{r_\pi}^N$ by the following diagram:

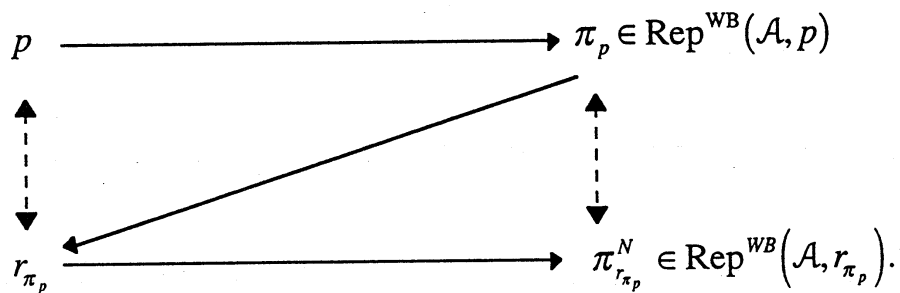


We have the following results for the relation of π and $\pi_{r_\pi}^N$:

Proposition 3.1. Suppose π is a $*$ -representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\pi$. Then the following statements hold:

- (1) $\pi_{r_\pi}^N \subset \tilde{\pi}$.
- (2) Suppose π_b is nondegenerate. Then $\pi_{r_\pi}^N \in \text{FNRep}(\mathcal{A}, r_\pi)$.
- (3) π is strongly nondegenerate if and only if $\pi_{r_\pi}^N \in \text{Rep}^{WB}(\mathcal{A}, r_\pi)$. If this is true, then $\pi_{r_\pi}^N$ is strongly nondegenerate with $\mathcal{A}_b^{\pi_{r_\pi}^N} = \mathcal{A}_b^\pi$, and r_π is weakly semifinite.
- (4) Suppose there exists a net $\{u_\alpha\}$ in $\mathfrak{N}_{r_\pi}$ such that $s\text{-}\lim_\alpha \pi(u_\alpha) = I$ and $s\text{-}\lim_\alpha \pi(au_\alpha) = \pi(a)$ for each $a \in \mathcal{A}$. Then $\tilde{\pi}_{r_\pi}^N = \tilde{\pi}$.

By Proposition 3.1 we have the following diagram:



We here investigate the relations of unbounded C^* -seminorms p and r_{π_p} and the $*$ -representation π_p and $\pi_{r_{\pi_p}}$. We first define an order relation among unbounded seminorms as follows:

Definition 3.2. Let p and q be unbounded seminorms on $\mathcal{A}$. We say that p is an extension of q (or q is a restriction of p) if $\mathcal{D}(q) \subset \mathcal{D}(p)$ and $q(x) = p(x)$ for each $x \in \mathcal{D}(q)$, and then denote by $q \subset p$.

We denote by $C^*N(\mathcal{A})$ the set of all unbounded C^* -seminorms on $\mathcal{A}$. Then $C^*N(\mathcal{A})$ is an ordered set with the order $\subset$. For any $p \in C^*N(\mathcal{A})$ we put

$$C^*N(p) = \{q \in C^*N(\mathcal{A}); p \subset q\}.$$

Then it follows from Zorn's lemma that $C^*N(p)$ has a maximal element. We show that if p is weakly semifinite then r_{π_p} is a maximal element of $C^*N(p)$.

Lemma 3.3. Let p and r be unbounded C^* -seminorms on $\mathcal{A}$. Suppose $p \subset r$. Then, for any $\pi_p \in \text{Rep}(\mathcal{A}, p)$ there exists an element π_r of $\text{Rep}(\mathcal{A}, r)$ such that $\pi_p \subset \pi_r$.

Proposition 3.4. Suppose p is a weakly semifinite unbounded C^* -seminorm on $\mathcal{A}$ and $\pi_p \in \text{Rep}^{\text{WB}}(\mathcal{A}, p)$. Then r_{π_p} is a maximal element of $C^*N(p)$ and $r_{\pi_p} = r_{\pi_p'}$ for each $\pi_p, \pi_p' \in \text{Rep}^{\text{WB}}(\mathcal{A}, p)$.

By Proposition 3.1, (3) and Proposition 3.4 we have the following

Corollary 3.5. Suppose π is a strongly nondegenerate $*$ -representation of $\mathcal{A}$. Then r_π is maximal.

For the relation of $*$ -representation π_p and $\pi_{r_{\pi_p}}^N$ we have the following

Proposition 3.6. Suppose p is a weakly semifinite unbounded C^* -seminorm on $\mathcal{A}$ and $\pi_p \in \text{Rep}^{\text{WB}}(\mathcal{A}, p)$. Then $\pi_p \subset \pi_{r_{\pi_p}}^N$ and $\tilde{\pi}_{r_{\pi_p}}^N = \tilde{\pi}_p$.

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