

On spacelike curve in nullcone 3-space

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1 Basic notions

The nullcone is one kind of pseudo-sphere of Minkowski space. Our aim in this article is to develop the study for spacelike curve in nullcone 3-space by Bruce and Giblin's singularity theory. In order to study the spacelike curve of nullcone 3-space, we need to develop differential geometry of spacelike curve in nullcone 3-space similarly as it was done for curves in Euclidean space [2].

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be a 4-dimensional vector space. For any two vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the *pseudo-scalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = -x_1y_1 + \sum_{i=2}^4 x_iy_i$. $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$ is called a *Minkowski 4-space* and denoted by $\mathbb{R}_1^4$. A vector $\mathbf{x}$ in $\mathbb{R}_1^4 \setminus \{0\}$ is called *spacelike*, *lightlike* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive, zero or negative respectively. The norm of a vector $\mathbf{x} \in \mathbb{R}_1^4$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}_1^4$, we say $\mathbf{x}$ *pseudo-perpendicular* to $\mathbf{y}$ if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. For a vector $\mathbf{v} \in \mathbb{R}_1^4$ and a real number c , we define a hyperplane with pseudo normal $\mathbf{v}$ by $HP(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}_1^4 | \langle \mathbf{x}, \mathbf{v} \rangle = c\}$. $HP(\mathbf{v}, c)$ is called a *timelike hyperplane*, a *spacelike hyperplane* or a *lightlike hyperplane* if $\mathbf{v}$ is timelike, spacelike or lightlike respectively. Now, define the *nullcone 3-space* by $NC^3 = \{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}_1^4 | x_1 \neq 0, \langle \mathbf{x}, \mathbf{x} \rangle = 0\}$, the *de Sitter 3-space* by $S_1^3 = \{\mathbf{x} \in \mathbb{R}_1^4 | \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$ and the *hyperbolic 3-space* by $H_1^3 = \{\mathbf{x} \in \mathbb{R}_1^4 | \langle \mathbf{x}, \mathbf{x} \rangle = -1\}$. If $\mathbf{x} = (x_1, x_2, x_3, x_4)$ is a lightlike vector, then $x_1 \neq 0$. Therefore $\tilde{\mathbf{x}} = (1, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1}) \in S_+^2 = \{\mathbf{x} \in \mathbb{R}_1^4 | \langle \mathbf{x}, \mathbf{x} \rangle = 0, x_1 = 1\}$. S_+^2 is called the *nullcone unit 2-sphere*.

For any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}_1^4$, we define a vector $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3$ by

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1^1 & x_1^2 & x_1^3 & x_1^4 \\ x_2^1 & x_2^2 & x_2^3 & x_2^4 \\ x_3^1 & x_3^2 & x_3^3 & x_3^4 \end{vmatrix},$$

where e_1, e_2, e_3, e_4 are the canonical basis of $\mathbb{R}_1^4$ and $\mathbf{x}_i = (x_i^1, x_i^2, x_i^3, x_i^4)$. It is easy to check that $\langle \mathbf{x}, \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \rangle = \det(\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, so that $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3$ is pseudo orthogonal to $\mathbf{x}_i (i = 1, 2, 3)$.

Let $\gamma : I \rightarrow NC^3$; $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ be a smooth regular curve in NC^3 (i.e., $\dot{\gamma}(t) \neq 0$ for any $t \in I$), where I is an open interval. The curve γ is called a *spacelike curve* if $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle$ is positive for any $t \in I$. The *arc-length* of a spacelike curve γ , measured from $\gamma(t_0)$, $t_0 \in I$ is $s(t) = \int_{t_0}^t \|\dot{\gamma}(t)\| dt$. Then a parameter s is determined such that $\|\gamma'(s)\| = 1$, where $\gamma'(s) = d\gamma/ds(s)$. We say that a spacelike curve γ is *parameterized by arc-length* if it satisfies that $\|\gamma'(s)\| = 1$. Throughout the reminder in this article, s will denote the arc-length parameter. Let $\mathbf{t}(s) = \gamma'(s)$. we call $\mathbf{t}(s)$ an *unit tangent vector* of γ at s . The *signature* of $\mathbf{x}$ is defined to be

$$\delta(\mathbf{x}) = \text{sign}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} : \text{spacelike}; \\ 0 & \mathbf{x} : \text{lightlike}; \\ -1 & \mathbf{x} : \text{timelike} . \end{cases}$$

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For any nonlightlike curve $\gamma : I \rightarrow NC^3$, which is parameterized by arc-length and satisfies $k_1(s) \neq 0$. We can construct a pseudo-orthogonal frame $\{\mathbf{t}(s), \mathbf{n}_1(s), \mathbf{n}_2(s), \mathbf{n}_3(s)\}$ of $\mathbb{R}_1^4$ along γ which satisfies the following Frenet-Serret type formulae:

$$\begin{cases} \mathbf{t}(s) &= \gamma'(s); \\ \mathbf{t}'(s) &= k_1(s)\mathbf{n}_1(s); \\ \mathbf{n}_1'(s) &= -\delta_1 k_1(s)\mathbf{t}(s) + k_2(s)\mathbf{n}_2(s); \\ \mathbf{n}_2'(s) &= \delta_3 k_2(s)\mathbf{n}_1(s) + k_3(s)\mathbf{n}_3(s); \\ \mathbf{n}_3'(s) &= \delta_1 k_3(s)\mathbf{n}_2(s), \end{cases}$$

where $\mathbf{n}_1 = \frac{\gamma''}{\|\gamma''\|} = \frac{\gamma''}{k_1}$, $\mathbf{n}_i = \frac{\mathbf{n}_{i-1}' + \delta_0 \delta_1 \dots \delta_{i-1} k_{i-1} \mathbf{n}_{i-2}}{\delta_0 k_i}$, $\delta_0 = \delta(\mathbf{t})$ and $\delta_i = \delta(\mathbf{n}_i)$ ($i = 1, 2, 3$).

Let $\mathbf{n}_2(s)$ be a timelike vector. Then $\mathbf{n}_j$ ($j \neq 2$) is a spacelike vector.

Define maps

$$NG_{2,j}^\pm : I \rightarrow S_+^2$$

by $NG_{2,j}^\pm(s) = \widetilde{\mathbf{n}_j \pm \mathbf{n}_2}(s)$ ($j = 1, 3$). Also define a map

$$\eta : S_+^2 \rightarrow S_+^2,$$

by $\eta(\widetilde{\mathbf{n}_1 \pm \mathbf{n}_2}(s)) = \widetilde{\mathbf{n}_2 \pm \mathbf{n}_3}(s)$, $\eta(\widetilde{\mathbf{n}_2 \pm \mathbf{n}_3}(s)) = \widetilde{\mathbf{n}_1 \pm \mathbf{n}_2}(s)$ and η is identity on the other elements of S_+^2 . Each one of $NG_{2,j}^\pm$ ($j = 1, 3$) is called the *nullsphere Gauss map* of γ .

2 Nullsphere height functions on spacelike curve in NC^3

Now the function

$$H_1 : I \times S_+^2 \rightarrow \mathbb{R}$$

is defined by $H_1(s, \mathbf{v}) = \langle \gamma(s), \mathbf{v} \rangle$ and the function

$$H_2 : I \times S_+^2 \rightarrow \mathbb{R}$$

is defined by $H_2(s, \mathbf{v}) = \langle \gamma(s), \eta(\mathbf{v}) \rangle$, H_1 and H_2 are called the *nullsphere height function* on the spacelike curve γ . For any fixed $\mathbf{v}_0 \in S_+^2$, we denote that $h_{1,\mathbf{v}_0}(s) = H_1(s, \mathbf{v}_0)$ and $h_{2,\mathbf{v}_0}(s) = H_2(s, \mathbf{v}_0)$, then we have the following theorem.

Theorem 2.1. Let $\gamma : I \rightarrow NC^3$ be an unit speed spacelike curve with $k_1(s) \neq 0$. Then we have the following assertions:

(1) $h_{1,\mathbf{v}_0}'(s_0) = 0$ (resp. $h_{2,\mathbf{v}_0}'(s_0) = 0$) if and only if there exist λ_1 and λ_2 such that $\mathbf{v} = \widetilde{\mathbf{n}}(s_0)$ (resp. $\eta(\mathbf{v}) = \widetilde{\mathbf{n}}(s_0)$), $\mathbf{n}(s_0) = (\lambda_1 \mathbf{n}_1 \pm \sqrt{\lambda_1^2 + \lambda_2^2} \mathbf{n}_2 + \lambda_2 \mathbf{n}_3)(s_0) \in NC^3$.

(2) $h_{1,\mathbf{v}_0}'(s_0) = h_{1,\mathbf{v}_0}''(s_0) = 0$ (resp. $h_{2,\mathbf{v}_0}'(s_0) = h_{2,\mathbf{v}_0}''(s_0) = 0$) if and only if $\mathbf{v} = \widetilde{\mathbf{n}_3 \pm \mathbf{n}_2}(s_0)$ (resp. $\mathbf{v} = \widetilde{\mathbf{n}_1 \pm \mathbf{n}_2}(s_0)$).

(3) $h_{1,\mathbf{v}_0}'(s_0) = h_{1,\mathbf{v}_0}''(s_0) = h_{1,\mathbf{v}_0}'''(s_0) = 0$ (resp. $h_{2,\mathbf{v}_0}'(s_0) = h_{2,\mathbf{v}_0}''(s_0) = h_{2,\mathbf{v}_0}'''(s_0) = 0$) if and only if $\mathbf{v} = \widetilde{\mathbf{n}_3 \pm \mathbf{n}_2}(s_0)$ (resp. $\mathbf{v} = \widetilde{\mathbf{n}_1 \pm \mathbf{n}_2}(s_0)$) and $k_2(s_0) = 0$.

(4) $h_{1,\mathbf{v}_0}'(s_0) = \dots = h_{1,\mathbf{v}_0}^{(4)}(s_0) = 0$ (resp. $h_{2,\mathbf{v}_0}'(s_0) = \dots = h_{2,\mathbf{v}_0}^{(4)}(s_0) = 0$) if and only if $\mathbf{v} = \widetilde{\mathbf{n}_3 \pm \mathbf{n}_2}(s_0)$ (resp. $\mathbf{v} = \widetilde{\mathbf{n}_1 \pm \mathbf{n}_2}(s_0)$) and $k_2(s_0) = k_2'(s_0) = 0$.

Theorem 2.2. Let $\gamma(s)$ be a spacelike curve in nullcone 3-space. Then:

(1) If $\mathbf{v}_0 = \widetilde{\gamma}(s_0)$, then $h_{1,\mathbf{v}_0}''(s_0)$ never equal to zero.

(2) If $\eta(\mathbf{v}_0) = \widetilde{\gamma}(s_0)$, then $h_{2,\mathbf{v}_0}''(s_0)$ never equal to zero.

Proposition 2.3. If $\gamma(s)$ is an unit speed spacelike curve, H_1 and H_2 are nullsphere height functions, $B_{H_1} = \{\mathbf{v} \in S_+^2 \mid h_{1,\mathbf{v}}'(s) = h_{1,\mathbf{v}}''(s) = 0\}$ and $B_{H_2} = \{\mathbf{v} \in S_+^2 \mid h_{2,\mathbf{v}}'(s) = h_{2,\mathbf{v}}''(s) = 0\}$, then the following conditions are equivalent:

(1) $h_{1,\mathbf{v}_0}'''(s_0) = 0$ for $\mathbf{v}_0 = (\widetilde{\mathbf{n}_3 \pm \mathbf{n}_2})(s_0)$ (resp. $h_{2,\mathbf{v}_0}'''(s_0) = 0$ for $\mathbf{v}_0 = (\widetilde{\mathbf{n}_1 \pm \mathbf{n}_2})(s_0)$);

(2) s_0 is a singularity of nullsphere Gauss map $NG_{2,3}^\pm$ (resp. $NG_{2,1}^\pm$) on γ ;

(3) $k_2(s_0) = 0$.

Consider now the particular case of a curve $\gamma \subset NC^3$. Given a vector $\mathbf{v} \in S_+^2$ (resp. S_+^3, H_+^3) and a number c , denote by $S(\mathbf{v}, c)$ the *null hyperhorosphere* (resp. *null hypersphere, null equidistant hyperplane*) determined by the intersection of the hyperplane $HP(\mathbf{v}, c)$ with NC^3 .

Proposition 2.4. *Suppose that $\tilde{\gamma}(s) = NG_{2,j}^\pm(s)$. If $NG_{2,j}^\pm$ is constant, then $\gamma(s)$ is a straight line.*

Proof. Since $\tilde{\gamma}(s) = NG_{2,j}^\pm(s)$, $\gamma(s) = \gamma_1(s)NG_{2,j}^\pm(s)$. $NG_{2,j}^\pm(s)$ is constant, so $\gamma(s)$ is a straight line. $\square$

For an unit speed spacelike curve $\gamma : I \rightarrow NC^3$, we now define *extended nullsphere height functions* $\tilde{H}_1 : I \times NC^3 \rightarrow \mathbb{R}$ by $\tilde{H}_1(s, \mathbf{v}) = H_1(s, \tilde{\mathbf{v}}) - \mathbf{v}_1 = \langle \gamma(s), \tilde{\mathbf{v}} \rangle - \mathbf{v}_1$ and $\tilde{H}_2 : I \times NC^3 \rightarrow \mathbb{R}$ by $\tilde{H}_2(s, \mathbf{v}) = H_2(s, \tilde{\mathbf{v}}) - \mathbf{v}_1 = \langle \gamma(s), \eta(\tilde{\mathbf{v}}) \rangle - \mathbf{v}_1$, where H_1 and H_2 are the nullsphere height function on γ . For any fixed $\mathbf{v}_0 \in NC^3$, we denote $\tilde{h}_{1,v_0}(s) = \tilde{H}_1(s, \mathbf{v}_0)$ and $\tilde{h}_{2,v_0}(s) = \tilde{H}_2(s, \mathbf{v}_0)$.

Let $F : NC^3 \rightarrow \mathbb{R}$ be a submersion and $\gamma : I \rightarrow NC^3$ be a spacelike curve. We say that γ and $F^{-1}(0)$ have *k-point contact* at t_0 if $g(t) = F \circ \gamma(t)$ satisfies $g(t_0) = g'(t_0) = \dots = g^{(k-1)}(t_0) = 0$, $g^{(k)}(t_0) \neq 0$. Then we have the following corollary.

Corollary 2.5. *Let $\gamma : I \rightarrow NC^3$ be an unit speed spacelike curve with $k_1(s) \neq 0$. Then γ and the null hyperhorosphere $S(\mathbf{v}_0^\pm, c_0^\pm)$ have 4-point contact at s_0 if and only if $k_2(s) = 0$ and $k_2'(s) \neq 0$, where $\mathbf{v}_0^\pm = \mathbf{n}_3 \pm \mathbf{n}_2(s_0)$, $c_0^\pm = \langle \gamma(s_0), \mathbf{v}_0^\pm \rangle$.*

This work is only a preparation for further studying, in the following, we will give the classification of singularities of nullsphere Gauss map and discuss some geometrical properties of spacelike curve from singularity theory viewpoint.

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