

On (sub)-holonomicity of
Some Modules and b-functions

By

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本記録において, $\mathcal{D}$ -Module の section u に対して,
 $\mathcal{N} = \mathcal{D}[s] f^s u$, $\mathcal{D} f^\alpha u$ ($\alpha \in \mathbb{C}$) を定義し, 性質をいべる.
 この $\mathcal{N}$ は $\mathcal{D}[t, s]$ -Module の構造をもち, t は injective
 になっている。 $\mathcal{N}$ の b 函数は常に存在し, 根によって,
 $\mathcal{D}u$ が holonomic の場合が重要であり, この場合 $\mathcal{D} f^\alpha u$ と
 $\mathcal{N}/(s-\alpha)\mathcal{N}$ の同型性を判別される。定理 1.17, 20, 21, 24
 が重要である。又, reduced b-function についての事項もあつた。
 詳細は下記を参照し, 又 付録につづきについで発表される。

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Chapter I Generalities

In this chapter, we study the basic features of general $\mathcal{D}[t,s]$ -Modules and b-functions associated with them, which are indispensable to later chapters. The author develops the general theory of such b-functions and Modules in [32].

§ 1. $\mathcal{D}[t,s]$ - Modules and b-functions.

Let $\mathbb{C}[t,s]$ be the associative algebra over $\mathbb{C}$ with generators s and t and defining relation

$$ts - st = t. \quad (1)$$

Set $\mathcal{D}[t,s] = \mathcal{D} \otimes_{\mathbb{C}} \mathbb{C}[t,s]$.

A $\mathcal{D}$ -Module $\mathcal{M}$ is called a $\mathcal{D}[s]$ -Module (respectively $\mathcal{D}[t,s]$ -Module), if $\mathcal{M} \otimes_{\mathcal{D}} \mathcal{D}[s]$ (respectively $\mathcal{M} \otimes_{\mathcal{D}} \mathcal{D}[t,s]$, $\mathcal{M} \otimes_{\mathcal{D}} \mathcal{D}[t,s]$) holds. In this chapter, all Modules are $\mathcal{D}[t,s]$ -Modules unless otherwise stated. Since $t^\nu s = (s + \nu)t^\nu$ in view of (1), $\text{Ker } t^\nu$, $\text{Coker } t^\nu$ and $\text{Im } t^\nu$ are $\mathcal{D}[t,s]$ -Modules along with $\mathcal{M}$ given $\mathcal{D}[t,s]$ -Module.

Definition 1.1 Let $\mathcal{L}$ be a $\mathcal{D}[s]$ -Module. If $s \in \text{End}_{\mathcal{L}}(\mathcal{L})$ has the non-zero minimal polynomial, we denote it by $d_{\mathcal{L}}(s)$, and say " $d_{\mathcal{L}}(s)$ exists." "b-functions" for a $\mathcal{D}[t,s]$ -Module $\mathcal{N}$ are defined by $b_{\mathcal{N},\nu}(s) = d_{\mathcal{N}/t^{\nu}\mathcal{N}}(s)$, $\nu=1,2,\dots$.

Usually, $b_{\mathcal{N},1}$ is abbreviated as $b_{\mathcal{N}}$. As is easily seen, $b_{\mathcal{N},\nu}$ exist if and only if $b_{\mathcal{N}}$ exists.

It should be remarked that if $\mathcal{L}$ is a holonomic $\mathcal{D}[t,s]$ -Module $d_{\mathcal{L}}(s)$ exists, since $\text{End}_{\mathcal{D}}(\mathcal{L})_x$ ($x \in X$) is finite dimensional and $\text{End}_{\mathcal{D}}(\mathcal{L})$ is coherent [13].

Standard example of $\mathcal{D}[t,s]$ -Module is constructed as follows. Let f be a holomorphic function on $U \subset X$, let $\mathcal{L}$ be a coherent $\mathcal{D}$ -Module and let u be its section over U . We denote the annihilator of u by $\mathcal{I}$, that is; $\mathcal{I} = \{Q \in \mathcal{D} \mid Qu=0\}$. Define the ideal $\mathcal{J}(s) \subset \mathcal{D}[s]$ by the condition that

$$\begin{aligned} P(s,x,D) \in \mathcal{J}(s) & \quad \text{if and only if} \\ f^m P(s,x,D + \frac{s}{f} \text{grad } f) \in \mathbb{C}[s] \mathcal{I} & \quad , \text{ for some } m. \end{aligned}$$

We denote by $\mathcal{N}$ the Module $\mathcal{D}[s]/\mathcal{J}(s)$ and by $f^s u$ the class $(1 \bmod \mathcal{J}(s))$. $\mathcal{N} = \mathcal{D}[s]f^s u$ is a $\mathcal{D}[t,s]$ -Module with actions of t and s given by,

$$t: P(s) \mapsto P(s+1)f, \quad s: P(s) \mapsto P(s)s.$$

The map t is injective in $\mathcal{N}$. In fact, if $P(s+1)f \in \mathcal{J}(s)$ then

$$f^{m+1}P(s+1, x, D + \frac{s}{f} \text{grad } f)f = \sum Q_j s^j$$

for some m and $Q_j \in \mathcal{O}$. The left-hand side equals to

$$f^{m+1}P(s+1, x, D + \frac{s+1}{f} \text{grad } f),$$

and the right-hand side can be rewritten in the form

$$\sum R_j (s+1)^j$$

for some $R_j \in \mathcal{O}$. Therefore,

$$f^{m+1}P(s, x, D + \frac{s}{f} \text{grad } f) = \sum R_j s^j,$$

which implies $P(s) \in \mathcal{J}(s)$.

The $\mathcal{L}$ -Module $\mathcal{L} f^s u$ is coherent, and if u is a holonomic section, $\mathcal{L} f^s u$ is subholonomic (See [32]).

Definition 1.2 With a non-zero polynomial $p(s)$, we associate a number $w(p) \in \mathbb{N}_0$ in the following manner ($w(p)$ is called the width of p .)

- then
- i) If $p(s) \in \mathbb{C}^*$ $w(p) = 0$,
 - ii) If $p(s) = \prod_{i=1}^k (s + \alpha_i + 1)^{\varepsilon_i}$, $\alpha_i \in \mathbb{C}$, $\varepsilon_i \in \mathbb{N}$, $\varepsilon_i \neq 0$ then $w(p) = k+1$,
 - iii) If $p(s)$ has the form
 $p(s) = \prod_{j=1}^k p_j(s)$, where each $p_j(s)$ is of the form
in ii), $p_j(s) = \prod (s + \alpha_j + 1)^{\varepsilon_j^{(i)}}$, and $\alpha_j \neq \alpha_{j'}$
mod $\mathbb{Z}$ ($j \neq j'$); then $w(p) = \max_j w(p_j)$.

Theorem 1.3 If $d_{\mathcal{L}}(s)$ exists, then $t^{w(d_{\mathcal{L}})} \mathcal{L} = 0$. Furthermore
if we assume that t is injective or surjective, then $\mathcal{L} = 0$.

Proof) we have

$$0 = d_{\mathcal{L}}(s) \mathcal{L} \supset d_{\mathcal{L}}(s) t^{w(d_{\mathcal{L}})} \mathcal{L},$$

and by virtue of (1),

$$0 = t^{w(d_{\mathcal{L}})} d_{\mathcal{L}}(s) \mathcal{L} = d_{\mathcal{L}}(s + w(d_{\mathcal{L}})) t^{w(d_{\mathcal{L}})} \mathcal{L}.$$

It follows from the definition of $w(d_{\mathcal{L}})$ that

$$\text{g.c.d.}(d_{\mathcal{L}}(s), d_{\mathcal{L}}(s + w(d_{\mathcal{L}}))) = 1.$$

Hence the assertion follows. When t is injective or
surjective, it is obvious that $\mathcal{L} = 0$. Q.E.D.

A coherent $\mathcal{G}$ -Module $\mathcal{L}$ is called holonomic (resp.
sub-holonomic) if $\mathcal{E}_{\mathcal{L}}^i(\mathcal{L}, s) = 0$ for $i < n$ (resp. $i < n-1$).

This condition is equivalent to $\text{codim } \check{SS}(\mathcal{L}) \geq n$ (resp. $\text{codim } \check{SS}(\mathcal{L}) \geq n-1$). $\mathcal{L}$ is called purely subholonomic if $\text{Ext}_{\mathcal{S}}^i(\mathcal{L}, \mathcal{S}) = 0$ for $i \neq n-1$. It is known that for any coherent $\mathcal{S}$ -Module, $\text{Ext}_{\mathcal{S}}^n(\mathcal{L}, \mathcal{S})$ (resp. $\text{Ext}_{\mathcal{S}}^{n-1}(\mathcal{L}, \mathcal{S})$) is holonomic (resp. sub-holonomic) and $\text{Ext}_{\mathcal{S}}^i(\mathcal{L}, \mathcal{S}) = 0, i > n$. Let W be an irreducible component of $\check{SS}(\mathcal{L})$. Then the multiplicity of $\mathcal{L}$ at a generic point x_0 of an irreducible component of $\check{SS}(\mathcal{L})$ can be defined (which is denoted by $m_{x_0}(\mathcal{L})$), and has the additivity, that is, if

$$0 \leftarrow \mathcal{L}_1 \leftarrow \mathcal{L}_2 \leftarrow \mathcal{L}_3 \leftarrow 0,$$

is an exact sequence of coherent $\mathcal{S}$ -Modules, $m_{x_0}(\mathcal{L}_2) = m_{x_0}(\mathcal{L}_1) + m_{x_0}(\mathcal{L}_3)$.

Corollary 1.4 Let $\mathcal{N}$ be a sub-holonomic $\mathcal{S}[t, s]$ -Module such that $t: \mathcal{N} \rightarrow \mathcal{N}$ is injective. Then, $\mathcal{N}$ is purely sub-holonomic.

Proof) Consider the exact sequence

$$0 \leftarrow \mathcal{N}/t\mathcal{N} \leftarrow \mathcal{N} \xleftarrow{t} \mathcal{N} \leftarrow 0.$$

Set $\mathcal{L} = \text{Ext}_{\mathcal{S}}^n(\mathcal{N}, \mathcal{S})$. Then $\mathcal{L}$ is holonomic and the long exact sequence of Ext gives us the surjection $\mathcal{L} \xrightarrow{t} \mathcal{L} \rightarrow 0$. Therefore $\mathcal{L} = 0$ by virtue of Theorem 1.3. Q.E.D.

Proposition 1.5 Upon the conditions in Corollary 1.4, $b_{\mathcal{N}}$ exists.

Proof) Consider an irreducible component W of $\check{SS}(\mathcal{N})$. Since t is injective, the multiplicity of $\mathcal{N}/t\mathcal{N}$ at a generic point of W vanishes. Therefore $\text{codim } \check{SS}(\mathcal{N}/t\mathcal{N}) \geq n$ which implies that $\mathcal{N}/t\mathcal{N}$ is holonomic. Thus $b_{\mathcal{N}}$ exists (and so does $b_{\mathcal{N},\nu}$, by the argument after Definition 1.1.). Q.E.D.

The conditions in Corollary 1.4 are satisfied for $\mathcal{N} = \mathcal{G}[s]f^s u$, if one of the following two conditions holds.

- i) f is arbitrary holomorphic function, $u = 1$.
- ii) f is quasi-homogeneous, $\mathcal{G}u$ is holonomic.

In the present paper, we restrict ourselves to case i). We investigate case ii) in [32], where the detailed structure of $b_{\mathcal{N},\nu}(s)$ and the relation between $\mathcal{N}_\alpha$ and $\mathcal{G}f^\alpha u$ ($\alpha \in \mathbb{C}$) are also discussed. The existence of $b_{\mathcal{N}}(s)$ for $\mathcal{N} = \mathcal{G}[s]f^s u$ with general f and $\mathcal{G}u$ being holonomic can be derived from that of case ii), following the technique in [14]. (See [32] §.).

以下は別の原稿よりとったもので、以上と直接はつながりない。

A. General structure of $\mathcal{L}[t,s]$ -Modules

In §1 we study the structure of $b_{\pi,\nu}(s)$ and define reduced b-functions. The relation between reduced b-function of π_1 and that of a sub-Module π_2 is studied in §2.

The key theorem is the following.

Theorem 1.1 (Theorem 1.3 in [Y])

Let $\mathcal{L}$ be a $\mathcal{L}[t,s]$ -Module such that $d_{\mathcal{L}}(s)$ exist.
Then $t^{w(d_{\mathcal{L}})} \mathcal{L} = 0$.

Here, $w(d_{\mathcal{L}})$ is the width of $d_{\mathcal{L}}$. We recall its meaning and add some more definitions.

Definition 1.2 For a non-zero polynomial $p(s)$, we associate a number $w(p)$, called the width of p , and polynomials $\hat{p}(s)$ and $\check{p}(s)$ in the following manner.

- i) $p(s) \in \mathbb{C}^*$; $w(p) = 0$, $\hat{p}(s) = \check{p}(s) = 1$
- ii) $p(s) = \prod_{i=0}^k (s+\alpha+i)^{\varepsilon_i}$, $\alpha \in \mathbb{C}$, $\varepsilon_0 \varepsilon_k = 0$; $w(p) = k+1$,
 $\hat{p}(s) = (s+\alpha)^{\varepsilon_0}$, $\check{p}(s) = (s+\alpha+k)^{\varepsilon_k}$.
- iii) $p(s) = \prod_{j=1}^k p_j(s)$, where each $p_j(s)$ is of the form in ii), $p_j(s) = \prod (s+\alpha_j+i)^{\varepsilon_i^{(j)}}$, and $\alpha_j \not\equiv \alpha_{j'} \pmod{\mathbb{Z}}$ ($j \neq j'$); $w(p) = \max_j w(p_j)$, $\hat{p}(s) = \prod \hat{p}_j(s)$,
 $\check{p}(s) = \prod \check{p}_j(s)$.

§1. Structure of $b_{\pi, \nu}(s)$.

We first note that $b_{\pi, \nu}(s)$ is of the significant structure. Given a rational function $p(s)$, we use the notation

$$[p(s)]_{\nu} = \prod_{i=0}^{\nu-1} p(s+i) \quad \nu > 0, \quad [p(s)]_0 = 1.$$

Theorem 1.3 i) There are a rational function $\bar{b}_{\pi}(s)$, polynomials $\bar{b}'_{\pi}(s)$ and $c_{\pi}(s)$, unique up to a constant multiple, and $\nu_0 \in \mathbb{N}_0$, such that for $\nu \geq \nu_0$,

$$b_{\pi, \nu}(s) = [\bar{b}_{\pi}(s)]_{\nu} c_{\pi}(s+\nu) \quad (2)$$

$$= c_{\pi}(s) [\bar{b}'_{\pi}(s)]_{\nu}, \quad (3)$$

$$\bar{b}_{\pi}(s) c_{\pi}(s+1) = c_{\pi}(s) \bar{b}'_{\pi}(s). \quad (4)$$

ii) If $t: \pi \rightarrow \pi$ is injective, $\bar{b}_{\pi}(s)$ is also a polynomial, and for $\nu \leq \nu_0$ there are polynomials $c_{\pi, \nu}(s)$ and $c'_{\pi, \nu}(s)$ such that

$$b_{\pi, \nu}(s) = [\bar{b}_{\pi}(s)]_{\nu} c_{\pi, \nu}(s+\nu)$$

$$\begin{aligned}
b_{\pi, \nu}(s) &= [\bar{b}_{\pi}(s)]_{\nu} c_{\pi, \nu}(s+\nu) \\
&= c'_{\pi, \nu}(s) [\bar{b}'_{\pi}(s)]_{\nu},
\end{aligned}$$

$$c_{\pi, \nu}(s) \mid c_{\pi, \nu'}(s), \quad c'_{\pi, \nu}(s) \mid c'_{\pi, \nu'}(s) \quad \text{for } \nu \leq \nu'.$$

$$\hat{b}_{\pi}(s) \mid \bar{b}_{\pi}(s), \quad \check{b}_{\pi}(s) \mid \bar{b}'_{\pi}(s). \quad (5)$$

Moreover, it is possible to take $\nu_0 = w(b_{\pi}) - 1$, and the following relations hold.

$$c_{\pi}(s) \mid [\bar{b}_{\pi}(s)]_{\nu_0}, [\bar{b}'_{\pi}(s-\nu_0)]_{\nu_0}, \quad (6)$$

$$b_{\pi}(s) \mid [\bar{b}_{\pi}(s)]_{\nu_0+1}, [\bar{b}'_{\pi}(s-\nu_0)]_{\nu_0+1}, \quad (7)$$

$$w(c_{\pi}) \leq \nu_0. \quad (8)$$

Corollary 1.4 As easily seen, $\bar{b}_{\pi}(s)$ and $\bar{b}'_{\pi}(s)$ can be so determined that

$$\bar{b}_{\pi}(s) = b_{\pi, \nu+1}(s)/b_{\pi, \nu}(s+1),$$

$$\bar{b}'_{\pi}(s) = b_{\pi, \nu+1}(s-\nu)/b_{\pi, \nu}(s-\nu), \quad \nu \geq \nu_0.$$

$\bar{b}_{\pi}(s)$ is called the reduced b-function of π . The special case of the part of this theorem is substantially due to M.Sato [2].

この証明は省略する.

Hereafter $R, \bar{R}, \bar{R}', R_\nu$ and C denote the set of the roots of equations $b_\pi(s) = 0, \bar{b}_\pi(s) = 0, \bar{b}'_\pi(s) = 0$ $b_{\pi,\nu}(s) = 0$ and $c_\pi(s) = 0$ respectively, when t is injective.

Proposition 1.5 $R \supset \bar{R}, \bar{R}', C; R \cap (R+1) \supset C$

$$R_{k+k} = \bigcup_{i=1}^k (R+i) = \bigcup_{i=1}^k (\bar{R}+i) \cup C, \quad R_k = \bigcup_{i=0}^{k-1} (R-i) = \bigcup_{i=0}^{k-1} (\bar{R}-i) \cup C.$$

Proof is straightforward.

We end this section by adding the following remarks when $t \in \text{End}(\pi)$ is not necessarily injective.

Definition 1.6 We define the $\mathcal{S}[t,s]$ -Module $\tilde{\pi}$ by $\pi / \bigcup_{\nu \geq 1} \text{Ker } t^\nu$. (Hence t is injective in $\tilde{\pi}$.)

We can prove $\bar{b}_\pi(s) = \frac{\tilde{c}(s)}{\tilde{c}(s+1)} \bar{b}_{\tilde{\pi}}(s)$, where $\tilde{c}(s) = c_\pi(s)/c_{\tilde{\pi}}(s)$ is a polynomial, and $\bar{b}'_\pi(s) = \bar{b}'_{\tilde{\pi}}(s)$. The proof is omitted.

Proposition 1.7 Let $0 \rightarrow \mathcal{Z} \hookrightarrow \pi \rightarrow \pi' \rightarrow 0$ be an exact sequence of $\mathcal{S}[t,s]$ -Modules, let $t \in \text{End}_{\mathcal{S}}(\pi')$ be injective, and let $d_{\mathcal{Z}}(s)$ exist. Then $\pi' \cong \tilde{\pi}$.

For, since $\pi \twoheadrightarrow \pi'$ and $t|_{\pi'}$ is injective, $\bigcup_{\nu} \text{Ker}(t|_{\pi'})^\nu \hookrightarrow \mathcal{Z}$. On the other hand, $t^{w(d_{\mathcal{Z}})} \mathcal{Z} = 0$ by Theorem 1.1. Therefore, $\mathcal{Z} = \text{Ker } t^{w(d_{\mathcal{Z}})} = \bigcup \text{Ker } t^\nu$, and $\tilde{\pi} \cong \pi'$.

§2. b-functions for a sub-Module

In terms of $b_{\pi}(s)$, we can estimate the b-function of a submodule of π .

Theorem 1.8 Let π_1 be $\mathcal{D}[t,s]$ -Module and let π_2 be its submodule. Further assume 1° $t \in \text{End}(\pi_1)$ is injective, 2° $d_{\pi_1/\pi_2}(s)$ exists and 3° $b_{\pi_1}(s)$ or $b_{\pi_2}(s)$ exists. Then, $\deg \bar{b}_{\pi_1} = \deg \bar{b}_{\pi_2} (= d)$ and there are polynomials $c(s)$ and $c'(s)$, unique up to a constant multiple, such that

$$c(s), c'(s) \mid d_{\pi_1/\pi_2}(s), \quad (15)$$

$$c_{\pi_1}(s)c'(s) = c_{\pi_2}(s)c(s), \quad (16)$$

$$\bar{b}_{\pi_1}(s) = \frac{c(s)}{c'(s+1)} \bar{b}_{\pi_2}(s), \quad \bar{b}'_{\pi_1}(s) = \frac{c'(s)}{c'(s+1)} \bar{b}'_{\pi_2}(s)$$

Corollary 1.9

$$b_{\pi_2}(s) \mid [b_{\pi_1}(s)]_{\nu_0+1}, \quad b_{\pi_1}(s) \mid [b_{\pi_2}(s-\nu_0)]_{\nu_0+1}, \quad (18)$$

$$\bar{b}_{\pi_2}(s) \mid [\bar{b}_{\pi_1}(s)]_{\nu_0+1}, \quad \bar{b}_{\pi_1}(s) \mid [\bar{b}_{\pi_2}(s-\nu_0)]_{\nu_0+1}, \quad (19)$$

$$b_{\pi_2}(s) \mid [\bar{b}_{\pi_1}(s)]_{\nu_0+\nu'}, \quad b_{\pi_1}(s) \mid [\bar{b}_{\pi_2}(s-\nu_0)]_{\nu_0+\nu'}, \quad (20)$$

$$|\deg c_{\pi_1} - \deg c_{\pi_2}| \leq \nu_0 d, \quad (21)$$

where $\nu_0 = w(d_{\pi_1/\pi_2})$, $\nu' = \min(w(b_{\pi_1}), w(b_{\pi_2}))$.

Proof of Theorem 1. 7) It follows from Thm.1.1 and condition 2 that $\pi_1 \supset t^{\nu_0} \pi_2$. Consider the following diagram for $\nu \geq \nu_0$,

$$\begin{array}{ccccc} & & t^{\nu_0} \pi_1 & \supset & \\ \pi_1 \supset \pi_2 & \supset & & \supset & t^{\nu} \pi_1 \supset t^{\nu} \pi_2 \\ & \supset & t^{\nu-\nu_0} \pi_2 & \supset & \end{array}$$

This immediately reads

$$i) \quad b_{2, \nu-\nu_0}(s) \mid b_{1, \nu}(s) \quad (22)$$

$$b_{1, \nu-\nu_0}(s+\nu_0) \mid b_{2, \nu}(s) \quad (23)$$

$$ii) \quad b_{2, \nu}(s) \mid b_{1, \nu}(s) d(s+\nu), \quad (24)$$

$$b_{1, \nu}(s) \mid d(s) b_{2, \nu}(s). \quad (25)$$

Here, we have used the notations, $b_{i, \nu}(s) = b_{\pi_i, \nu}(s)$, $c_i(s) = c_{\pi_i}(s)$, $d(s) = d_{\pi_1/\pi_2}(s)$. (22) and (23) tell us that the existence of b_1 and that of b_2 are equivalent. In particular, setting $\nu = \nu_0 + 1$, we have (18).

i) gives also,

$$(\nu - \nu_0) \deg b_2 + \deg c_2 \leq \deg b_1 + \deg c_1,$$

$$(\nu - \nu_0) \deg b_1 + \deg c_1 \leq \deg b_2 + \deg c_2,$$

and letting ν tend to infinity, we have $\deg b_1 = \deg b_2$.

This implies (21).

Because of (22) and (23), we can assume, $b_i(s) = \prod_{j=1}^d (s+n_j^{(i)})$, and $n_j^{(1)} \leq n_{j+1}^{(1)}$ for $n_j^{(1)} \in \mathbb{Z}$. Setting $\nu \gg 0$ in formula (22), we have $s+n_1^{(2)} \mid [\bar{b}_1(s)]_\nu$, hence $n_1^{(2)} \geq n_1^{(1)}$. Similarly by (23), $n_1^{(1)} + \nu_0 \geq n_1^{(2)}$. Therefore $r_{1,\nu}(s) = [s+n_1^{(1)}]_\nu / [s+n_1^{(2)}]_{\nu-\nu_0}$ is a polynomial. Then the relation

$$[\prod_{j=2}^d (s+n_j^{(2)})]_{\nu-\nu_0} c_2(s+\nu-\nu_0) \mid r_{1,\nu}(s) [\prod_{j=2}^d (s+n_j^{(1)})]_\nu c_1(s+\nu),$$

for $\nu \gg 0$ yields $n_2^{(2)} \geq n_2^{(1)}$, and similarly, $n_2^{(1)} + \nu_0 \geq n_2^{(2)}$.

Proceeding in this way, we have

$$n_j^{(1)} + \nu_0 \geq n_j^{(2)} \geq n_j^{(1)} \quad j = 1, \dots, d. \quad (26)$$

Set $c(s) = \prod_{j=1}^d [s+n_j^{(1)}]_{n_j^{(2)}-n_j^{(1)}}$. Then clearly $c(s)$ is a polynomial and the first of (17) holds. Uniqueness of $c(s)$ is obvious. We apply (17) to (25) and have, after cancellation,

$$c(s) c_1(s+\nu) \mid d(s) c_2(s+\nu) c(s+\nu),$$

taking $\nu \gg 0$, $c(s) \mid d(s)$.

Statements about $c'(s)$ can be proved analogously, and equation (17) applied to (2) and (3) gives (16). From equation (17), we have

$$c(s) [\bar{b}_2(s)]_\nu = [\bar{b}_1(s)]_\nu c(s+\nu). \quad (27)$$

(with $\nu = \nu_0 + 1$)

The definition of $c(s)$, together with (26) and (27) gives

(19). Analogously, (18), (26) and (27) (with $\nu = \nu_0 + \nu'$) prove (20).

Q.E.D.

Remark 1. $b_{\pi_2}(s) \mid [b_{\pi_1}(s)]_{\nu_0+1}$ (28)

holds even when t is not injective.

2. (27), (16) and (2) give

$$c(s)b_{2,\nu}(s) = b_{1,\nu}(s)c'(s+\nu). \quad (29)$$

3. Let h and k be the minimum and the maximum of the indices which satisfy $n_i^{(1)} < n_i^{(2)}$ respectively. Then by (15) $n_k^{(2)} - n_h^{(1)} \leq \nu_0$. It should be noted that this inequality improves (26).

Theorem 1.10 Let X' and X be complex analytic and let $\pi: X' \rightarrow X$ be projective holomorphic map.

For an $f(x) \in \mathcal{O}_X$, we set $f' = f \cdot \pi$. We assume $X' = \pi^{-1}(f^{-1}(0)) \cong X - f^{-1}(0)$. Then, $\mathcal{R} = \mathcal{D}_X[s]f^s$ is a sub-Module of $\mathcal{R}'' = \int \mathcal{R}'$, $\mathcal{R}' = \mathcal{D}_{X'}[s]f'^s$, and

$$b_{f',X}(s) \mid [b_{f',\pi^{-1}(x)}(s)]_{\nu_0+1}, \quad [\overline{b}_{f',\pi^{-1}(x)}(s)]_{\nu_0+w(b_{f'})}. \quad (30)$$

Here $\nu_0 = w(d \mathcal{R}''/\mathcal{R})$.

この証明は省略する。

B. Structure of $\mathcal{L}[s](f^s u)$

In the following sections, we investigate the structure of special $\mathcal{G}[t,s]$ -Module $\mathcal{N} = \mathcal{L}[s](f^s u)$. It is to be proved that if u is a holonomic section, $\mathcal{G}f^s u$ is subholonomic and $\mathcal{G}f^s u$ is holonomic. The characterization of reduced b-function is also given.

In the sequel, $\mathcal{N}$ always denote a $\mathcal{G}[t,s]$ -Module $\mathcal{L}[s](f^s u)$ which is defined in §1 [Y]. Recall that the operation $t: P(s)(f^s u) \mapsto P(s+1)f(f^s u)$ is injective in $\mathcal{N}$.

We denote by $\mathcal{A}$ the annihilator of u . Basic concept and notations are same with S-K-K and [3]. Especially, a coherent $\mathcal{G}$ -Module is called a System.

§3. Preliminary results on Systems

We define some general concepts and collect propositions which we shall need.

Definition 1.1 For a system $\mathcal{L}$, we define

$$\text{hol}(\mathcal{L}) = \begin{cases} \dim X - \text{codim } \check{\text{SS}}(\mathcal{L}) & , \mathcal{L} \neq 0, \\ -\infty & , \mathcal{L} = 0. \end{cases}$$

Note that $\text{Ext}_{\mathcal{G}}^i(\mathcal{L}, \mathcal{G}) = 0$ for $i < \dim X - \text{hol}(\mathcal{L})$.

Definition 1.12 1. Let $\varphi: Y \rightarrow X$ be a holomorphic map and let $\mathcal{L}$ be a system on X . We define the induced Module of $\mathcal{L}$ on Y by

$$\varphi^* \mathcal{L} = \mathcal{S}_{Y \rightarrow X} \bigotimes_{\mathcal{S}_X} \mathcal{L}.$$

2. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be systems on X_1 and X_2 , respectively. The product Module of $\mathcal{L}_1$ and $\mathcal{L}_2$ on $X_1 \times X_2$ is defined by:

$$\mathcal{L}_1 \hat{\otimes} \mathcal{L}_2 = \mathcal{S}_{X_1 \times X_2} \bigotimes_{\mathcal{S}_{X_1} \otimes \mathcal{S}_{X_2}} (\mathcal{L}_1 \otimes_{\mathbb{C}} \mathcal{L}_2).$$

3. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be systems on X . The product Module of them on X is defined to be

$$\mathcal{L}_1 \boxtimes \mathcal{L}_2 = \Delta^* (\mathcal{L}_1 \hat{\otimes} \mathcal{L}_2),$$

where $\Delta: X \rightarrow X \times X$ is a diagonal embedding.

For the Definition 1.12, 1. and 2. and the following Theorem, we refer the reader to S-K-K and M. Kashiwara [3], [].

Theorem 1.13 1. Assume that for $V \subset P^*Y$, the map induced from the canonical projection is proper.

$$\varphi^{-1}(V) \cap \omega^{-1}(\text{SS}(\mathcal{L})) \rightarrow V.$$

Then, $\varphi^* \mathcal{L}$ is a system on Y and the following isomorphism holds

$$\varphi^* \mathbb{R} \text{Hom}_{\mathcal{G}_X}(\mathcal{L}, \mathcal{G}_X)[\dim X] \simeq \mathbb{R} \text{Hom}_{\mathcal{G}_Y}(\varphi^* \mathcal{L}, \mathcal{G}_Y)[\dim Y].$$

2. $\mathcal{L}_1 \hat{\otimes} \mathcal{L}_2$ is always a system and $(\mathcal{L}_1, \mathcal{L}_2) \mapsto \mathcal{L}_1 \hat{\otimes} \mathcal{L}_2$ is an exact functor.
3. If $\check{\text{SS}}(\mathcal{L}_1) \cap \check{\text{SS}}(\mathcal{L}_2) \subset X$, $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ is a system.

Statement 3 is derived from 1 and 2 easily.

Proposition 1.14 Upon the conditions in Theorem 1.13 ,

1. $\text{hol}(\varphi^* \mathcal{L}) \leq \text{hol}(\mathcal{L})$
2. $\text{hol}(\mathcal{L}_1 \hat{\otimes} \mathcal{L}_2) = \text{hol}(\mathcal{L}_1) + \text{hol}(\mathcal{L}_2),$
3. $\text{hol}(\mathcal{L}_1 \boxtimes \mathcal{L}_2) \leq \text{hol}(\mathcal{L}_1) + \text{hol}(\mathcal{L}_2).$

Since this is an easy Corollary of Theorem 1.13, we omit the proof.

We note that Prof. Bernstein considered above theorems under a little different situation in [4]. The notation $\boxtimes$ is borrowed from it.

4. Holonomicity and subholonomicity of some Modules

In this section, we study the structure of $\mathcal{G}[s]f^s u$ and $\mathcal{G}f^s u$ when $\mathcal{G}u$ is holonomic.

We define the Modules $\mathcal{N}_\alpha$ and $\mathcal{S}f^\alpha u$ for $\alpha \in \mathbb{C}$ as follows.

Definition 1.15

$$\mathcal{N}_\alpha = \mathcal{N} / (s-\alpha)\mathcal{N}.$$

We use the notation

$$\mathcal{J}(\alpha) = \{ P \in \mathcal{L} \mid P=Q(\alpha) \text{ for some } Q(s) \in \mathcal{J}(s) \}.$$

Then $\mathcal{N}_\alpha$ is isomorphic to $\mathcal{S} / \mathcal{J}(\alpha)$. Let $v \in \mathcal{N}$. Then, $v \bmod (s-\alpha)\mathcal{N}$ is denoted by $(v)_\alpha$. Especially, $(f^s u)_\alpha$ is the class $1 \bmod \mathcal{J}(\alpha)$.

We define

$$\mathcal{J}_\alpha = \{ P \in \mathcal{S} \mid f^m P(x, D + \frac{\alpha}{f} df) \in \mathcal{J} \text{ for some } m. \}$$

Consider the Module $\mathcal{S} / \mathcal{J}_\alpha$ and denote $1 \bmod \mathcal{J}_\alpha$ by $f^\alpha u$. Thus $\mathcal{S} f^\alpha u = \mathcal{S} / \mathcal{J}_\alpha$.

We also define

$$\mathcal{J}^{(0)} = \mathcal{J}(s) \cap \mathcal{D}.$$

The following inclusions hold.

$$\mathcal{J}^{(0)} \subset \mathcal{J}(\alpha) \subset \mathcal{J}_\alpha.$$

Proposition 1.16 Ideals $\mathcal{I}^{(0)}$, and $\mathcal{I}_\alpha$ are coherent.

Proof. The proof relies on the following theorem of M. Kashiwara.

"Let $\mathcal{I}$ be an ideal of $\mathcal{D}$ with filtration:
 $\mathcal{I} = \bigcup_m \mathcal{I}_m$, $\mathcal{D}^{(h)} \mathcal{I}_m \subset \mathcal{I}_{m+h}$. In order to be coherent for $\mathcal{I}$ over $\mathcal{D}$, it is necessary and sufficient that each $\mathcal{I}_m$ is coherent over $\mathcal{O}$."

From this, the coherency of $\mathcal{I}^{(0)}$ and $\mathcal{I}_\alpha$ follows.

Q.E.D.

Thus we have three systems with canonical surjections.

$$\mathcal{D}_f^s u \rightarrow \mathcal{N}_\alpha \rightarrow 0, \quad \mathcal{N}_\alpha \rightarrow \mathcal{D}_f^\alpha u \rightarrow 0.$$

We study (sub-)holonomicity of these Modules in the following.

Theorem 1.17 $\mathcal{L}^s u$ is subholonomic, when $\mathcal{L}u$ is holonomic.

Proof. Since $\mathcal{L}u$ is holonomic, M. Kashiwara's theorem in [5] says that $\check{SS}(\mathcal{L}u) \subset \bigcup T_{X_j}^* X$ for some Whitney stratification $X = \bigcup X_j$. We first prove

Lemma 1.13 $\mathcal{L}f^s \boxtimes \mathcal{L}u$ is subholonomic outside $f^{-1}(0)$.

Proof. It is sufficient to show $\check{SS}(\mathcal{L}f^s) \cap \check{SS}(\mathcal{L}u) \subset X$ outside $f^{-1}(0)$ by 3. of Theorem 1.14. We refine the stratification, if necessary, such that each X_j is contained in $f^{-1}(0)$ or disjoint to it. Assume that there exists (x_0, ξ_0) which has the following properties: $x_0 \in f^{-1}(0)$, and there is an analytic path $x(t)$ in some X_j such that $x(0) = x_0$, $(x(t), \xi(t)) \in W$ for $t > 0$ and $\lim_{t \rightarrow 0} \xi(t) = \xi_0$. Since the tangent of the curve $x = x(t)$ is $(\dot{x}_1(t), \dots, \dot{x}_n(t))$, we have

$$0 = \sum_{i=1}^n \dot{x}_i(t) \frac{f_i}{f} = \frac{d}{dt} f(x(t))$$

from the definition of W . Therefore, the path $x = x(t)$ is included in $f^{-1}(0)$, and so is X_j . q.e.d.

Owing to the canonical surjection and injection

$$\mathcal{L}f^s \boxtimes \mathcal{L}u \leftarrow \mathcal{L}(f^s \boxtimes u) \rightarrow \mathcal{L}f^s u \rightarrow 0,$$

$\mathcal{L}f^s u$ is subholonomic outside $f^{-1}(0)$. We use the

argument in of [3]. Take the subholonomic part of $\mathcal{D}f^s u$ and denote it by $\mathcal{L}$. Lemma 1.18 shows that the support of the Module $\mathcal{D}f^s u / \mathcal{L} = \overline{\mathcal{D}f^s u}$ is contained in $f^{-1}(0)$. Therefore, considering the coherent $\mathcal{O}$ -Module $\overline{\mathcal{O}f^s u}$, we have a natural number k such that $f^k \cdot f^s u \in \mathcal{L}$. Since $\mathcal{D}f^k \cdot f^s u$ and $\mathcal{D}f^s u$ are isomorphic, the subholonomicity of $\mathcal{D}f^s u$ is derived from that of $\mathcal{D}f^k \cdot f^s u$. Q.E.D.

We note that the holonomicity of $\mathcal{D}f^s u / \mathcal{D}f^k \cdot f^s u$ is an easy consequence of the above theorem and injectivity of t , considering multiplicity of each Module in the following exact sequence along irreducible components of $\check{SS}(\mathcal{D}f^s u)$.

$$0 \rightarrow \mathcal{D}f^s u \xrightarrow{t^k} \mathcal{D}f^s u \rightarrow \mathcal{D}f^s u / \mathcal{D}f^k \cdot f^s u \rightarrow 0.$$

When f is quasi-homogeneous, $\pi \cong \mathcal{D}/f^{(0)}$ and hence subholonomic. Thus $b_\pi(s)$ exists, by Proposition 1.5 in [Y]. In the general cases, we use the technique of adding a parameter. Define $f'(t, x) = tf(x)$. Then $\pi' \cong \mathcal{D}_{\mathbb{C} \times X} / f'^{(0)}$ and hence there exists $b'(s)$ and $Q(t, x, D_t, D_x)$ such that

$$Q(t, x, D_t, D_x) f'^{s+1}_u = b'(s) f'^s_u.$$

Let $Q_0(tD_t, x, D_x)D_t = \sum_j a_j(x, D_x)(tD_t)^j \cdot D_t$ be the homogeneous part of degree -1 in t of Q . Then, defining P by

$$P(s, x, D) = Q_0(s, x, D),$$

we have

$$P(s, x, D) f^{s+1}_u = \frac{b'(s)}{s+1} f^s_u$$

Thus b -function always exists. We denote by R the set of roots of the equation $b(s)=0$.

Theorem 1.20 $\mathcal{D} f^\alpha u$ is holonomic, when $\mathcal{D} u$ is holonomic.

Proof) As in the proof of Theorem 1.17, one can see that

$\mathcal{D} f^k(f^\alpha u)$ is holonomic for sufficiently large k . Then the following diagram proves the holonomicity of $\mathcal{D} f^\alpha u$.

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{D} f^k \cdot f^s u & \hookrightarrow & \mathcal{D} f^s u & \rightarrow & \mathcal{D} f^s u / \mathcal{D} f^k \cdot f^s u \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathcal{D} f^k \cdot f^\alpha u & \hookrightarrow & \mathcal{D} f^\alpha u & \rightarrow & \mathcal{D} f^\alpha u / \mathcal{D} f^k \cdot f^\alpha u \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Theorem 1.21 $\mathcal{N}_\alpha \cong \mathcal{B}^{\text{fin}}_\alpha$ if and only if $\alpha \notin \mathbb{R} + \mathbb{N}$.

Proof. Let $P \in \mathcal{J}_\alpha$, $\text{ord } P = m$. Then, there is $Q(s) \in \mathcal{G}[s]$ such that

$$Pf^m \equiv (s+m-1)Q(s, x, D) \pmod{\mathcal{J}(s)}. \quad (40)$$

To prove (40), we prepare

Lemma 1.22 For any $R \in \mathcal{L}$, $\text{ord } R = m$, the following equality holds for some $S(s, x, D) \in \mathcal{L}[s]$ with $\text{ord } S < m$.

$$\left\{ R(x, D + s \frac{f'}{f} df) - R(x, D + \gamma \frac{f'}{f} df) \right\} f^m = (s - \gamma) S(s, x, D + s \frac{f'}{f} df).$$

Proof. The proof is carried out by induction on m . When $m=0$, the result is trivial. Let $m \geq 1$. For the simplicity of the notation, we explain the case of one variable. General case is similar.

By the hypothesis of induction,

$$\left\{ (D + s \frac{f'}{f})^{m-1} - (D + \gamma \frac{f'}{f})^{m-1} \right\} f^{m-1} = (s - \gamma) Q_{m-2}(s, x, D + s \frac{f'}{f}).$$

Then,

$$\begin{aligned} & \left\{ (D + s \frac{f'}{f})^m - (D + \gamma \frac{f'}{f})^m \right\} f^m \\ &= \left\{ (D + s \frac{f'}{f})^{m-1} - (D + \gamma \frac{f'}{f})^{m-1} \right\} (D + s \frac{f'}{f}) f^m + (s - \gamma) (D + \gamma \frac{f'}{f})^{m-1} f, f^{m-1} \\ &= (s - \gamma) Q_{m-2}(s, x, D + s \frac{f'}{f}) (f(D + s \frac{f'}{f}) + m f') + (s - \gamma) Q'_{m-1}(s, x, D + s \frac{f'}{f}) f' \end{aligned}$$

where

$$Q'_{m-1}(s, x, D) = (fD + (\gamma - s + 1)f')(fD + (\gamma - s + 2)f') \dots (fD + (\gamma - s + m - 1)f').$$

This yields the case m.

q.e.d.

We apply this lemma for $R = P$, $\gamma = \lambda - m$. Then, we have

$$P(x, D + \frac{s}{f} df) f^m - f^m P(x, D + \frac{\alpha}{f} df) = (s + m - \alpha) Q(s, x, D + \frac{s}{f} df),$$

which proves (4c).

Lemma 1.23 $t^m \mathcal{N} \cap (s + m - \lambda) \mathcal{N} \subset (s + m - \lambda) t^m \mathcal{N}$.

Because of the condition on λ , $(s + m - \lambda)$ is not a factor of $b_{\mathcal{N}, m}(s)^*$. Hence we have an isomorphism

$$\mathcal{N} / t^m \mathcal{N} \xrightarrow{s + m - \lambda} \mathcal{N} / t^m \mathcal{N}. \quad (41)$$

Now take an element $v = (s + m - \lambda)w \in t^m \mathcal{N} \cap (s + m - \lambda) \mathcal{N}$.

If we consider $w \bmod t^m \mathcal{N}$ in the left-hand side of (41), it turns out to be 0 in the right-hand side. Hence $w \in t^m \mathcal{N}$, that is $v \in (s + m - \lambda) t^m \mathcal{N}$. q.e.d.

Owing to this lemma and (40), we obtain

$$Pf^m \equiv (s + m - \lambda) Q'(s, x, D) f^m \bmod f(s),$$

Note that $R + \mathcal{N} = (\bar{R} + \mathcal{N}) \cup C$. 25

and hence

$$P \equiv (s-\alpha)Q'(s-\alpha, x, D) \pmod{f(s)},$$

which proves the "if part".

("only if" part) Suppose $\alpha \in R + \mathbb{N}$. Then, $\exists \nu > 0$, such that $b_{n, \nu}(\alpha - \nu) = 0$. It follows from the definition of $b_{n, \nu}(s)$ that there exists $P_\nu(s) \in \mathcal{S}[s]$, such that

$$P_\nu(s+\nu)f^\nu \equiv b_{n, \nu}(s) \pmod{f(s)}.$$

Therefore $P_\nu(\alpha) \in \mathcal{I}_\alpha$. If $\mathcal{N}_\alpha \simeq \mathcal{S}f^\alpha$ were valid, we should have

$$P_\nu(\alpha) = Q(s) + (s-\alpha)R(s), \quad \exists Q(s) \in \mathcal{I}(s).$$

Then, if we set $R_\nu(s) = (P_\nu(s) - P_\nu(\alpha))/(s-\alpha) + R(s)$,

$$R_\nu(s+\nu)f^\nu \equiv b_{n, \nu}(s)/(s+\nu-\alpha) \pmod{f(s)}$$

This contradicts the minimality of $b_{n, \nu}(s)$. Q.E.D.

There is a canonical map

$$\tau : \mathcal{N}_{\alpha+1} \rightarrow \mathcal{N}_\alpha, \quad (\hat{f}^a u)_{\alpha+1} \mapsto \hat{f} \cdot (\hat{f}^a u)_\alpha.$$

As a map between $\mathcal{S}[s]$ -Modules, this is $Q(s)\bar{1} \mapsto Q(s+1)f \cdot \bar{1}$.

Since $(f(s+1) + \mathcal{S}[s](s-\alpha))f \subset f(s) + \mathcal{S}[s](s-\alpha)$,

this map is well-defined. There is also a map $\mathcal{S}f^{\alpha+1}\bar{u} \rightarrow \mathcal{S}f\bar{u}$,

defined by $f^{\alpha+1}_u \rightarrow f \cdot f^\alpha_u$. These maps are compatible with the surjection (35).

Theorem 1.24 i) $\mathcal{N}_{\alpha+1} \simeq \mathcal{N}_\alpha$ if and only if $\alpha \notin R$.

ii) When $\alpha \notin R$, $\mathcal{G}f^{\alpha+1}_u \simeq \mathcal{G}f^\alpha_u$.

Proof) We first prove "if part" of i) and ii). If $\alpha \notin R_f$, we can define the map $\rho: \mathcal{N}_\alpha \rightarrow \mathcal{N}_{\alpha+1}$, by $\bar{1} \mapsto b(\alpha)^{-1}P(\alpha)\bar{1}$. As a map between $\mathcal{G}[s]$ -Modules, ρ is $R(s)\bar{1} \mapsto b(\alpha)^{-1}R(s-1)P(\alpha)\bar{1}$. Then this is a well-defined homomorphism, since if $R(s) \in \mathcal{J}(s) + \mathcal{G}[s](s-\alpha)$, $R(s-1)P(\alpha)f^s_u \equiv 0 \pmod{(s-\alpha-1)\mathcal{G}[s] \cdot f^s_u}$. Similarly,

$$\begin{aligned} Q(s)fP(\alpha)f^s_u &= Q(s)f\{P(s-1) + (P(\alpha) - P(s-1))\}f^s_u \\ &\equiv Q(s)b(s-1)f^s_u \\ &\equiv b(\alpha)Q(s)f^s_u \pmod{(s-\alpha-1)\mathcal{G}[s]f^s_u}. \end{aligned}$$

Therefore, $\rho\tau(Q(s)\bar{1}) = \rho(Q(s+1)f \cdot \bar{1}) = b(\alpha)^{-1}Q(s)fP(\alpha)\bar{1} = Q(s)\bar{1}$. Analogously, $\tau\rho(R(s)\bar{1}) = R(s)\bar{1}$. Thus, ρ is the inverse of τ . The proof of ii) can be given in the same manner. ("only if" part) $\mathcal{N}_{\alpha+1} \simeq \mathcal{N}_\alpha$ implies

$$(\mathcal{J}(s+1) + \mathcal{G}[s](s-\alpha))f = \mathcal{J}(s) + \mathcal{G}[s](s-\alpha).$$

Hence, if $R(s)f^s_u = (s-\alpha)Q(s)f^s_u$, then there is $Q'(s)$ such that $R(s)f^s_u = (s-\alpha)Q'(s)f \cdot f^s_u$. Therefore, if $\alpha \in R$ were valid, the relation

$$P(s)f \cdot f^s u = (s-x) \frac{b(s)}{s-x} f^s u = (s-x)Q'(s)f \cdot f^s u$$

shows $Q'(s)f \cdot f^s u = \frac{b(s)}{s-x} f^s u$. This contradicts the minimality of $b(s)$. Q.E.D.

Corollary 1.25

i) When $x \notin \mathbb{R} + \mathbb{N}$, the following commutative diagram exists for any $k \in \mathbb{N}_0$.

When $x \notin \mathbb{R} + \mathbb{Z}$, it holds for $k = \mathbb{Z}$.

$$\begin{array}{ccc} \mathcal{N}_x & \simeq & \mathcal{N}_{x-\mathbb{R}} \\ \downarrow & & \downarrow \\ \mathcal{L}_f^x u & \simeq & \mathcal{L}_f^{x-\mathbb{R}} u \end{array}$$

$$ii) \lim_{\rightarrow} \mathcal{N}_{x-k} \cong \lim_{\rightarrow} \mathcal{L}_f^{x-k} u$$

is holonomic for $\forall x \in \mathbb{C}$.

Proof) i) is ³direct consequence of Theorems 1.24 and 1.21.

ii) follows from i).

§5. Reduced b-functions

We can realize a reduced b-function as a b-function of some $\mathcal{G}[t,s]$ -Modules. The characterization of these Modules are also given. We are indebted to M.Sato[2] for basic ideas in this section.

Definition 1.26

$$\mathcal{N}_{\#} = \bigcup_{\nu \geq 0} [\bar{b}(s-\nu)]_{\nu} t^{\nu} \mathcal{N},$$

$$\mathcal{N}^{\#} = \left\{ v(s) \in \bigcup_{\nu \geq 0} \mathcal{G}[s] t^{\nu} \mathcal{N} \mid \exists m, [\bar{b}(s)]_m v(s) \in t^m \mathcal{N}_{\#} \right\}.$$

Proposition 1.27

i) $\mathcal{N}_{\#}$ and $\mathcal{N}^{\#}$ are $\mathcal{G}[t,s]$ -Modules. If $\mathcal{N}$ is coherent, $\mathcal{N}_{\#}$ is also coherent.

$$\text{ii) } b_{\mathcal{N}_{\#}} = b_{\mathcal{N}^{\#}} = \bar{b}.$$

Proof) i) $\mathcal{N}_{\#}$ and $\mathcal{N}^{\#}$ are easily seen to be $\mathcal{G}[t,s]$ -Modules. To see the coherency of $\mathcal{N}_{\#}$, we use the operators $P_{\nu}(s)$ which satisfy

$$P_{\nu}(s+\nu) f^{\nu} \equiv b_{\mathcal{N},\nu}(s) \pmod{\mathcal{G}(s)}.$$

Since $\mathcal{G}[s]$ is a noetherian ring, there is $m \in \mathbb{N}$ such that

$$P_m(s) + A_1(s)P_{m-1}(s) + \dots + A_m(s)P_0(s) = 0.$$

for some $A_\nu(s) \in \mathcal{L}[s]$. Since

$$P_h(s)P_n(s+n) \equiv c(s)P_{h+n}(s+n) \pmod{f(s+n)},$$

multiplying $P_n(s+n)$ from the right, cancelling $c(s)$ and rewriting s to $s-n$, we have

$$P_{m+n}(s) + A_1(s-n)P_{m+n-1}(s) + \dots + A_m(s-n)P_n(s) \equiv 0 \pmod{f(s)}.$$

Therefore, $\mathcal{N}_\# = \bigcup_{\nu=0}^{m-1} [\bar{b}(s-\nu)]_\nu t^\nu \mathcal{N} \subset t^{-m+1} \mathcal{N}$.

ii) Obviously, $[\bar{b}(s)]_\nu \mathcal{N}_\# \subset t^\nu \mathcal{N}_\#$. Set $\bar{b}_\nu(s) = b_{\mathcal{N}_\#}(s)$. If $\bar{b}_\nu(s) \neq [\bar{b}(s)]_\nu$ for some ν , there is $k' < k = \deg \bar{b}(s)$, such that $\deg \bar{b}_\nu(s) < \nu k'$ for $\nu \gg 0$. But the following diagram shows $\nu k \leq (\nu+m-1)k'$. This is a contradiction. Thus we proved $b_{\mathcal{N}_\#} = \bar{b}$.

$$\begin{array}{ccc} & 0 & \\ & \downarrow & \\ \mathcal{N} / t^{\nu+m-1} \mathcal{N}_\# & \rightarrow & \mathcal{N} / t^\nu \mathcal{N} \rightarrow 0 \\ & \downarrow & \\ \mathcal{N}_\# / t^{\nu+m-1} \mathcal{N}_\# & & \end{array}$$

It follows from the definition of $\mathcal{N}^\#$ that $\mathcal{N}^\# \supset \mathcal{N}_\#$ and $\bar{b}(s)\mathcal{N}^\# \subset t\mathcal{N}^\#$. Set $\bar{\bar{b}}(s) = b_{\mathcal{N}^\#}(s)$ and assume $\bar{\bar{b}}(s) \neq \bar{b}(s)$. Then for $v(s) \in \mathcal{N}_\#$, $\bar{\bar{b}}(s)v(s) \in t\mathcal{N}^\#$ yields $[\bar{b}(s)]_m \bar{\bar{b}}(s-1)v(s-1) \in t^m \mathcal{N}_\#$.

This relation is equivalent to $\bar{\bar{b}}(s)[b(s+1)]_m v(s) \in t^{m+1} \mathcal{N}_\#$.

Since $\mathcal{N}_\#$ is finitely generated over $\mathcal{L}[s]$, we see that

$b_{\mathcal{N}_\#,m}(s)$ is a strict divisor of $[\bar{b}(s)]_m$ for sufficiently large m . That is a contradiction. Hence $b_{\mathcal{N}^\#}(s) = \bar{\bar{b}}(s)$. Q.E.D.

It is not for certain whether $\pi^\#$ is coherent or not when π is coherent. We have, however, the following characterization.

Theorem 1.28 Let π' be a $\mathbb{Z}[t,s]$ -Module satisfying,

$t^k \pi' \supset \pi' \supset \pi$ for some k . Then $b_{\pi'}(s) = \bar{b}(s)$ if and only if

$$\pi^\# \supset \pi' \supset \pi_\#.$$

Proof) ("only if" part) Since $b_{\pi'}(s) = \bar{b}(s)$, we have relations

$$\pi' \supset \bar{b}(s-1)t^{-1}\pi' \supset \bar{b}(s-1)\bar{b}(s-2)t^{-2}\pi' \supset [\bar{b}(s-h)]_h t^{-h}\pi'.$$

Therefore,

$$\pi' \supset \bigcup_{h \geq 0} [\bar{b}(s-h)]_h t^{-h}\pi' \supset \bigcup_{h \geq 0} [\bar{b}(s-h)]_h t^{-h}\pi = \pi_\#.$$

Then the following diagram

$$\pi' \supset \pi_\# \supset t^m \pi' \supset t^m \pi_\# \quad (46)$$

shows that $d_{\pi'/t^m \pi_\#}(s)$ divides both $d'(s)[\bar{b}(s)]_m$ and $[\bar{b}(s)]_m d'(s+m)$ (where we set $d'(s) = d_{\pi'/\pi_\#}(s)$), and hence one of $[\bar{b}(s)]_m$ for $m \gg 0$. But $[\bar{b}(s)]_m$ is the best possible for the pair $\pi_\# \supset t^m \pi_\#$. Therefore, $d_{\pi'/t^m \pi_\#}(s) = [\bar{b}(s)]_m$. Thus the definition of π proves $\pi^\# \supset \pi'$.

("if" part) Consider the following diagram for $m \gg 0$.

$$\pi^\# \supset \pi' \supset \pi_\# \supset t^m \pi' \supset t^m \pi_\# .$$

Then the definition of $\pi^\#$ implies $[\bar{b}_{\pi'}(s)]_m c_{\pi'}(s+m) \mid [\bar{b}(s)]_m$.

On the other hand, equation (17) of Theorem 1.8 shows

$$\bar{b}_{\pi'}(s) = (c'(s)/c'(s+1))\bar{b}(s). \text{ From these formulae,}$$

we have $c' = c_{\pi'} = 1$, and then $b_{\pi'}(s) = \bar{b}(s)$. Q.E.D.

Corollary 1.29 Assume that $w(\bar{b}) = 1$ in addition to the condition on π' in Theorem 1.28. Then,

$$b_{\pi'} = \bar{b}, \text{ if and only if } \pi' = \pi_\#$$

Proof) The "if" part is trivial. Consider the diagram (46) when $b_{\pi'} = \bar{b}$. $d_{\pi'/t^m \pi_\#}(s) = [\bar{b}(s)]_m$ is shown in the proof of Theorem 1.28. Therefore, $d'(s) = d_{\pi'/\pi_\#}(s)$ and $d'(s+m)$ are divisors of $[\bar{b}(s)]_m$ for large m . Since $w(\bar{b}) = 1$, this is actually possible only when $d'(s) = 1$, that is, $\pi' = \pi_\#$. Q.E.D.