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ON PSEUDO-MERIDIANS OF THE TREFOIL KNOT GROUP

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1. INTRODUCTION

Let $G(K)$ be the knot group of a knot K. We call a word $w\in G(K)$ a pseudo-meridian if $G(K)$ is normally generated by w, that is, $G(K)/\langle w\rangle$ is trivial where $\langle w\rangle$ is the normal closure of w in $G(K)$. For example, a meridian of each knot group is a pseudo-meridian. Moreover, the image of a meridian under any automorphism of $G(K)$ is also a pseudo-meridian.

Silver-Whitten-Williams showed in [2] that the knot group $G(K)$ contains infinitely many non-equiavalent pseudo-meridians if K is a non-trivial two bridge knot or a torus knot, or a hyperbolic knot with unknotting number one. Furthermore, they conjectured that every knot group has infinitely many non-equivalent pseudo-meridians.

In this short note, we will consider the trefoil knot 3_1 and determine which word of $G(3_1)$ is a pseudo-meridian up to a certain word length.

2. CRITERION

First, we fix the following presentation of the knot group of the trefoil:

$$G(3_1)=\langle x,y\,|\,xyx=yxy\rangle.$$

The generators x and y are meridians. Under this presentation, we investigate which word of $G(3_1)$ is a pseudo-meridian.

If x or y can be written as a product of conjugates of a word w and the inverse $\bar{w}$ in $G(3_1)$, then x and y belong to the normal closure $\langle w\rangle$. Therefore w is a pseudo-meridian. For example, $xx\bar{y}$ is a pseudo-meridian, since

$$x(xx\bar{y})\bar{x}\cdot\bar{y}(xx\bar{y})y\cdot\bar{x}\overline{(xx\bar{y})}x=xxx\bar{y}\bar{x}\bar{y}xy\bar{x}=xxx\bar{x}\bar{y}\bar{x}xy\bar{x}=x.$$

Here $\bar{z}$ is the inverse of z.

On the other hand, if the exponent sum of a word w is neither 1 nor -1, then x and y cannot be written as a product of conjugates of w and $\bar{w}$ in $G(3_1)$. Hence w is not a pseudo-meridian. In addition, the follewing is a useful criterion to show that a word is not a pseudo-meridian.

Lemma 2.1. *Let w be a word of $G(3_1)$. If there exists a non-trivial representation ρ : $G(3_1)\to SL(2;\mathbb{Z}/p\mathbb{Z})$ such that $\rho(w)$ is the identity matrix, then w is not a pseudo-meridian.*

Proof. By the assumption that $\rho(w)$ is the identity matrix, ρ factors through $G(3_1)/\langle w\rangle$. Namely, ρ induces a representation

$$\bar{\rho}:G(3_1)/\langle w\rangle\longrightarrow SL(2;\mathbb{Z}/p\mathbb{Z}).$$

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	pseudo-meridians
1	x, y
3	$xx\bar y, \bar x yy$
5	$xxx\bar y\bar y, xxy\bar x\bar y, xx\bar y x\bar y, xx\bar y\bar x y, xyx\bar y\bar y, xy\bar x\bar x y, xyy\bar x\bar y, x\bar y\bar x yy, \bar x\bar x yyy, \bar x y\bar x yy$
7	$xxxx\bar y\bar x\bar y, xxxx\bar y\bar y\bar y, xxxy\bar x\bar x\bar y, xxxy\bar x\bar y\bar y, xxx\bar y x\bar y\bar y, xxx\bar y\bar x\bar x y, xxx\bar y\bar y x\bar y, xxx\bar y\bar y\bar x y,$ $xxyx\bar y\bar x\bar y, xxyx\bar y\bar y\bar y, xxy\bar x y\bar x\bar y, xxy\bar x\bar y x\bar y, xxy\bar x\bar y\bar x y, xxyy\bar x\bar x\bar y, xxyy\bar x\bar y\bar y, xx\bar y xx\bar y\bar y,$ $xx\bar y xy\bar x\bar y, xx\bar y x\bar y x\bar y, xx\bar y x\bar y\bar x y, xx\bar y\bar x\bar x yy, xx\bar y\bar x yx\bar y, xx\bar y\bar x y\bar x y, xx\bar y\bar x\bar y xy, xx\bar y\bar y\bar x yy,$ $xx\bar y\bar y\bar y xy, xyxy\bar x\bar x\bar y, xyxy\bar x\bar y\bar y, xyx\bar y x\bar y\bar y, xyx\bar y\bar x\bar x y, xyx\bar y\bar y x\bar y, xyx\bar y\bar y\bar x y, xy\bar x\bar x\bar x yy,$ $xy\bar x\bar x y\bar x y, xy\bar x y\bar x\bar x y, xy\bar x yy\bar x\bar y, xy\bar x\bar y\bar x yy, xyyx\bar y\bar x\bar y, xyy\bar x\bar x\bar x y, xyy\bar x y\bar x\bar y, xyy\bar x\bar y x\bar y,$ $xyy\bar x\bar y\bar x y, xyyy\bar x\bar x\bar y, xyyy\bar x\bar y\bar y, x\bar y x\bar y\bar x yy, x\bar y\bar x\bar x yyy, x\bar y\bar x y\bar x yy, x\bar y\bar x yy\bar x y, x\bar y\bar y\bar x yyy,$ $\bar x\bar x\bar x yyyy, \bar x\bar x y\bar x yyy, \bar x\bar x yy\bar x yy, \bar x\bar x yyy\bar x y, \bar x y\bar x y\bar x yy, \bar x yyyy\bar x\bar y$
	non-pseudo-meridians
7	$xxy\bar x\bar x\bar x y, xy\bar x\bar x yx\bar y, xy\bar x yx\bar y\bar y, xyyx\bar y\bar y\bar y$

TABLE 1. pseudo-meridians and non-pseudo-meridians

Since ρ is a non-trivial representation, $\rho(x)$ and $\rho(y)$ are not the identity matrix and then $\bar\rho(x), \bar\rho(y)$ are not the identity matrix too. Therefore $\bar\rho$ is also a non-trivial representation and $G(3_1)/\langle w\rangle$ is not trivial. This completes the proof. □

For example, there exists a non-trivial representation

$$\rho : G(3_1) \longrightarrow SL(2; \mathbb{Z}/5\mathbb{Z})$$

defined by

$$\rho(x) = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}.$$

It is easy to see that $\rho(xxy\bar x\bar x\bar x y)$ is the identity matrix. Then $xxy\bar x\bar x\bar x y$ is not a pseudo-meridian, though the exponent sum is 1.

3. Main result

By using the method shown in Section 2, we obtain Table 1 which shows pseudo-meridians and non-pseudo-meridians up to word length 7. The first column on Table 1 is word length.

All words whose exponential sum are not ± 1 are not pseudo-meridians and then we enumerate only the words whose exponential sum are ± 1. If a word is a pseudo-meridian, then the cyclic words and the inverses are also pseudo-meridians. For instance, $xx\bar y$ is a pseudo-meridian and then $x\bar y x, \bar y xx, y\bar x\bar x, \bar x\bar x y, \bar x y\bar x$ are so. The converse statement is also true. Therefore one of them is listed in Table 1. Besides, $xxyx\bar y\bar x\bar y$ is same as x for example. However, both of them are listed.

4. Problems

In Section 3, we determined which words of $G(3_1)$ up to the word length 7. Next, we want to consider the following.

Problem 4.1. *Determine which word of $G(3_1)$ is a pseudo-meridian under the fixed presentation.*

In this note, we deal only with the trefoil. However, we would like to consider all knot groups.

Problem 4.2. *Characterize the words of pseudo-meridians for given knot groups. In other words, find a useful criterion to determine whether a word is a pseudo-meridian or not.*

References

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