

HYPOELLIPTICITY OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we study the hypoellipticity problems for fully nonlinear partial differential equations of order m . For a solution $u \in C_{loc}^p(\Omega)$, if the linearized operator for the nonlinear equation on u satisfies some subelliptic conditions, we can deduce $u \in C^\infty(\Omega)$ by using the paradifferential operator theory of J.-M. Bony.

§0 Introduction

Let us consider the following equation:

$$F[u] = F(x, u(x), \dots, \partial^\beta u(x), \dots)_{|\beta| \leq m} = 0 \quad (0.1)$$

where $x \in \Omega$, $\Omega \subset \mathbb{R}^n$ open, F is a real C^∞ function.

If $u \in C_{loc}^p(\Omega)$ ($p > m$) is a real solution for the equ-

ation (0.1), we define an associate linearized operator:

$$P(x, D) = \sum_{|\alpha| \geq 2m-p} a_\alpha(x) \partial_x^\alpha \quad (0.2)$$

where $a_\alpha(x) = \frac{\partial F}{\partial u_\alpha}(x, u(x), \dots, \partial^\beta u(x), \dots) \in C_{loc}^{p-m}(\Omega)$. Its symbol

$$p(x, \xi) = \sum_{|\alpha| \geq 2m-p} a_\alpha(x) (i\xi)^\alpha. \quad (0.3)$$

Then we obtain the following main theorem.

Theorem 0.1. Suppose that $u \in C_{loc}^p(\Omega)$ is a real solution of equation (0.1), $0 \leq m' \leq m$, $0 \leq \delta < \frac{1}{2}$ and $p > m+1 + \frac{1}{1-2\delta}(m-m')$, and the symbol defined by (0.3) satisfies:

$H_1)$ $\forall K \subset \subset \Omega$, $\exists R > 0$, $C_1 > 0$, $C_2 > 0$, s.t.

$$C_1 |\xi|^{m'} \leq |p(x, \xi)| \leq C_2 |\xi|^m,$$

$$\forall x \in K, \xi \in \mathbb{R}^n, |\xi| \geq R.$$

$H_2)$ $\forall K \subset \subset \Omega$, $\forall \alpha, \beta \in \mathbb{N}^n$, $|\beta| < p-m$, $\exists R > 0$, $C_{\alpha, \beta, K} > 0$ s.t.

$$|\partial_x^\beta \partial_\xi^\alpha p(x, \xi)| \leq C_{\alpha, \beta, K} |p(x, \xi)| |\xi|^{-|\alpha| + \delta |\beta|},$$

$$\forall x \in K, \xi \in \mathbb{R}^n, |\xi| \geq R.$$

Then $u \in C^\infty(\Omega)$.

§1 Nonhomogeneous symbolic calculus

First we recall that for any constants $0 < \varepsilon_1 < \varepsilon_2 < 1$, $R > 0$, there exists a function $\psi(\eta, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, such that $\psi = 0$ if $|\eta| \geq \varepsilon_2 |\xi|$, $\psi = 1$ if $|\eta| \leq \varepsilon_1 |\xi|$ and $|\xi| \geq R$, and for any $\alpha, \beta \in \mathbb{N}^n$, there exists a constant $C_{\alpha, \beta} > 0$, such that

$$|\partial_\eta^\alpha \partial_\xi^\beta \psi(\eta, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{-|\alpha| - |\beta|}. \quad (1.1)$$

Definition 1.1 For constants $0 \leq s < 1$, $m \in \mathbb{R}$, $r > 0$ ($r \notin \mathbb{Z}$), we define the symbol space

$$\Sigma_{r, s}^m = \left\{ p(x, \xi) \left| \begin{array}{l} \text{defined on } \mathbb{R}^n \times \mathbb{R}^n, \ C^\infty \text{ in } \xi, \ C^r \text{ in } x; \\ \forall \alpha, \beta \in \mathbb{N}^n, \ |\alpha| < r, \ \exists C_{\alpha, \beta} > 0, \text{ s.t.} \\ |\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - |\beta| + s|\alpha|}, \ \forall \xi \in \mathbb{R}^n. \end{array} \right. \right\}.$$

$$\sigma_p(x, \xi) = (2\pi)^{-n} \int e^{i\eta x} \psi(\eta, \xi) \hat{p}(\eta, \xi) d\eta.$$

where $\psi(\eta, \xi)$ is a cut-off function in (1.1).

We have the following properties for the symbol class defined above.

Proposition 1.2 If $p(x, \xi) \in \Sigma_{r, s}^m$, then $\sigma_p(x, \xi) \in S_{1, 1}^m$.

Proposition 1.3 (Composition of symbols)

Let $p \in \Sigma_{r, s}^m$, $q \in \Sigma_{r, s}^{m'}$ ($r > 1$), then

$$(1) \quad p \# q = \sum_{|\alpha| < r-1} \frac{1}{\alpha!} \partial_\xi^\alpha p(x, \xi) \mathcal{D}_x^\alpha q(x, \xi) \in \Sigma_{r-[r]+1, s}^{m+m'}.$$

$$(2) \quad \sigma_{p \# q} - \sigma_p \# \sigma_q \in S_{1, 1}^{m+m' - (1-2s)[r]}, \text{ where}$$

$$\sigma_p \# \sigma_q = \sum_{|\alpha| < r-1} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_p(x, \xi) \mathcal{D}_x^\alpha \sigma_q(x, \xi).$$

§2 The proof of Theorem 0.1

By the para-linearization process, one can find the following para-linearization theorem in [6, 12].

Theorem 2.1 Let $u \in C_{loc}^p(\Omega) \cap H_{loc}^s(\Omega)$, $p > m$, $s > 0$, be a

real solution for the equation (0.1), $P(x, D) \in Op(\Sigma_{p-m}^m(\Omega))$ is the paradifferential operator whose symbol $\sigma(P) = p(x, \xi)$ is defined by (0.3). Then there exists a function $f \in C_{loc}^{2p-2m}(\Omega) \cap H_{loc}^{s+p-2m}(\Omega)$ such that $Pu = f$.

Theorem 2.2 Let $u \in C_{loc}^p(\Omega) \cap H_{loc}^s(\Omega)$ ($s > 0$) satisfy the assumptions of Theorem 0.1, then there exists a constant $\varepsilon > 0$ (independent of s), such that $u \in C_{loc}^p(\Omega) \cap H_{loc}^{s+\varepsilon}(\Omega)$.

The proof of Theorem 0.1:

From the fact that $u(x) \in C_{loc}^p(\Omega)$ and the assumptions of Theorem 0.1, we can deduce $u(x) \in H_{loc}^{m+1}(\Omega)$. From Theorem 2.2, we know $u(x) \in C_{loc}^p(\Omega) \cap H_{loc}^{m+1+\varepsilon}(\Omega)$. By induction, repeating the process k times, we can obtain $u(x) \in C_{loc}^p(\Omega) \cap H_{loc}^{m+1+k\varepsilon}(\Omega)$. This implies $u(x) \in C_{loc}^p(\Omega) \cap \bigcap_{k=1}^{+\infty} H_{loc}^{m+1+k\varepsilon}(\Omega)$. Finally by the Sobolev embedding theorem, we know $\bigcap_{k=1}^{+\infty} H_{loc}^{m+1+k\varepsilon}(\Omega) = C^\infty(\Omega)$. Thus we have proved $u(x) \in C^\infty(\Omega)$.

§3 An example

Let us consider

$$F[u] = (\log u)^{16} \left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \right) - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - 2u(\log u)^{16} + 2u = 0,$$

where $(x, y) \in \Omega \subset \mathbb{R}^2$, $(0, 0) \in \Omega$.

Conclusion: Suppose $u(x, y) \in C_{loc}^p(\Omega)$, $u(x, y) > 0$, $p > 9$ is

a real solution for the equation above, then $u(x, y) \in C^\infty(\Omega)$.

In fact, $u(x, y) = e^{x+y}$ is a solution for the equation, but at $(0, 0)$, the linearized operator is degenerately elliptic.

References

- [1] Boutet de Monvel, L., Hypoelliptic operators with double characteristics and related pseudodifferential operators, *Comm. Pure Appl. Math.*, 27 (1974), 585-639.
- [2] 陈恕行, 仇庆久, 李成章, 伪微分算子引论, 科学出版社, 1990.
- [3] Chin-Hung Ching, Pseudodifferential operators with nonregular symbols, *J. Diff. Eqs.*, 11, 436-447, 1972.
- [4] Friedrichs, K. O., On the differentiability of solutions of linear elliptic differential equations, *Comm. Pure Appl. Math.*, Vol. 6 (1953), 299-326.
- [5] Hörmander, L., Hypoelliptic second order differential equations, *Acta Math.*, 119 (1967), 147-171.
- [6] J.-M. Bony, Calcul symbolique et propagation des

- des singularités pour les équations aux dérivées partielles non linéaires, *Ann. Sci. Ec. Norm. Sup.*, 14 (1981), 209-246.
- [7] J. Marschall, Pseudo-differential operators with nonregular symbols of the class $S_{r,s}^m$, *Comm. in P.D.E.*, 12 (8), 921-965 (1987).
- [8] 齐民友, 线性偏微分算子引论 (上册), 科学出版社, 1986.
- [9] 齐民友, 徐超江, 线性偏微分算子引论 (下册), 科学出版社, 1992.
- [10] Treves, F., A new method of proof of the subelliptic estimates, *Comm. Pure Appl. Math.*, 24 (1971), 71-115.
- [11] Treves, F., Hypoelliptic partial differential equations of principal type. Sufficient conditions and necessary conditions, *Comm. Pure Appl. Math.*, 24 (1971), 631-670.
- [12] Xu Chao-Jiang, Hypoellipticity of nonlinear second order partial differential equations, *Journal of P.D.E.*, (1) 1988.