

Solutions of Ginzburg-Landau type systems with Higher-dimensional Zero Sets

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1 Introduction

In this paper, we consider the following elliptic system of diagonal type:

$$\Delta V + \lambda(1 - |V|^2)V = 0 \quad (1)$$

where $V = (V^1, \dots, V^n)$ is defined on some domain in $\mathbf{R}^{n+k}$, $n \geq 2$, $k \geq 1$, and $\lambda \in \mathbf{R}$ is a parameter.

Here, we construct some solutions of (1) on certain domains in $\mathbf{R}^{n+k}$, with boundary values, invariant under the action of a k -parameter group of isometries of $\mathbf{R}^{n+k}$, and having nontrivial k -dimensional zero sets.

When $n = 2$, the equation (1) is the Ginzburg-Landau system (GLS), which is used as a mathematical model for many physical phenomena, such as super-conductivity and super-fluidity. In the theory of super-conductivity, the unknown V represents an order parameter which has two degrees of freedom, and its zero set, called *vortices*, corresponds to the region of the normal state in super-conductors. So, especially our result produces an example of solutions of the GLS in $\mathbf{R}^3$ with curved vortex lines.

Some results concerning the isolated zeros of solutions of the GLS in $\mathbf{R}^2$ are known ([1], [2]), however there seems to be no explicit example of solutions with higher-dimensional nontrivial zero sets.

Our proof is based on the “equivariant construction” method due to N. Smale [9], in which the examples of minimal hypersurfaces in Euclidean spaces with higher-dimensional *singularities* are shown. Later, the same method was used to construct examples with higher-dimensional singularities, of harmonic maps [4], and of solutions of a certain nonlinear elliptic equation [6].

Main result of this paper can be extended to equations with other type of nonlinearities, but we do not pursue here for simplicity of description.

2 Notations and statement of the main result

We follow the setting of “equivariant construction” method described in the papers [9], [4] and [6]: Let $n \geq 2$, $k \geq 1$ be two integers. Let $\mathcal{U} \subset \mathbf{R}^k$ be an open set containing $\{0\} \in \mathbf{R}^k$ and assume that there is a C^∞ group action

$$\Phi : t \in \mathcal{U} \longrightarrow \Phi(t) \in \text{Isom}(\mathbf{R}^{n+k}),$$

here $\text{Isom}(\mathbf{R}^{n+k})$ means the group of isometries of $\mathbf{R}^{n+k}$. We will denote $\Phi(t)$ by G_t .

We define

$$\begin{aligned}\Gamma &= \{G_t(0) : t \in \mathcal{U}\}, \\ \tilde{\mathbf{B}}^n &= B_1^n(0) \times \{0\}_k = \{\tilde{x} = (x, 0) \in \mathbf{R}^n \times \mathbf{R}^k, |x| < 1\}, \\ \Omega &= \{G_t(\tilde{\mathbf{B}}^n) : t \in \mathcal{U}\}.\end{aligned}$$

So, Γ is the orbit of $\{0\} \in \mathbf{R}^{n+k}$ of the group action Φ , and Ω is the unit n -disc bundle over Γ obtained by moving $\tilde{\mathbf{B}}^n$ along Γ by G_t , $t \in \mathcal{U}$. On the group action Φ , we make the following assumptions: Γ is a properly embedded k -dimensional submanifold in $\mathbf{R}^{n+k}$ and whenever $G_t(0) = 0$, we must have $G_t(\tilde{\mathbf{B}}^n) = \tilde{\mathbf{B}}^n$ for any $t \in \mathcal{U}$, that is, the isotropy group of 0 is the same as the one of $\tilde{\mathbf{B}}^n$. Furthermore, when $G_t = O(t) + v_t$ is the decomposition of the element of $\text{Isom}(\mathbf{R}^{n+k})$, where $O(t) \in O(n+k)$, the orthogonal group of $\mathbf{R}^{n+k}$, and $v_t \in \mathbf{R}^{n+k}$, we define the group action

$$\Phi_\epsilon : t \in \mathcal{U} \longmapsto G_t^\epsilon \in \text{Isom}(\mathbf{R}^{n+k}),$$

and

$$\begin{aligned}\Gamma_\epsilon &= \{G_t^\epsilon(0) : t \in \mathcal{U}\} = \left(\frac{1}{\epsilon}\right) \Gamma, \\ \Omega_\epsilon &= \{G_t^\epsilon(\tilde{\mathbf{B}}^n) : t \in \mathcal{U}\},\end{aligned}$$

where $G_t^\epsilon = O(t) + \frac{1}{\epsilon}v_t$. Note that under the assumption of the group action Φ , Ω_ϵ is well-defined and then Ω_ϵ is the unit n -disc bundle over Γ_ϵ obtained by moving $\tilde{\mathbf{B}}^n$ along Γ_ϵ by G_t^ϵ , $t \in \mathcal{U}$. Note also that when $\epsilon > 0$ is sufficiently small, Ω_ϵ is close locally the trivial product bundle $B_1^n(0) \times \mathbf{R}^k$ over $\{0\}_n \times \mathbf{R}^k$. Finally, for a map $U : \mathbf{R}^{n+k} \rightarrow \mathbf{R}^n$, we denote by $\Gamma(U)$ the set of zeros of U , namely, $\Gamma(U) = \{x : U(x) = 0 \in \mathbf{R}^n\}$.

Now we state the main result of this paper.

Theorem *For any $\lambda \in \mathbf{R}$, there exists an open domain $\tilde{\Omega} \subset \mathbf{R}^{n+k}$ containing Γ , on which there are infinitely many solutions of (1) with boundary values, whose zero set is Γ .*

In the proof of the theorem, we will show that there exists $\bar{\epsilon} > 0$ sufficiently small, such that for any $0 < \epsilon < \bar{\epsilon}$, there is a solution U of

$$\Delta U + \lambda \epsilon^2(1 - |U|^2)U = 0 \quad \text{in } \Omega_\epsilon, \quad (2)$$

$$\Gamma(U) = \Gamma_\epsilon \quad (3)$$

with a boundary data fixed up to a finite dimensional space, and U is invariant under the action Φ_ϵ , i.e., $U(G_t^\epsilon(\tilde{x})) = U(\tilde{x})$ for all $\tilde{x} \in \tilde{\mathbf{B}}^n$ and $t \in \mathcal{U}$.

We will find a solution U of (2) by solving the appropriate fixed point problem. We make essential use of the invariant condition of U , thanks to which, we can think of (2) as a PDE on each fibers of the disc bundle Ω_ϵ , especially on $\tilde{\mathbf{B}}^n$ for $t = 0$. Note that the nonlinear term of (2) is well controlled when ϵ is small enough, so we can get a solution as a perturbation of the $\mathbf{R}^n$ -valued harmonic function $v_0 : B_1^n(0) \rightarrow \mathbf{R}^n$, $v_0(x) = x$. Taking $\tilde{\Omega} = \epsilon \cdot \Omega_\epsilon$, and $V(y) = U\left(\frac{y}{\epsilon}\right)$ for $y \in \tilde{\Omega}$ will give the desired result. The domain $\tilde{\Omega}$ so obtained, is the bundle over Γ of the n -dimensional discs of radius ϵ , so looks like locally a thin perturbed tube of radius ϵ with center axis Γ .

Now we describe our coordination of Ω_ϵ : For $y \in \Omega_\epsilon$, there exists $x \in B_1^n(0)$ and $t \in \mathcal{U}$ such that $y = G_t^\epsilon(\tilde{x})$, then let us denote $F : B_1^n(0) \times \mathcal{U} \rightarrow \Omega_\epsilon$, $F(x, t) = G_t^\epsilon(\tilde{x})$. we will introduce the local coordinate system by this map, and identify y with (r, θ, t) where (r, θ) are polar coordinates for $x \in B_1^n(0)$. So, functions defined on Ω_ϵ can naturally be

considered as functions on $B_1^n(0) \times \mathcal{U}$ by F . Note for $\varepsilon > 0$ sufficiently small, r also is the distance to Γ_ε .

In the sequel we use the following function spaces: For $\nu \in \mathbf{R}, \alpha \in (0, 1), m = 0, 1, 2$, define

$$C^{m, \alpha, \nu}(\Omega_\varepsilon; \mathbf{R}^n) = \{u \in C_{loc}^{m, \alpha}(\Omega_\varepsilon \setminus \Gamma_\varepsilon; \mathbf{R}^n) : |u|_{m, \alpha, \nu} < +\infty\},$$

where $|\cdot|_{m, \alpha, \nu}$ is the norm

$$|u|_{m, \alpha, \nu} = \sup_{0 < s \leq 1/2} \left(\sum_{j=0}^m |\nabla^j u|_{[0, [s, 2s]]} s^{j-\nu} + \sum_{j=0}^m |\nabla^j u|_{(\alpha), [s, 2s]} s^{j+\alpha-\nu} \right).$$

Here, ∇ and ∇^2 denote the gradient and Hessian respectively on Ω_ε , and $|\eta|_{[0, [s, 2s]]}$ and $|\eta|_{(\alpha), [s, 2s]}$ are the sup norm and the α -th Hölder seminorm of a function (or a section) η on Ω_ε over the set $\{y = y(r, \theta, t) \in \Omega_\varepsilon : s \leq r \leq 2s\}$. These are Banach spaces under the norm $|\cdot|_{m, \alpha, \nu}$, and if $u \in C^{m, \alpha, \nu}(\Omega_\varepsilon; \mathbf{R}^n)$, then $|u|$ decays like r^ν near Γ_ε .

Furthermore, let us define the closed subspace of $C^{m, \alpha, \nu}(\Omega_\varepsilon; \mathbf{R}^n)$ as

$$C_G^{m, \alpha, \nu}(\Omega_\varepsilon; \mathbf{R}^n) = \{u \in C^{m, \alpha, \nu}(\Omega_\varepsilon; \mathbf{R}^n) : u(G_t^\varepsilon(\tilde{x})) = u(\tilde{x}) \text{ for all } x \in B_1^n(0), t \in \mathcal{U}\},$$

that is, maps in $C^{m, \alpha, \nu}$ which are Φ_ε -invariant. We also denote $C_G^{m, \alpha}(\partial\Omega_\varepsilon; \mathbf{R}^n)$ for the space of Φ_ε -invariant boundary data in $C^{m, \alpha}(\partial\Omega_\varepsilon; \mathbf{R}^n)$.

Weighted Hölder spaces like above are now widely used for other nonlinear problems, see [9], [10], [4], [6], [8], [5], [3].

3 Proof of the Theorem

In this section, we seek for a solution of (2) satisfying (3) by the same technique as in [9], [4], [6]: linearization and solving the appropriate fixed point problem. First, we construct the approximate solution. We fix $\varepsilon > 0$. Let $v_0 : B_1^n(0) \rightarrow \mathbf{R}^n$ be the identity map $v_0(x) = x$; so evidently $\Gamma(v_0) = \{0\} \in \mathbf{R}^n$ and $\Delta_{B^n} v_0 = 0$, where Δ_{B^n} means the Laplace operator on $B_1^n(0)$. Now we define the approximate solution $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathbf{R}^n$ by

$$u_\varepsilon(G_t^\varepsilon(\tilde{x})) = v_0(x) \text{ for } x \in B_1^n(0), t \in \mathcal{U}$$

where $\tilde{x} = (x, 0) \in \tilde{\mathbf{B}}^n$. By definition of Ω_ε and by our assumption on the group action Φ , u_ε is well-defined and invariant under the action Φ_ε . The zero set of u_ε satisfies $\Gamma(u_\varepsilon) = \Gamma_\varepsilon$.

We wish to find a solution of (2) of the form

$$U(u) = u_\varepsilon + u$$

where the perturbation u is assumed to be invariant under the action Φ_ε and to decay rapidly near Γ_ε , so as to ensure that $\Gamma(U(u)) = \Gamma_\varepsilon$.

Let $N(u)$ be the left hand side of (2) for $U(u)$, that is,

$$N(u) = \Delta U(u) + \lambda \varepsilon^2 (1 - |U(u)|^2) U(u).$$

We make a Taylor expansion of $N(u)$ about $u = 0$ to get

$$N(u) = N(0) + Lu + Q(u),$$

where

$$\begin{aligned}
N(0) &= \Delta u_\varepsilon + \lambda \varepsilon^2 (1 - |u_\varepsilon|^2) u_\varepsilon, \\
Lu &= \frac{d}{dt} N(tu)|_{t=0} \\
&= \Delta u + \lambda \varepsilon^2 \{ (1 - |u_\varepsilon|^2) u - 2(u_\varepsilon \cdot u) u_\varepsilon \}, \\
Q(u) &= \int_0^1 (1-t) \frac{d^2}{dt^2} N(tu) dt \\
&= (-2\lambda \varepsilon^2) \int_0^1 (1-t) \{ |u|^2 u_\varepsilon + 2(u_\varepsilon \cdot u) u + 3t|u|^2 u \} dt \\
&= (-\lambda \varepsilon^2) \{ |u|^2 u_\varepsilon + 2(u_\varepsilon \cdot u) u + |u|^2 u \},
\end{aligned}$$

here Δ means the Laplace operator on Ω_ε . Now, if we define the linear operators

$$R = \Delta - \Delta_{B^n}$$

and

$$\xi u = \lambda \varepsilon^2 \{ (1 - |u_\varepsilon|^2) u - 2(u_\varepsilon \cdot u) u_\varepsilon \},$$

then the equation $N(u) = 0$ can be rewritten as

$$\Delta_{B^n} u = -N(0) - Ru - \xi u - Q(u) \quad (4)$$

which we solve by contraction mapping argument on some weighted Hölder space. Note that if u is invariant under the action Φ_ε , all of the terms in (4) are also Φ_ε -invariant, so we can consider (4) as a PDE on the slice $\tilde{B}^n$. This is crucial for our subsequent arguments.

To estimate the terms in the right hand side of (4), we need the following lemma due to R. Mazzeo and N. Smale [5].

Lemma1 *Under the local coordination by F , we have*

$$\Delta = \Delta_{B^n} + \Delta_{R^*} + e_1 \nabla^2 + e_2 \nabla, \quad (5)$$

where Δ and ∇ are the Laplace operator and gradient on Ω_ε , $e_1 \in C^\infty((\text{Sym}^2 \Omega_\varepsilon)^*)$, $e_2 \in C^\infty(T^* \Omega_\varepsilon)$ are smooth sections and satisfy

$$\begin{aligned}
|e_1(x, t)| &\leq C_0 r \varepsilon, & |e_2(x, t)| &\leq C_0 \varepsilon, \\
|e_1|_{(\alpha), [s, 2s]} s^\alpha &\leq C_0 s \varepsilon, & |e_2|_{(\alpha), [s, 2s]} s^\alpha &\leq C_0 \varepsilon
\end{aligned}$$

for some constant C_0 independent of $\varepsilon > 0$ and $\alpha \in (0, 1)$.

For functions u invariant under Φ_ε , the factor Δ_{R^*} in (5) drops out.

Using this lemma, we have

Lemma2 *If $\varepsilon > 0$, $1 < \nu < 2$, and $u \in C_G^{2, \alpha, \nu}(\Omega_\varepsilon; \mathbf{R}^n)$, then $N(0), Ru, \xi u, Q(u)$ are all in $C_G^{0, \alpha, \nu-2}(\Omega_\varepsilon; \mathbf{R}^n)$ and the following estimates hold:*

$$\begin{aligned}
|N(0)|_{0, \alpha, \nu-2} &\leq C_1 \varepsilon (1 + |\lambda| \varepsilon), \\
|Ru|_{0, \alpha, \nu-2} &\leq C_1 \varepsilon |u|_{2, \alpha, \nu}, \\
|\xi u|_{0, \alpha, \nu-2} &\leq C_1 |\lambda| \varepsilon^2 |u|_{2, \alpha, \nu}, \\
|Q(u)|_{0, \alpha, \nu-2} &\leq C_1 |\lambda| \varepsilon^2 (|u|_{2, \alpha, \nu}^2 + |u|_{2, \alpha, \nu}^3)
\end{aligned}$$

for some constant $C_1 > 0$ independent of ε and λ .

Proof Since u_ε and u are Φ_ε -invariant, so are also all terms appeared in the right hand side of (4), and can be considered as functions of $B_1^n(0)$. By definition, the map u_ε satisfies $\Delta_{B^n} u_\varepsilon = 0$, so we have

$$\begin{aligned} N(0) &= \Delta u_\varepsilon + \lambda \varepsilon^2 (1 - |u_\varepsilon|^2) u_\varepsilon \\ &= (\Delta - \Delta_{B^n}) u_\varepsilon + \lambda \varepsilon^2 (1 - |u_\varepsilon|^2) u_\varepsilon. \end{aligned}$$

Then using Lemma 1 and the fact that $|\nabla u_\varepsilon(x)| + |\nabla^2 u_\varepsilon(x)| \leq C$ and $|u_\varepsilon(x)| \leq 1$ for some constant C independent of ε and $x \in B_1^n(0)$, we have

$$\begin{aligned} |N(0)(x)| &\leq |e_1 \nabla^2 u_\varepsilon(x)| + |e_2 \nabla u_\varepsilon(x)| + |\lambda| \varepsilon^2 (1 - |u_\varepsilon|^2) |u_\varepsilon| \\ &\leq C s \varepsilon + C \varepsilon + |\lambda| \varepsilon^2 \end{aligned}$$

for $s \leq |x| \leq 2s$. Taking the supremum over the set $\{x : s \leq |x| \leq 2s\}$ and multiplying $s^{2-\nu}$, we get

$$\begin{aligned} |N(0)|_{0,[s,2s]} s^{2-\nu} &\leq s^{2-\nu} \cdot C \varepsilon (1 + |\lambda| \varepsilon) \\ &\leq C \varepsilon (1 + |\lambda| \varepsilon), \end{aligned}$$

since $1 < \nu < 2$ and $0 < s \leq 1/2$. Hölder seminorm estimate for $N(0)$ has the same form, then by taking the supremum over $s \leq 1/2$, we have the first estimate of the lemma.

Similarly by Lemma 1,

$$Ru = (\Delta - \Delta_{B^n})u = e_1 \nabla^2 u + e_2 \nabla u,$$

so we have

$$\begin{aligned} |Ru|_{0,[s,2s]} s^{2-\nu} &\leq C s \varepsilon |\nabla^2 u|_{0,[s,2s]} s^{2-\nu} + C \varepsilon |\nabla u|_{0,[s,2s]} s^{1-\nu} \cdot s \\ &\leq C \varepsilon (|\nabla^2 u|_{0,[s,2s]} s^{2-\nu} + |\nabla u|_{0,[s,2s]} s^{1-\nu}), \end{aligned}$$

for $0 < s \leq 1/2$. Hölder seminorm estimate is also similar, then taking the supremum over $s \leq 1/2$ yields the estimate for Ru .

As for the estimates for ξu and $Q(u)$, by using the basic properties of the Hölder seminorm

$$|\mu + \eta|_{(\alpha)} \leq |\mu|_{(\alpha)} + |\eta|_{(\alpha)},$$

and

$$|\mu \eta|_{(\alpha)} \leq |\mu|_{(0)} |\eta|_{(\alpha)} + |\mu|_{(\alpha)} |\eta|_{(0)},$$

as in the above computation, we can derive the following bounds:

$$|\xi u(x)| \leq C |\lambda| \varepsilon^2 |u(x)|, \quad x \in B_1^n(0) \quad (6)$$

$$|\xi u|_{(\alpha),[s,2s]} \leq C |\lambda| \varepsilon^2 (|u|_{0,[s,2s]} + |u|_{(\alpha),[s,2s]}), \quad (7)$$

$$|Q(u)(x)| \leq C |\lambda| \varepsilon^2 (|u(x)|^2 + |u(x)|^3), \quad x \in B_1^n(0) \quad (8)$$

$$|Q(u)|_{(\alpha),[s,2s]} \leq C |\lambda| \varepsilon^2 (|u|_{0,[s,2s]} |u|_{(\alpha),[s,2s]} + |u|_{0,[s,2s]}^2 + |u|_{0,[s,2s]}^2 |u|_{(\alpha),[s,2s]}). \quad (9)$$

If we multiply both sides of (6) and (8) by $s^{2-\nu}$, or of (7) and (9) by $s^{2-\nu+\alpha}$ and take the supremum over $s \leq 1/2$, we immediately have

$$\sup_{0 < s \leq 1/2} (|\xi u|_{0,[s,2s]} s^{2-\nu} + |\xi u|_{(\alpha),[s,2s]} s^{2-\nu+\alpha}) \leq C |\lambda| \varepsilon^2 |u|_{0,\alpha,\nu}$$

$$\sup_{0 < s \leq 1/2} (|Q(u)|_{0,[s,2s]} s^{2-\nu} + |Q(u)|_{(\alpha),[s,2s]} s^{2-\nu+\alpha}) \leq C|\lambda|\varepsilon^2 (|u|_{0,\alpha,\nu}^2 + |u|_{0,\alpha,\nu}^3)$$

which complete the proof of the lemma. $\square$

Now, to find solutions of (4), we first recall the unique solvability result for the linear problem $\Delta_{B^n} u = f$ on $B_1^n(0)$, for $f \in C_G^{0,\alpha,\nu-2}(\Omega_\varepsilon; \mathbf{R}^n)$ with some appropriate boundary conditions.

Let us take the sequence of eigenvalues of $\Delta_{S^{n-1}}$ acting on $C^\infty(S^{n-1}; \mathbf{R}^n)$, μ_j , $0 = \mu_1 \leq \mu_2 \leq \dots$, (counting multiplicity), $\mu_j \rightarrow \infty$, and corresponding sequence of L^2 -normalized eigenmaps $\phi_j \in C^\infty(S^{n-1}; \mathbf{R}^n)$ such that $\Delta_{S^{n-1}} \phi_j + \mu_j \phi_j = 0$, $j = 1, 2, \dots$. Let λ_j and $\lambda_j(-)$ be two real solutions of the equation $\lambda^2 + (n-2)\lambda - \mu_j = 0$, that is

$$\lambda_j = \frac{2-n}{2} + \sqrt{\frac{(n-2)^2}{4} + \mu_j} \quad \text{and} \quad \lambda_j(-) = \frac{2-n}{2} - \sqrt{\frac{(n-2)^2}{4} + \mu_j}.$$

We now fix ν so that $1 < \nu < 2$ and choose an positive integer J such that $\lambda_J < \nu < \lambda_{J+1}$.

For this J , we define

$$\Pi_J : L^2(S^{n-1}; \mathbf{R}^n) \rightarrow \{\phi_1, \phi_2, \dots, \phi_J\}^\perp$$

be the orthogonal projection.

Then we have :

Lemma3 *If $f \in C_G^{0,\alpha,\nu-2}(\Omega_\varepsilon; \mathbf{R}^n)$ and $\psi \in C_G^{2,\alpha}(\partial\Omega_\varepsilon; \mathbf{R}^n)$ with $0 < \alpha < 1$, then there exists a unique $u \in C_G^{2,\alpha,\nu}(\Omega_\varepsilon; \mathbf{R}^n)$ such that*

$$\begin{cases} \Delta_{B^n} u &= f \quad \text{on } \Omega_\varepsilon \setminus \Gamma_\varepsilon, \\ \Pi_J(u|_{\partial\Omega_\varepsilon}) &= \Pi_J(\psi). \end{cases} \quad (10)$$

Furthermore, we have the estimate

$$|u|_{2,\alpha,\nu} \leq C_2 (|f|_{0,\alpha,\nu-2} + |\psi|_{2,\alpha})$$

for some constant C_2 depending only on α .

Proof The proof of this is done by separation of variables and now quite standard (see [3], [9], [4], [6]), so we make only few comments.

If we write

$$\begin{aligned} u(r, \theta) &= \sum_{j=1}^{\infty} u_j(r) \phi_j(\theta), \quad u_j(r) = \langle u(r, \cdot), \phi_j(\cdot) \rangle_{L^2(S^{n-1}; \mathbf{R}^n)}, \\ f(r, \theta) &= \sum_{j=1}^{\infty} f_j(r) \phi_j(\theta), \quad f_j(r) = \langle f(r, \cdot), \phi_j(\cdot) \rangle_{L^2(S^{n-1}; \mathbf{R}^n)}, \\ \psi(\theta) &= \sum_{j=1}^{\infty} \psi_j \phi_j(\theta), \quad \psi_j = \langle \psi, \phi_j \rangle_{L^2(S^{n-1}; \mathbf{R}^n)}, \end{aligned}$$

then each u_j must be the solution of the following ODE with boundary conditions:

$$\begin{cases} a''(r) + \frac{n-1}{r} a'(r) - \frac{\mu_j}{r^2} = f_j(r), \\ a(1) = \psi_j \quad \text{for } j > J, \\ |a(r)| \leq Cr^\nu. \end{cases}$$

By elementary ODE argument, Caffarelli, Hardt and Simon [3] showed that

$$\begin{aligned} u_j(r) &= r^{\lambda_j} \int_0^r s^{1-n-2\lambda_j} \int_0^s \tau^{n-1+\lambda_j} f_j(\tau) d\tau ds, \quad (j = 1, 2, \dots, J) \\ u_j(r) &= \psi_j r^{\lambda_j} - r^{\lambda_j} \int_r^1 s^{1-n-2\lambda_j} \int_0^s \tau^{n-1+\lambda_j} f_j(\tau) d\tau ds, \quad (j \geq J+1) \end{aligned}$$

are the unique solutions. Thus the map $\sum_{j=1}^{\infty} u_j \phi_j$ formally solves the equation $\Delta_{B^n} u = f$ on $B_1^n(0)$ with $\Pi_J(u|_{\partial\Omega_\epsilon}) = \Pi_J(\psi)$, and in fact C^2 classical sense on $B_1^n(0) \setminus \{0\}$.

To prove the estimate, note that we are dealing with the system of PDE, but in the same situation this was done in [4] using the local supremum estimates of [8] and the standard Schauder estimates in [7]. $\square$

We now apply Lemma2 and Lemma3 to find fixed points of (4). Fix $\alpha \in (0, 1)$ and $\nu \in (1, 2)$ as before. For $K > 0$ and $\varepsilon > 0$, let us define

$$B_{K\varepsilon, \alpha, \nu} = \left\{ u \in C_G^{2, \alpha, \nu}(\Omega_\epsilon; \mathbf{R}^n) : |u|_{2, \alpha, \nu} \leq K\varepsilon \right\}.$$

Then we prove

Lemma4 *For any $\lambda \in \mathbf{R}$, there exists $K > 0$ and $0 < \bar{\varepsilon} < 1$ such that if $\varepsilon < \bar{\varepsilon}$, $v \in B_{K\varepsilon, \alpha, \nu}$ and $\psi \in C_G^{2, \alpha}(\partial\Omega_\epsilon; \mathbf{R}^n)$ satisfying $|\psi|_{2, \alpha} \leq \varepsilon$, then the problem: to find $u \in B_{K\varepsilon, \alpha, \nu}$ such that*

$$\begin{cases} \Delta_{B^n} u &= -N(0) - Rv - \xi v - Q(v) \\ \Pi_J(u|_{\partial\Omega_\epsilon}) &= \Pi_J(\psi) \end{cases} \quad (11)$$

has a unique solution.

Proof The problem above has a unique solution $u \in C_G^{2, \alpha, \nu}(\Omega_\epsilon; \mathbf{R}^n)$ by Lemma2 and Lemma3. Furthermore according to Lemma2, Lemma3 and $|v|_{2, \alpha, \nu} \leq K\varepsilon$, we have

$$\begin{aligned} |u|_{2, \alpha, \nu} &\leq C_2 (|\psi|_{2, \alpha} + |N(0)|_{0, \alpha, \nu-2} + |Rv|_{0, \alpha, \nu-2} + |\xi v|_{0, \alpha, \nu-2} + |Q(v)|_{0, \alpha, \nu-2}) \\ &\leq C_2 (\varepsilon + C_1 \varepsilon (1 + |\lambda| \varepsilon) + C_1 \varepsilon \cdot K\varepsilon + |\lambda| \varepsilon^2 \cdot K\varepsilon + |\lambda| \varepsilon^2 (K^2 \varepsilon^2 + K^3 \varepsilon^3)) \\ &\leq C_3 (\varepsilon + |\lambda| \varepsilon^2 + K\varepsilon^2 + |\lambda| \varepsilon^2 (K\varepsilon + K^2 \varepsilon^2 + K^3 \varepsilon^3)) \end{aligned}$$

for some constant $C_3 > 0$.

So, if we can take K and ε such that

$$C_3 \left(\frac{1 + |\lambda| \varepsilon}{K} + \varepsilon + |\lambda| \varepsilon^2 (1 + K\varepsilon + K^2 \varepsilon^2) \right) \leq 1,$$

then the proof will be completed. This can be done as follows: First, fix $K > 0$ sufficiently large so that

$$\frac{(1 + |\lambda|)}{K} < \frac{1}{2C_3},$$

and then, fix $\bar{\varepsilon} \in (0, 1)$ sufficiently small so that

$$\bar{\varepsilon} + |\lambda| \bar{\varepsilon}^2 (1 + K\bar{\varepsilon} + K^2 \bar{\varepsilon}^2) < \frac{1}{2C_3}.$$

$\square$

Now, fix $\psi \in C_G^{2,\alpha}(\partial\Omega_\epsilon; \mathbf{R}^n)$ so that $|\psi|_{2,\alpha} \leq \epsilon (< \bar{\epsilon})$. Let us denote $T(v)$ the unique solution of (11) for $v \in B_{K\epsilon, \alpha, \nu}$. Then, by Lemma4, T defines a self-map of $B_{K\epsilon, \alpha, \nu}$. To show that T is indeed a contraction, we need

Lemma5 *There is a constant $C_4 > 0$ independent of $u, v \in C_G^{2,\alpha,\nu}(\Omega_\epsilon; \mathbf{R}^n)$, ϵ and λ such that*

$$|Q(u) - Q(v)|_{0, \alpha, \nu-2} \leq C_4 |\lambda| \epsilon^2 |u - v|_{2, \alpha, \nu} \left[|u|_{2, \alpha, \nu} + |v|_{2, \alpha, \nu} + (|u|_{2, \alpha, \nu} + |v|_{2, \alpha, \nu})^2 \right]$$

holds.

Proof This is obtained quite easily by elementary computation and basic property of Hölder seminorms, if we write

$$Q(u) - Q(v) = (-\lambda \epsilon^2) (I_1 + 2I_2 + I_3),$$

where

$$\begin{aligned} I_1 &= [(u - v) \cdot (u + v)] u_\epsilon, \\ I_2 &= (u_\epsilon \cdot u)(u - v) + [u_\epsilon \cdot (u - v)] v, \\ I_3 &= |u|^2(u - v) + [(u - v) \cdot (u + v)] v. \end{aligned}$$

□

Let $u_1 = T(v_1)$ and $u_2 = T(v_2)$ are the unique solution of (11) for fixed ψ , given by Lemma4. Then by Lemma3, Lemma2 and Lemma5, we have

$$\begin{aligned} &|T(v_1) - T(v_2)|_{2, \alpha, \nu} \\ &\leq C_2 |R(v_1 - v_2)|_{0, \alpha, \nu-2} + C_2 |\xi(v_1 - v_2)|_{0, \alpha, \nu-2} + C_2 |Q(v_1) - Q(v_2)|_{0, \alpha, \nu-2} \\ &\leq C_2 C_1 \epsilon |v_1 - v_2|_{2, \alpha, \nu} + C_2 C_1 |\lambda| \epsilon^2 |v_1 - v_2|_{2, \alpha, \nu} \\ &+ C_2 C_4 |\lambda| \epsilon^2 |v_1 - v_2|_{2, \alpha, \nu} \left[|v_1|_{2, \alpha, \nu} + |v_2|_{2, \alpha, \nu} + (|v_1|_{2, \alpha, \nu} + |v_2|_{2, \alpha, \nu})^2 \right] \\ &\leq C_5 [\epsilon + |\lambda| \epsilon^2 + |\lambda| \epsilon^2 (K\epsilon + K^2 \epsilon^2)] |v_1 - v_2|_{2, \alpha, \nu}. \end{aligned}$$

So if we retake $\bar{\epsilon}$ small enough so that $C_5 [\bar{\epsilon} + |\lambda| \bar{\epsilon}^2 + |\lambda| \bar{\epsilon}^2 (K\bar{\epsilon} + K^2 \bar{\epsilon}^2)] < 1$, the map T defines a contraction on the closed subset of a complete metric space, then it has a fixed point u . Thus we have found a map $U = u + u_\epsilon$ satisfying (2), at least in $\Omega_\epsilon \setminus \Gamma_\epsilon$.

Note that when $1 < \nu < 2$, we can extend $u \in C_G^{2,\alpha,\nu}(\Omega_\epsilon; \mathbf{R}^n)$ (as thought of a map defined on $B_1^n(0) \setminus \{0\}$) to $0 \in B_1^n(0)$ so that $u(0) = 0$ and $|\nabla u(0)| = 0$, then the map U is indeed a smooth solution of (2) on each fibers of Ω_ϵ . Moreover if we require $\bar{\epsilon}$ small enough such that $K\bar{\epsilon} \leq 1/2$, then $|U(x)| \geq |u_\epsilon(x)| - |u(x)| \geq (1/2)|x|$ for any $x \in B_1^n(0)$, so $\Gamma(U) = \Gamma(u_\epsilon) = \Gamma_\epsilon$. As noted earlier, simple rescaling by a factor of ϵ completes the proof of Theorem. □

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