

L^p estimates for some Schrödinger type operators

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Abstract

We consider the Schrödinger operator $L = -\Delta + V$ with non-negative potentials V on $\mathbf{R}^n$, $n \geq 3$. We assume that the potential V belongs to the reverse Hölder class which includes non-negative polynomials. We show the L^p estimates for the operators $V^k L^{-k}$ and $V^{k-1/2} \nabla L^{-k}$, where k is a positive integer.

1 Introduction

In this paper we consider the Schrödinger operator $L = -\Delta + V$ on $\mathbf{R}^n$, $V \geq 0$, $n \geq 3$. When V is a non-negative polynomial, Zhong ([Zh]) proved that the operators $V^k L^{-k}$ and $V^{k-1/2} \nabla L^{-k}$, $k \in \mathbf{N}$, are bounded on L^p , $1 < p \leq \infty$. For the potential V which belongs to the reverse Hölder class, which includes non-negative polynomials, Shen ([Sh]) generalized Zhong's results. Actually, he proved that the operators $V L^{-1}$ and $V^{1/2} \nabla L^{-1}$ are bounded on $L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$.

For the operator L with potentials V which belong to the reverse Hölder class, Kurata and the author generalized Shen's results as follows. In [KS1], we replace Δ by the second order uniformly elliptic operator $L_0 = -\sum_{i,j=1}^n (\partial/\partial x_i) \{a_{ij}(x) (\partial/\partial x_j)\}$ and assume certain assumptions for a_{ij} . Then we showed that the operators $V(L_0 + V)^{-1}$ and $V^{1/2} \nabla (L_0 + V)^{-1}$ are bounded on weighted L^p space ($1 < p < \infty$) and Morrey spaces. Moreover, in [Su], the author showed weighted L^p - L^q estimates of the operators $V^\alpha L^{-\beta}$ and $V^\alpha \nabla L^{-\beta}$ ($\alpha, \beta \in (0, 1]$) and their boundedness on Morrey spaces.

The purpose of this paper is to show the L^p boundedness of the operators $V^k L^{-k}$ and $V^{k-1/2} \nabla L^{-k}$, $k \in \mathbf{N}$, where V belongs to the reverse Hölder class.

We shall repeat the definitions of the reverse Hölder class (e.g. [Sh]). Throughout this paper we denote by $B_r(x)$ the ball centered at x with radius r , and the letter C stands for a constant not necessarily the same at each occurrence.

Definition 1 (Reverse Hölder class) *Let $V \geq 0$.*

(1) *For $1 < p < \infty$ we say $V \in (RH)_p$, if $V \in L^p_{loc}(\mathbf{R}^n)$ and there exists a constant C such that*

$$\left(\frac{1}{|B_r(x)|} \int_{B_r(x)} V(y)^p dy \right)^{1/p} \leq \frac{C}{|B_r(x)|} \int_{B_r(x)} V(y) dy \quad (1)$$

holds for every $x \in \mathbf{R}^n$ and $0 < r < \infty$.

(2) *We say $V \in (RH)_\infty$, if $V \in L^\infty_{loc}(\mathbf{R}^n)$ and there exists a constant C such that*

$$\|V\|_{L^\infty(B_r(x))} \leq \frac{C}{|B_r(x)|} \int_{B_r(x)} V(y) dy \quad (2)$$

holds for every $x \in \mathbf{R}^n$ and $0 < r < \infty$.

Remark 1 If $P(x)$ is a polynomial and $\alpha > 0$, then $V(x) = |P(x)|^\alpha$ belongs to $(RH)_\infty$ ([Fe]). For $1 < p < \infty$, it is easy to see $(RH)_\infty \subset (RH)_p$.

In [Zh], Zhong proved the L^p estimates of the operators $V^k L^{-k}$ and $V^{k-1/2} \nabla L^{-k}$ with non-negative polynomials V by using the k times composition of the Hardy-Littlewood maximal operator M . In [KS1] we considered the uniformly elliptic operators L_0 and proved a pointwise bound $|Tf(x)| \leq CM(|f|)(x)$ where Mf is Hardy-Littlewood maximal function and T is either $V(L_0 + V)^{-1}$ or $V^{1/2} \nabla (L_0 + V)^{-1}$. Pointwise estimates are also used by Zhong in the polynomial case. Once we have these pointwise estimates the boundedness of these operators in any spaces on which the Hardy-Littlewood maximal operator is known to be bounded. Examples are weighted L^p space and Morrey spaces.

In this paper we establish pointwise estimates (see Lemma 3) which generalize Zhong's estimates we mentioned above. By using them we show the L^p boundedness of these operators (see Theorem 1).

We denote by $\Gamma(x, y)$ the fundamental solution for L . The operator L^{-1} is the integral operator with $\Gamma(x, y)$ as its kernel. Let $f \in C_0^\infty(\mathbf{R}^n)$. Then we have $L^{-1}f \in L^p(\mathbf{R}^n)$ for $1 \leq p \leq \infty$. For any integer $k \geq 2$, we define L^{-k} as follows.

$$L^{-k}f(x) = \int_{\mathbf{R}^n} \Gamma(x, y) L^{-(k-1)}f(y) dy.$$

Now we state our theorem.

Theorem 1 *Suppose $V \in (RH)_\infty$. Then there exist constants C, C' such that*

$$\|V^k L^{-k}f\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)} \quad \text{for } f \in C_0^\infty(\mathbf{R}^n), \quad (3)$$

$$\|V^{k-1/2}\nabla L^{-k}f\|_{L^p(\mathbf{R}^n)} \leq C'\|f\|_{L^p(\mathbf{R}^n)} \quad \text{for } f \in C_0^\infty(\mathbf{R}^n), \quad (4)$$

where $1 < p \leq \infty$ and $k \in \mathbf{N}$.

Remark 2 In Theorem 1 the case $k = 1$ was shown in [Sh, Remark 2.9, Theorem 4.13].

The plan of this paper is as follows. In section 2, we recall Shen's lemmas which we use to prove Theorem 1. In section 3, we prove Theorem 1.

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2 Preliminaries

In [Sh], Shen defined the auxiliary function $m(x, V)$ and established the estimates of the fundamental solution of L (see Lemma 1). By using the estimates he proved L^p boundedness of the operators VL^{-1} and $V^{1/2}\nabla L^{-1}$. We also need them to prove our theorem.

We recall the definition of the function $m(x, V)$.

Definition 2 ([Sh, Definition 1.3]) *Let $V \in (RH)_{n/2}$ and $V \not\equiv 0$. Then it is well-known that there exists $\epsilon > 0$ such that $V \in (RH)_{n/2+\epsilon}$ ([Ge]). Then the function $m(x, V)$ is well-defined by*

$$\frac{1}{m(x, V)} = \sup \left\{ r > 0 : \frac{r^2}{|B_r(x)|} \int_{B_r(x)} V(y) dy \leq 1 \right\}$$

and satisfies $0 < m(x, V) < \infty$ for every $x \in \mathbf{R}^n$.

Remark 3 If $V \in (RH)_\infty$ then there exists a constant C such that $V(x) \leq Cm(x, V)^2$ ([Sh, Remark 2.9]).

We recall the estimates of the fundamental solution for L .

Lemma 1 ([Sh])

(1) *Suppose $V \in (RH)_{n/2}$. Then for any positive integer N there exists a constant C_N such that*

$$(0 \leq) \Gamma(x, y) \leq \frac{C_N}{\{1 + m(x, V)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-2}}.$$

(2) Suppose $V \in (RH)_n$. Then for any positive integer N there exists a constant C_N such that

$$|\nabla_x \Gamma(x, y)| \leq \frac{C_N}{\{1 + m(x, V)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-1}}.$$

The following Lemma is also needed to prove our theorem.

Lemma 2 ([Sh, Lemma 1.4(c)]) Suppose $V \in (RH)_{n/2}$. Then there exist positive constants C, k_0 such that

$$m(y, V) \geq \frac{Cm(x, V)}{\{1 + m(x, V)|x - y|\}^{k_0/(k_0+1)}}.$$

3 Proof

Theorem 1 is easily proved by the following pointwise estimates. These estimates generalize the results in [Zh, Lemma 3.2] to the Schrödinger operators with reverse Hölder class potentials.

Lemma 3 Let k be a positive integer. The operator M^k stands for the k times composition of the Hardy-Littlewood maximal operator M .

(1) Suppose $V \in (RH)_{n/2}$. Then there exist a constant C such that

$$|m(x, V)^{2k} L^{-k} f(x)| \leq CM^k(|f|)(x) \quad \text{for } f \in C_0^\infty(\mathbf{R}^n). \quad (5)$$

(2) Suppose $V \in (RH)_n$. Then there exist a constant C such that

$$|m(x, V)^{2k-1} \nabla L^{-k} f(x)| \leq CM^k(|f|)(x) \quad \text{for } f \in C_0^\infty(\mathbf{R}^n). \quad (6)$$

Remark 4 In Lemma 3 the case $k = 1$ was shown in [KS, Theorem 1.3].

Proof of Theorem 1. Since $V(x) \leq Cm(x, V)^2$, estimate (3) immediately follows from (5) and the fact that the Hardy-Littlewood maximal operator is bounded on $L^p(\mathbf{R}^n)$, $1 < p \leq \infty$. The proof of (4) can be done in the same way as above by using (6). $\square$

Proof of Lemma 3. Let $f \in C_0^\infty(\mathbf{R}^n)$. We prove estimate (5) by induction on k . For the proof of the case $k = 1$, see [KS1, Theorem 1.3]. We assume it is true for $k = l$, that is, there exists a constant C such that

$$|m(x, V)^{2l} L^{-l} f(x)| \leq CM^l(|f|)(x) \quad (7)$$

and show the case $k = l + 1$. It follows from Lemma 1 (1) and Lemma 2 that

$$\begin{aligned} & |m(x, V)^{2(l+1)} L^{-(l+1)} f(x)| \\ & \leq \left| C m(x, V)^2 \int_{\mathbf{R}^n} \Gamma(x, y) m(x, V)^{2l} L^{-l} f(y) dy \right| \\ & \leq CC_N m(x, V)^2 \int_{\mathbf{R}^n} \frac{\{1 + m(x, V)|x - y|\}^{2lk_0/(k_0+1)} |m(y, V)^{2l} L^{-l} f(y)|}{\{1 + m(x, V)|x - y|\}^N |x - y|^{n-2}} dy. \end{aligned}$$

Therefore we obtain the desired estimate in the same way as the case $k = 1$.

The proof of (6) can be done in the same way as the proof of (5) by using Lemma 1 (2). $\square$

Remark 5 Let $s \in (0, \infty)$. We can obtain the estimate for the operator $V^s L^{-s}$ as follows. Suppose $V \in (RH)_{n/2}$ and $\alpha \in (0, 1]$. Then there exists a constant C such that

$$|m(x, V)^{2\alpha} L^{-\alpha} f(x)| \leq CM(|f|)(x) \quad \text{for } f \in C_0^\infty(\mathbf{R}^n) \quad (8)$$

(see [Su Theorem 1]). Combining (8) and the argument in the proof of Lemma 3, we arrive at the following pointwise estimate:

$$|m(x, V)^{2s} L^{-s} f(x)| \leq CM^{s^*}(|f|)(x) \quad \text{for } f \in C_0^\infty(\mathbf{R}^n), \quad (9)$$

where $s \in (0, \infty)$ and

$$s^* = \begin{cases} s, & \text{if } s \text{ is an integer,} \\ [s] + 1, & \text{otherwise,} \end{cases}$$

where $[s]$ is the largest integer smaller than or equal to s . We should remark that, for the case V is a non-negative polynomial, Zhong proved the L^p boundedness (only for $1 < p < \infty$) of the operator $V^s L^{-s}$, $s \in (0, \infty)$ ([Zh, Corollary 1.5]).

Remark 6 Zhong also showed that the L^p estimate of the operator $V^{k-q/2} \Delta^{q/2} L^{-k}$ with non-negative polynomials V , where q and k are positive integers and $2 \leq q \leq 2k$ ([Zh, Theorem 1.3]). He proved this results by using the fact that the functions

$m(x, V)^{2k} L^{-k} f(x)$ and $m(x, V)^{2k-1} \nabla L^{-k} f(x)$ are bounded by the k times composition of the Hardy-Littlewood maximal function and there exists a constant C such that

$$|\Delta^{q/2} V(x)| \leq C m(x, V)^{q+2} \quad (10)$$

which holds for non-negative polynomials V . Hence if we assume the inequality (10), we can obtain the L^p estimate of the operator $V^{k-q/2} \Delta^{q/2} L^{-k}$ with potentials V which belong to the reverse Hölder class in the same way as for polynomial potentials by using Lemma 3 and the assumption (10).

Remark 7 Shen proved that the operator $\nabla^2 L^{-1}$ is bounded on L^p , $1 < p < \infty$ ([Sh]). In [KS1] Kurata and the author extended this result to the uniformly elliptic operators. They also showed that the estimate for the kernel of the operator $\nabla^2 L^{-1}$ ([KS2]). However, it is not known that whether the operator $V^{k-1} \nabla^2 L^{-k}$, $k \geq 2$ is bounded on L^p or not.

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