

Some starlikeness conditions concerned with the second coefficient

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Abstract

Let $\mathcal{A}$ be the class of analytic functions $f(z) = z + a_2z^2 + \cdots$ in the open unit disk $\mathbb{U}$. Some starlikeness conditions for $f(z) \in \mathcal{A}$ missing the second coefficient a_2 were given by V. Singh (Math. Math. Sci. **23** (2000), 855-857). By considering starlikeness of order α for $f(z) \in \mathcal{A}$ with $a_2 \neq 0$, some starlikeness conditions concerned with the second coefficient a_2 are discussed.

1 Introduction

Let $\mathcal{H}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n , let $\mathcal{A}_n$ be the class of functions $f(z) \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

with $\mathcal{A}_1 = \mathcal{A}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$ is denoted by $\mathcal{S}$. In 1972, Ozaki and Nunokawa [4] proved a univalence criterion for $f(z) \in \mathcal{A}$ as follows.

Lemma 1.1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 \quad (z \in \mathbb{U}),$$

then $f(z)$ is univalent in $\mathbb{U}$, which means that $f(z) \in \mathcal{S}$.

Moreover, let $\mathcal{T}_n(\mu)$ denote the class of functions $f(z) \in \mathcal{A}_n$ which satisfy the inequality

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \mu \quad (z \in \mathbb{U})$$

for some real number μ with $0 < \mu \leq 1$ and $\mathcal{T}_n(1) = \mathcal{T}_n$. The assertion in Lemma 1.1 gives us that $\mathcal{T}_n(\mu) \subset \mathcal{T}_n \subset \mathcal{S}$.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies

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$$(1.1) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

for some real number α with $0 \leq \alpha < 1$. This class is denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^*(0) = \mathcal{S}^*$. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S}$.

For a positive integer n , we define by $\mathcal{B}_n$ the class of functions $w(z) \in \mathcal{H}$ of the form

$$w(z) = \sum_{k=n}^{\infty} c_k z^k$$

which satisfy the inequality $|w(z)| < 1$ ($z \in \mathbb{U}$). The following lemma is well-known as Schwarz's lemma (see [1]).

Lemma 1.2 *If $w(z) \in \mathcal{B}_n$, then*

$$(1.2) \quad |w(z)| \leq |z|^n$$

for each point $z \in \mathbb{U}$. The equality in (1.2) is attended for $w(z) = e^{i\varphi} z^n$ ($\varphi \in \mathbb{R}$).

Applying Lemma 1.2 with $n = 2$, Singh [5] discussed starlikeness for $f(z) \in \mathcal{T}_2(\mu)$.

Lemma 1.3 *If $f(z) \in \mathcal{A}_2$ satisfies*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \frac{1}{\sqrt{2}} \quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}^*$. This means that $\mathcal{T}_2(\mu)$ is a subclass of $\mathcal{S}^*$ for $0 < \mu \leq \frac{1}{\sqrt{2}}$.

Furthermore, Kuroki, Hayami, Uyanik and Owa [3] deduced some sufficient condition for $f(z) \in \mathcal{A}_n$ to be starlike of order α in $\mathbb{U}$.

Lemma 1.4 *If $f(z) \in \mathcal{A}_n$ with $n \neq 1$ satisfies*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \frac{(n-1)(1-\alpha)}{\sqrt{(n-1+\alpha)^2 + (1-\alpha)^2}} \quad (z \in \mathbb{U})$$

for some real number α with $0 \leq \alpha < 1$, then $f(z) \in \mathcal{S}^*(\alpha)$.

In view of Lemma 1.3, Singh [5] discussed some starlikeness condition for $f(z) \in \mathcal{A}$ missing the second coefficient a_2 . In the present paper, we consider starlikeness of order α for $f(z) \in \mathcal{A}$ with $a_2 \neq 0$.

2 Main result 1

By using a certain method of the proof of Lemma 1.4 which was discussed by Kuroki, Hayami, Uyanik and Owa [3], we deduce some sufficient condition for $f(z) \in \mathcal{A}$ to be starlike of order α in $\mathbb{U}$ (see [2]).

Theorem 2.1 *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$ satisfies*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \frac{(1-\alpha)\sqrt{2(1+\alpha^2) - |a_2|^2} - (1-\alpha+2\alpha^2)|a_2|}{2(1+\alpha^2)} \quad (z \in \mathbb{U})$$

for some real number α with $0 \leq \alpha < 1$, then $f(z) \in \mathcal{S}^(\alpha)$.*

Remark 2.1 If we take $a_2 = 0$ in Theorem 2.1, then we obtain the assertion of Lemma 1.4 with $n = 2$.

Letting $\alpha = 0$ in Theorem 2.1, we obtain

Corollary 2.1 *If $f(z) = z + a_2 z^2 + \dots \in \mathcal{A}$ satisfies*

$$(2.1) \quad \left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \frac{\sqrt{2 - |a_2|^2} - |a_2|}{2} \quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}^$.*

Example 2.1 Noting that

$$\frac{\sqrt{2 - |a_2|^2} - |a_2|}{2} = \frac{1}{2} \quad \text{when} \quad a_2 = \frac{\sqrt{3} - 1}{2},$$

let us consider the function $f(z)$ given by

$$(2.2) \quad f(z) = \frac{z}{1 - \frac{\sqrt{3}-1}{2}z - \frac{1}{2}z^2} = z + \frac{\sqrt{3}-1}{2}z^2 + \frac{3-\sqrt{3}}{2}z^3 + \dots \quad (z \in \mathbb{U})$$

in Corollary 2.1. It follows from (2.2) that

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| = \left| \frac{1}{2}z^2 \right| < \frac{1}{2} \quad (z \in \mathbb{U}).$$

Thus, we find that $f(z)$ given by (2.2) satisfies the inequality (2.1) with $a_2 = \frac{\sqrt{3}-1}{2}$.

On the other hand, a simple check gives us that

$$\begin{aligned} \operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) &= \operatorname{Re} \left(\frac{1 + \frac{1}{2}z^2}{1 - \frac{\sqrt{3}-1}{2}z - \frac{1}{2}z^2} \right) \\ &> \frac{33 + (5\sqrt{3} - 3)\sqrt{12\sqrt{3} - 6}}{198} = 0.2765 \dots > 0 \quad (z \in \mathbb{U}). \end{aligned}$$

This leads that $f(z)$ given by (2.2) belongs to the class $\mathcal{S}^*$.

Furthermore, putting $\alpha = \frac{1}{2}$ in Theorem 2.1, we have

Corollary 2.2 *If $f(z) = z + a_2 z^2 + \dots \in \mathcal{A}$ satisfies*

$$(2.3) \quad \left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \frac{\sqrt{\frac{5}{2} - |a_2|^2} - 2|a_2|}{5} \quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}^ \left(\frac{1}{2} \right)$.*

Example 2.2 Noting that

$$\frac{\sqrt{\frac{5}{2} - |a_2|^2} - 2|a_2|}{5} = \frac{1}{10} \quad \text{when } a_2 = \frac{1}{2},$$

let us consider the function $f(z)$ given by

$$(2.4) \quad f(z) = \frac{z}{1 - \frac{1}{2}z - \frac{1}{10}z^2} = z + \frac{1}{2}z^2 + \frac{1}{20}z^3 + \dots \quad (z \in \mathbb{U})$$

in Corollary 2.2. It is easy to check that

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| = \left| \frac{1}{10} z^2 \right| < \frac{1}{10} \quad (z \in \mathbb{U}).$$

Then, we see that $f(z)$ given by (2.4) satisfies the inequality (2.3) with $a_2 = \frac{1}{2}$. Moreover, we can observe that

$$\begin{aligned} \operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) &= \operatorname{Re} \left(\frac{1 + \frac{1}{10} z^2}{1 - \frac{1}{2} z - \frac{1}{10} z^2} \right) \\ &> \frac{10153 + 792\sqrt{71}}{24534} = 0.6858 \dots > \frac{1}{2} \quad (z \in \mathbb{U}), \end{aligned}$$

which implies that $f(z)$ given by (2.4) belongs to the class $\mathcal{S}^* \left(\frac{1}{2} \right)$.

3 Main result 2

Suppose that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{T}_1(\mu)$. It is easy to see that

$$\frac{z^2 f'(z)}{(f(z))^2} - 1 = (a_3 - a_2^2) z^2 + 2(a_4 - 2a_2 a_3 + a_2^3) z^3 + \dots \quad (z \in \mathbb{U}).$$

If we define the function $w(z)$ by

$$(3.1) \quad w(z) = \frac{1}{\mu} \left(\frac{z^2 f'(z)}{(f(z))^2} - 1 \right) \quad (z \in \mathbb{U}),$$

then since $f(z) \in \mathcal{T}_1(\mu)$, we see that $w(z) \in \mathcal{B}_2$. On the other hand, let us consider the function $f(z)$ given by

$$(3.2) \quad f(z) = z + \sum_{k=2}^n a_2^{k-1} z^k + \sum_{k=n+1}^{\infty} a_k z^k \quad (z \in \mathbb{U}),$$

where n is positive integer with $n \neq 1$. Noting that

$$\frac{f(z)}{z} = \frac{1 - (a_2 z)^n}{1 - a_2 z} + \sum_{k=n}^{\infty} a_{k+1} z^k \quad (z \in \mathbb{U}),$$

we have

$$\begin{aligned} \frac{z}{f(z)} &= \frac{1 - a_2 z}{1 - (a_2 z)^n + (1 - a_2 z) \sum_{k=n}^{\infty} a_{k+1} z^k} \\ &= \frac{1 - a_2 z}{1 - (a_2^n - a_{n+1})z^n - \sum_{k=n+1}^{\infty} (a_2 a_k - a_{k+1})z^k} \\ &= \frac{1 - a_2 z}{1 - \sum_{k=n}^{\infty} (a_2 a_k - a_{k+1})z^k} \quad (a_n = a_2^{n-1}) \\ &= (1 - a_2 z) + (1 - a_2 z) \left\{ \sum_{k=n}^{\infty} (a_2 a_k - a_{k+1})z^k \right\} \\ &\quad + (1 - a_2 z) \left\{ \sum_{k=n}^{\infty} (a_2 a_k - a_{k+1})z^k \right\}^2 + \dots \\ &= 1 - a_2 z + (a_2^n - a_{n+1})z^n + (2a_2 a_{n+1} - a_2^{n+1} - a_{n+2})z^{n+1} \\ &\quad + (2a_2 a_{n+2} - a_2^2 a_{n+1} - a_{n+3})z^{n+2} + \dots \end{aligned}$$

for $z \in \mathbb{U}$. Therefore, we obtain that

$$\begin{aligned} \frac{z^2 f'(z)}{(f(z))^2} - 1 &= \frac{z}{f(z)} - z \left(\frac{z}{f(z)} \right)' - 1 \\ &= (n-1)(a_{n+1} - a_2^n)z^n + n(a_{n+2} - 2a_2 a_{n+1} + a_2^{n+1})z^{n+1} + \dots \end{aligned}$$

for $z \in \mathbb{U}$. This gives that $w(z)$ defined by (3.1) belongs to the class $\mathcal{B}_n$ if $f(z) \in \mathcal{T}_1(\mu)$. Hence by applying Lemma 1.2, we deduce some sufficient condition for $f(z)$ given by (3.2) to be starlike of order α in $\mathbb{U}$.

Theorem 3.1 *Let n be a positive integer with $n \neq 1$. If $f(z) = z + \sum_{k=2}^n a_2^{k-1} z^k + \sum_{k=n+1}^{\infty} a_k z^k \in \mathcal{A}$ satisfies*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \mu_n(\alpha) \quad (z \in \mathbb{U})$$

for some real number α with $0 \leq \alpha < 1$, where

$$\mu_n(\alpha) = \frac{(n-1) \left[(1-\alpha) \sqrt{A - (n-1)^2 |a_2|^2} - \left\{ A - (n-1)(n-1+\alpha) \right\} |a_2| \right]}{A} \quad \left(A = (1-\alpha)^2 + (n-1+\alpha)^2 \right),$$

then $f(z) \in \mathcal{S}^*(\alpha)$.

Remark 3.1 If we take $a_2 = 0$ in Theorem 3.1, then we obtain Lemma 1.4 proven by Kuroki, Hayami, Uyanik and Owa [3].

Remark 3.2 Setting $n = 2$ in Theorem 3.1, we find the assertion of Theorem 2.1.

Letting $n = 3$ in Theorem 3.1, we obtain

Corollary 3.1 *If $f(z) = z + a_2 z^2 + a_2^2 z^3 + \sum_{k=4}^{\infty} a_k z^k \in \mathcal{A}$ satisfies*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \mu_3(\alpha) \quad (z \in \mathbb{U})$$

for some real number α with $0 \leq \alpha < 1$, where

$$\mu_3(\alpha) = \frac{2 \left[(1-\alpha) \sqrt{(1-\alpha)^2 + (2+\alpha)^2 - 4|a_2|^2} - \left\{ (1-\alpha)^2 + \alpha(2+\alpha) \right\} |a_2| \right]}{(1-\alpha)^2 + (2+\alpha)^2},$$

then $f(z) \in \mathcal{S}^*(\alpha)$.

Furthermore, taking $\alpha = 0$ in Corollary 3.1, we get

Corollary 3.2 *If $f(z) = z + a_2 z^2 + a_2^2 z^3 + \sum_{k=4}^{\infty} a_k z^k \in \mathcal{A}$ satisfies*

$$(3.3) \quad \left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \frac{2\sqrt{5 - 4|a_2|^2} - 2|a_2|}{5} \quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}^*$.

Example 3.1 Noting that

$$\frac{2\sqrt{5-4|a_2|^2}-2|a_2|}{5} = \frac{3}{5} \quad \text{when } a_2 = \frac{1}{2},$$

let us consider the function $f(z)$ given by

$$(3.4) \quad f(z) = \frac{z}{1 - \frac{1}{2}z - \frac{3}{10}z^3} = z + \frac{1}{2}z^2 + \left(\frac{1}{2}\right)^2 z^3 + \frac{17}{40}z^4 + \cdots \quad (z \in \mathbb{U})$$

in Corollary 3.2. It follows from (3.4) that

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| = \left| \frac{3}{5}z^3 \right| < \frac{3}{5} \quad (z \in \mathbb{U}).$$

Thus, we find that $f(z)$ given by (3.4) satisfies the inequality (3.3) with $a_2 = \frac{1}{2}$. On the other hand, a simple check gives us that

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) = \operatorname{Re} \left(\frac{1 + \frac{3}{5}z^3}{1 - \frac{1}{2}z - \frac{3}{10}z^3} \right) > \frac{2}{9} > 0 \quad (z \in \mathbb{U}).$$

This leads that $f(z)$ given by (3.4) belongs to the class $\mathcal{S}^*$.

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