

Fluctuation-Induced First-Order Transition and Dynamic Scaling in Rayleigh-Bénard Convection

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The Rayleigh-Bénard (RB) convection provides a canonical example of bifurcations and patterns in nonlinear dissipative system. In the present report we study the ordering dynamics of the RB convective patterns using both numerical and analytical techniques.

A simplified model of the RB instability was introduced by Swift and Hohenberg (SH)¹ to study the critical behavior associated with fluctuation effects at the bifurcation. It is a two-dimensional theory involving a real order parameter, $\psi(\mathbf{r}, t)$, which describes the slow (spatial and temporal) variation of the vertical component of the velocity and the temperature. The variable ψ is also related to the Nusselt number, $\mathcal{N}$, in such a way that $\tilde{\mathcal{N}} \equiv \mathcal{N} - 1 = S^{-1} \int d\mathbf{r} \psi^2(\mathbf{r})$, S being the area of the system. The model is derived from the Oberbeck-Boussinesq equations in the limit of large Prandtl number, supplemented by the Langevin noise terms. It reads

$$\tau_0 \partial_t \psi(\mathbf{r}, t) = -\delta H\{\psi\} / \delta \psi(\mathbf{r}, t) + f(\mathbf{r}, t) ,$$

$$H\{\psi\} = - \int d\mathbf{r} \{ (1/2) \epsilon \psi^2 - (1/4) \tilde{g} \psi^4 - (1/2) \tilde{\xi}_0^4 [(\nabla^2 + q_0^2) \psi]^2 \} , \quad (1)$$

with a Gaussian noise satisfying $\langle f(\mathbf{r}, t) f(\mathbf{r}', t') \rangle = 2F \tau_0 \delta(t - t') \tilde{\xi}_0^2 \delta(\mathbf{r} - \mathbf{r}')$. The quantities τ_0 , $\tilde{\xi}_0$, q_0 , $\tilde{g}$ and F are constants determined for appropriate horizontal

boundary conditions, and $\epsilon \equiv (R - R_c)/R_c$ is the reduced Rayleigh number with the convective threshold R_c .

An important conclusion of the SH theory is the prediction that because of its high degree of degeneracy of the ordered state the RB convective instability belongs to the Brazovskii universality class², where a first-order transition would occur as a result of fluctuations. In ordinary physical systems, however, the strength F of noise (thermal fluctuations) is many orders of magnitude smaller than the typical macroscopic dissipative energies of the system, and the induced fluctuations are extremely weak even for sufficiently small $|\epsilon|$. Since the scale of the predicted jump (say, in $\tilde{N}$) is set by $F^{2/3}$, it is unobservably small, and in fact the prediction has never been confirmed by the experiments on fluid systems. However, in numerical simulations it is possible to realize a system with the increased effective thermal noise strength. In this manner it is expected that we can make direct observation of this fluctuation-induced first-order transition.

We have performed simulations of the SH model (1) using the cell-dynamical-system (CDS) approach. The CDS scheme provides an efficient algorithm for numerical simulations and has proved valuable in studying the late stage of phase-ordering processes such as domain growth in binary alloys³. In CDS scheme the SH equation is replaced by the following equation⁴:

$$\psi(n, t+1) = A \tanh \psi(n, t) - L[[\psi(n, t)]_c] + B\eta(n, t), \quad (2)$$

with $[\psi]_c \equiv \ll \psi \gg - c\psi$, where $\psi(n, t)$ is the order parameter in the n th “cell” at time t . The positive constants A , L , c and B are parameters of our model, B being the noise amplitude; the noise field $\eta(n, t)$ is a random number (uniformly distributed in the interval $[-1, 1]$) assigned at each time t to each cell site n . The operator $\ll \gg$ is the isotropic spatial average, and is defined on the square lattice by $\ll \psi \gg = (1/6) \sum \psi(\text{nearest-neighbor cells}) +$

$(1/12) \sum \psi(\text{next-nearest-neighbor cells})$. In our numerical investigation we have studied a system on a square lattice of size 100×100 with periodic boundary conditions and with parameters $A = 1.0015$, $L = 0.8$, $c = 0.7$ and $B = 0.01$. The initial distribution of ψ 's is specified by a random uniform distribution in the range $[-0.01, 0.01]$.

Given the computationally efficient model, it should be reminded that to estimate the order of transition in numerical computations is a delicate question. We have thus followed Binder⁵ who suggested that the only possibility to distinguish the first-order from the second-order transitions comes from a kinetic analysis by which metastability will show up as a relaxation occurring in two steps. This is because in the second-order transition the rate of relaxation is monotonic. The result is displayed in Fig. 1, where the double-plateau evolution of the Nusselt number $\tilde{N}$ is clearly seen. Also shown in the figure are the actual patterns observed at the plateaus (at 7000 and 30000 time steps); the bright regions show downflow while the dark ones upflow. Since the SH theory could not address the pattern achieved at the metastable state, the observed essentially hexagonal pattern will be of help in further elucidating the nature of the transition.

Strange enough in view of the current intense interest in the study of ordering kinetics from initially unstable states (e.g., the so-called scaling behavior of domain growth in order-disorder transitions), little attention has yet been paid to the comparable study of the growth kinetics in RB systems⁶. We have decided to undertake such investigation. In so doing we should bear in mind the two possible distinctive features of the RB case. Firstly, the interplay with the mode-selection process; for $\epsilon > 0$ there is a finite bandwidth of modes (wavevectors) of width $\sim \epsilon$ which are stable against small-amplitude perturbations. However, ultimately a well-defined mode seems to be selected as a final stationary state. Secondly,

the presence of the critical fluctuations. In the ordinary first-order transitions the noise effect is unimportant for late-stage ordering kinetics. This is because a zero-temperature fixed point (in the renormalization-group sense) controls the domain growth for all temperatures below a critical temperature. In the case of the noise-induced first-order transition, however, irrelevance of noise in the growth kinetics is not self-evident. As a preparatory study towards understanding the complicated interplay of those effects, we consider the deterministic SH model.

In recent work, Elder and Grant⁷ have generalized a singular perturbation technique⁸ (to be referred to as KYG), developed to study the non-conserved ordering kinetics, to a broad class of other ordering dynamics problems of which RB convection is one example. The KYG approach is presumably valid for times long enough that the influence of the initial conditions has decayed, but still short enough that the correlation is not time-independent. We will refer to such a time domain as one exhibiting intermediate asymptotics. The quantity of primary interest for our study of evolving order parameter field is the equal-time correlation function, $g(\mathbf{r} - \mathbf{r}', t) \equiv \langle \psi(\mathbf{r}, t) \psi(\mathbf{r}', t) \rangle$ at intermediate asymptotics. We have calculated g with the extended KYG method, assuming that the initial profile of ψ is spatially uncorrelated and Gaussian distributed, to find

$$g(r, t) = (2/\pi) \arcsin[J_0(q_0 r) \exp(-r^2/4l^2(t))] , \quad (3)$$

where $l(t) = \xi_0 \sqrt{2t/\tau_0}$ and $\xi_0 \equiv 2q_0 \tilde{\xi}_0^2$, with J_0 being the Bessel function of the first kind. Within KYG approach one may take the limit $q_0 r \rightarrow \infty$, $t \rightarrow \infty$ such that r/l remains finite. Hence the result (3) implies the structure factor $S(q, t)$, which is the Fourier transform of $g(r, t)$, has the scaling form $S(q, t) = l^2(t) f((q - q_0)l(t))$, apart from the dependence upon q_0 itself; $f(x)$ is a scaling function.

We have used the same CDS scheme (2) as before, but now with $A = 1.01$

and $B = 0$. We computed the circularly averaged (and normalized) correlation function. The length scales $l(t)$ and q_0^{-1} were then calculated by fitting the KYG-like result (3) to the data. The value of q_0 determined from the matching could not be distinguished within numerical uncertainty from the value for the most unstable mode of the linear dispersion of (2). Our results for $l(t)$ are shown in Fig. 2. The least-squares fit to our data gives an exponent 0.50 ± 0.02 , which supports the predicted power-law growth $l(t) \sim t^{1/2}$ in the intermediate asymptotics. At late times a crossover takes place which is not described by the KYG-like solution. Monitoring the heat current $\tilde{N}$, we observed that the system still continued to evolve. This late-stage phenomena is of interest in connection with the mode selection, and will be the subject of a future study.

In summary, the numerical evidence of fluctuation-induced first-order transition is presented. The simulation result confirms the theoretical prediction to exhibit the dynamic scaling with a characteristic length scale $l(t) \sim t^{1/2}$.

References

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Fig. 1.

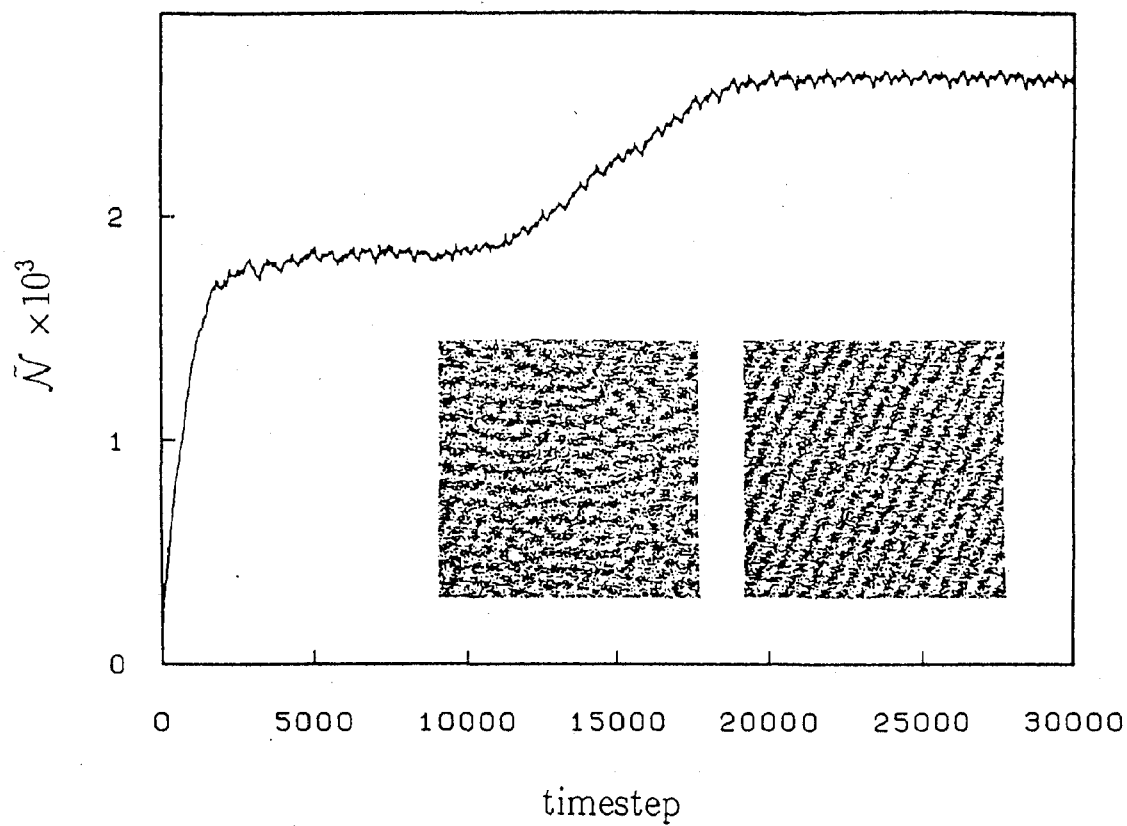


Fig. 2.

