

Some Criteria for univalence of certain integral operators

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Abstract

In this work, we derive some criteria for univalence of certain integral operators for analytic functions in the open unit disk.

1 Introduction

Let $\mathcal{A}$ be the class of the functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z) \in \mathcal{A}$ which are univalent in $\mathbb{U}$. Miller and Mocanu [1] have considered many integral operators for functions $f(z)$ belonging to the class $\mathcal{A}$. In this paper, we consider the following integral operators.

$$F_{\alpha}(z) = \left\{ \frac{1}{\alpha} \int_0^z f(u)^{\frac{1}{\alpha}} u^{-1} du \right\}^{\alpha} \quad (z \in \mathbb{U}) \quad (1.1)$$

for $f(z) \in \mathcal{A}$ and for some $\alpha \in \mathbb{C}$. It is well-known that $F_{\alpha}(z) \in \mathcal{S}$ for $f(z) \in \mathcal{S}^*$ and $\alpha > 0$, where $\mathcal{S}^*$ denotes the subclass of $\mathcal{S}$ consisting of all starlike functions $f(z)$ in $\mathbb{U}$.

2 PRELIMINARY RESULTS

To discuss about our integral operators, we need the following theorems.

Theorem 2.1 ([3]) *Let α be a complex number with $\operatorname{Re}(\alpha) > 0$, and $f(z) \in \mathcal{A}$. If $f(z)$ satisfies*

$$\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad (2.1)$$

for all $z \in \mathbb{U}$, then the following integral operator

$$G_{\alpha}(z) = \left\{ \alpha \int_0^z u^{\alpha-1} f'(u) du \right\}^{\frac{1}{\alpha}} \quad (2.2)$$

is in the class $\mathcal{S}$.

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Theorem 2.2 ([4]) *Let α be a complex number with $\operatorname{Re}(\alpha) > 0$ and $f(z) \in \mathcal{A}$. If $f(z)$ satisfies*

$$\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (2.3)$$

for all $z \in \mathbb{U}$, then, for any complex number β with $\operatorname{Re}(\beta) \geq \operatorname{Re}(\alpha)$, the integral operator

$$G_\beta(z) = \left\{ \beta \int_0^z u^{\beta-1} f'(u) du \right\}^{\frac{1}{\beta}} \quad (2.4)$$

is in the class $\mathcal{S}$.

Example 2.3 Defining the function $f(z)$ by

$$f(z) = \int_0^z \left(\frac{1 + u^{\operatorname{Re}(\alpha)}}{1 - u^{\operatorname{Re}(\alpha)}} \right)^{\frac{1}{2}} du$$

with $\operatorname{Re}(\alpha) \geq 1$, we have that

$$\frac{1 - z^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left(\frac{zf''(z)}{f'(z)} \right) = z^{\operatorname{Re}(\alpha)-1}.$$

Thus the function $f(z)$ satisfies the condition of Theorem 2.2. Therefore, for $\operatorname{Re}(\beta) \geq \operatorname{Re}(\alpha)$,

$$G_\beta(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{1 + u^{\operatorname{Re}(\alpha)}}{1 - u^{\operatorname{Re}(\alpha)}} \right)^{\frac{1}{2}} du \right\}^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Theorem 2.4 [2] *If the function $g(z)$ is regular in $\mathbb{U}$, then, for all $\xi \in \mathbb{U}$ and $z \in \mathbb{U}$, $g(z)$ satisfies*

$$\left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \overline{z}\xi} \right| \quad (2.5)$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}. \quad (2.6)$$

The equalities hold only in the case $g(z) = \varepsilon \frac{z + u}{1 + \overline{u}z}$, where $|\varepsilon| = 1$ and $|u| < 1$.

Remark 2.5 ([2]) For $z = 0$, from inequality (2.5)

$$\left| \frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)} \right| \leq |\xi| \quad (2.7)$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|}. \quad (2.8)$$

Considering $g(0) = a$ and $\xi = z$, we see that

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a||z|} \quad (2.9)$$

for all $z \in \mathbb{U}$.

Schwarz Lemma ([2]) *If the function $g(z)$ is regular in $\mathbb{U}$, $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in \mathbb{U}$, then*

$$|g(z)| \leq |z|, \quad (2.10)$$

for all $z \in \mathbb{U}$, and $|g'(0)| \leq 1$. The equality in (2.10) for $z \neq 0$ holds only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3 Main results

Theorem 3.1 *Let α be a complex number with $\operatorname{Re}\left(\frac{1}{\alpha}\right) = a > 0$ and the function $g(z) \in \mathcal{A}$ satisfy*

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| \leq 1 \quad (z \in \mathbb{U}). \quad (3.1)$$

Then, for

$$|\alpha| \geq \frac{2}{(2a+1)^{\frac{2a+1}{2a}}}, \quad (3.2)$$

the integral operator

$$F_\alpha(z) = \left\{ \frac{1}{\alpha} \int_0^z g(u)^{\frac{1}{\alpha}} u^{-1} du \right\}^\alpha \quad (3.3)$$

is in the class S .

Proof Let $\frac{1}{\alpha} = \beta$. Then we have

$$F_{\frac{1}{\beta}}(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{g(u)}{u} \right)^\beta du \right\}^{\frac{1}{\beta}}. \quad (3.4)$$

Let us consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u} \right)^\beta du. \quad (3.5)$$

Then the function

$$h(z) = \left(\frac{1}{|\beta|} \right) \frac{zf''(z)}{f'(z)} \quad (3.6)$$

is regular in $\mathbb{U}$ and the constant $|\beta|$ satisfies the inequality

$$|\beta| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}. \quad (3.7)$$

From (3.5) and (3.6), we have that

$$h(z) = \frac{\beta}{|\beta|} \left(\frac{zg'(z)}{g(z)} - 1 \right). \quad (3.8)$$

Using (3.8) and (3.1), we obtain

$$|h(z)| \leq 1 \quad (z \in \mathbb{U}). \quad (3.9)$$

Noting that $h(0) = 0$ and applying Schwarz - Lemma for $h(z)$, we get

$$\frac{1}{|\beta|} \left| \frac{zf''(z)}{f'(z)} \right| \leq |z| \quad (z \in \mathbb{U}), \quad (3.10)$$

and hence, we obtain

$$\frac{1 - |z|^{2a}}{a} \left| \frac{zf''(z)}{f'(z)} \right| \leq |\beta| \left(\frac{1 - |z|^{2a}}{a} \right) |z| \quad (z \in \mathbb{U}). \quad (3.11)$$

Because

$$\max_{|z| \leq 1} \left(\frac{1 - |z|^{2a}}{a} |z| \right) = \frac{2}{(2a + 1)^{\frac{2a+1}{2a}}}$$

from (3.11) and (3.7), we have

$$\frac{1 - |z|^{2a}}{a} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (3.12)$$

for $z \in \mathbb{U}$. From (3.12) and Theorem 2.1, it follows that

$$G_\beta(z) = \left\{ \beta \int_0^z u^{\beta-1} f'(u) du \right\}^{\frac{1}{\beta}} \quad (3.13)$$

belongs to the class $\mathcal{S}$.

By means of (3.13) and (3.5), we have the integral operator $F_{\frac{1}{\beta}}(z)$ is in the class $\mathcal{S}$, and hence, we conclude that the integral operator $F_\alpha(z)$ is in the class $\mathcal{S}$.

Example 3.2 If we take the function $g(z) = ze^z$ and $\alpha = \frac{1}{a} > 0$, then

$$g(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

is analytic in $\mathbb{U}$ and

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| = |z| < 1 \quad (z \in \mathbb{U}).$$

Since the function $g(z)$ satisfies the condition of Theorem 3.1, we have

$$T_\alpha(z) = \left\{ \frac{1}{\alpha} \int_0^z e^{\frac{1}{\alpha} u} u^{\frac{1}{\alpha}-1} du \right\}^\alpha \in \mathcal{S}.$$

Theorem 3.3 Let α, β be complex numbers with $\operatorname{Re}(\beta) \geq \operatorname{Re}(\alpha) > 0$ and the function $g(z) \in \mathcal{A}$ satisfy

$$\left| \frac{zg'(z) - g(z)}{zg(z)} \right| \leq 1 \quad (z \in \mathbb{U}). \quad (3.14)$$

Then, for

$$|\alpha| \geq \max_{|z| \leq 1} \left\{ \left(\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \right) |z| \left(\frac{|z| + |a_2|}{1 + |a_2||z|} \right) \right\}, \quad (3.15)$$

the integral operator

$$F_{\alpha, \beta}(z) = \left\{ \beta \int_0^z g(u)^{\frac{1}{\alpha}} u^{\beta - \frac{1}{\alpha} - 1} du \right\}^{\frac{1}{\beta}} \quad (3.16)$$

is in the class S .

Proof We have

$$F_{\alpha, \beta}(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{g(u)}{u} \right)^{\frac{1}{\alpha}} du \right\}^{\frac{1}{\beta}}. \quad (3.17)$$

Let us consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u} \right)^{\frac{1}{\alpha}} du. \quad (3.18)$$

which is regular in $\mathbb{U}$. The function

$$p(z) = |\alpha| \frac{f''(z)}{f'(z)}, \quad (3.19)$$

where the constant $|\alpha|$ satisfies the inequality (3.15), is regular in $\mathbb{U}$. From (3.19) and (3.18), we obtain

$$p(z) = \frac{|\alpha|}{\alpha} \left\{ \frac{zg'(z) - g(z)}{zg(z)} \right\} \quad (3.20)$$

and using (3.14) we have

$$|p(z)| < 1 \quad (z \in \mathbb{U}) \quad (3.21)$$

and $|p(0)| = |a_2|$. Applying Remark 2.5, we obtain

$$\left| \alpha \frac{f''(z)}{f'(z)} \right| \leq \frac{|z| + |a_2|}{1 + |a_2||z|} \quad (z \in \mathbb{U}). \quad (3.22)$$

It follows that

$$\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq \left(\frac{1}{|\alpha|} \right) \left(\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \right) |z| \left(\frac{|z| + |a_2|}{1 + |a_2||z|} \right) \quad (3.23)$$

for all $z \in \mathbb{U}$. Let us consider the function

$$Q(x) = \left(\frac{1 - x^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \right) x \left(\frac{x + |a_2|}{1 + |a_2|x} \right) \quad (x = |z|; x \in [0, 1]).$$

Because $Q\left(\frac{1}{2}\right) > 0$, $Q(x)$ satisfies

$$\max_{x \in [0, 1]} Q(x) > 0 \quad (3.24)$$

Using this fact, (3.23) gives us that

$$\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1}{|\alpha|} \max_{|z| \leq 1} \left\{ \left(\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \right) |z| \left(\frac{|z| + |a_2|}{1 + |a_2||z|} \right) \right\}. \quad (3.25)$$

From (3.25) and (3.15), we obtain

$$\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (z \in \mathbb{U}). \quad (3.26)$$

Using (3.26) and Theorem 2.2, we obtain that the integral operator

$$G_\beta(z) = \left\{ \beta \int_0^z u^{\beta-1} f'(u) du \right\}^{\frac{1}{\beta}} \quad (3.27)$$

belongs to the class $\mathcal{S}$. Therefore, it follows from (3.27) and (3.18), that $F_{\alpha, \beta}(z)$ is in the class $\mathcal{S}$.

Corollary 3.4 *Let α be a complex number with $\operatorname{Re}(\alpha) > 0$ and the function $g(z) \in \mathcal{A}$ satisfy*

$$\left| \frac{zg'(z) - g(z)}{zg(z)} \right| \leq 1 \quad (z \in \mathbb{U}). \quad (3.28)$$

Then, for

$$\max_{|z| \leq 1} \left\{ \left(\frac{1 - |z|^{2\operatorname{Re}(\alpha)}}{\operatorname{Re}(\alpha)} \right) |z| \left(\frac{|z| + |a_2|}{1 + |a_2||z|} \right) \right\} \leq |\alpha| \leq 1, \quad (3.29)$$

the integral operator

$$F_{\alpha}(z) = \left\{ \frac{1}{\alpha} \int_0^z g(u)^{\frac{1}{\alpha}} u^{-1} du \right\}^{\alpha} \quad (3.30)$$

is in the class S .

Proof From Theorem 3.3 for $\beta = \frac{1}{\alpha}$, the condition $\operatorname{Re}(\beta) \geq \operatorname{Re}(\alpha) > 0$, is identical with $|\alpha| < 1$ and we have $F_{\alpha,\beta}(z) = F_{\alpha}(z)$.

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