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A simple expression for the Casimir operator in Iwasawa co-odinates

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Introduction.

Let G be a connected semisimple Lie group and $G = KAN$ an Iwasawa decomposition of G. In this paper, we give a formula for the Casimir operator Ω on G in Iwasawa coodinates $(k,a,n) \in K\times A\times N$ (Theorem 2.3).

This formula was first obtained by Anderson [1] and then by Williams [2] in its corrected form. But in the second paper the formula contains extra terms, which can be cancelled out. We obtain this formula through a simpler computation, using a linear transform of differential operators on G: $D \to D^{\dagger}$, which was suggested by Yamashita. He utilized in [3, part I] a similar operation as $\dagger$ in order to get an expression of Ω in Bruhat coodinates.

Our map $\dagger$ carries left G-invariant differential operators on G to left K-invariant and right AN-invariant ones, and the operator Ω is fixed by $\dagger$: $\Omega = \Omega^{\dagger}$ (see Lemma 2.2). So we compute $\Omega^{\dagger}$ instead of Ω itself. This enables us to simplify the proof of Anderson-Williams' formula to a large extent. Actually, Williams makes in Lemmas 2.2 and

2.6 of [2] an elementary but long computation by using k_{ij}. matrix elements of the adjoint representation of K, but we can derive this formula without this calculation by considering the operator $\Omega^{\dagger}$.

§1. Preliminaries.

1.1. Let $\underline{g}$ be a non-compact real semisimple Lie algebra with a Cartan decomposition $\underline{g} = \underline{k} + \underline{p}$. We denote by θ the corresponding Cartan involution of $\underline{g}$. The Killing form B of $\underline{g}$ is negative definite on $\underline{k}$ and positive definite on $\underline{p}$. The formula

$$\langle x, y\rangle = - B(x, \theta y), \qquad x, y \in \underline{g},$$

defines a real positive definite inner product $\langle , \rangle$ on $\underline{g}$. Let $\underline{a} \subset \underline{p}$ be a maximal abelian subspace of $\underline{p}$. For $\alpha \in \underline{a}^*$ (the real dual space of $\underline{a}$), we put

$$\underline{g}_{\alpha} = \{x \in \underline{g};\ [H,x] = \alpha(H)x \quad \text{for every} \quad H \quad \text{in} \quad \underline{a}\}.$$

An element α is called a restricted root of $\underline{g}$ (relative to $\underline{a}$) if $\alpha \neq 0$ and $\underline{g}_{\alpha} \neq (0)$. Let $\Sigma \subset \underline{a}^*$ be the set of restricted roots, Σ^+ be a choice of a positive system of Σ, and $\underline{n}$ denote the sum of the positive root spaces $\underline{g}_{\alpha}$ $(\alpha \in \Sigma^+)$. Put for $\alpha \in \Sigma$,

$$m_{\alpha} = \dim \underline{g}_{\alpha}, \qquad \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_{\alpha}\alpha.$$

Then $\underline{g}$ has an Iwasawa decomposition $\underline{g} = \underline{k}+\underline{a}+\underline{n}$ and another decomposition $\underline{g} = \theta\underline{n}+\underline{m}+\underline{a}+\underline{n}$, where $\underline{m}$ is the centralizer of $\underline{a}$ in $\underline{k}$. Let G be a connected Lie group with Lie algebra $\underline{g}$, and $K \subset G$ the analytic Lie subgroup of G with Lie algebra $\underline{k}$. Then G has an Iwasawa decomposition $G = KAN$, where $A = \exp \underline{a}$, $N = \exp \underline{n}$.

Let $\underline{g}_C$ be the complexification of $\underline{g}$, and $U(\underline{g}_C)$ the universal enveloping algebra of $\underline{g}_C$. We regard elements of $\underline{g}$ as left-invariant vector fields on G: $(Xf)(g) = \frac{d}{dt} f(g(\exp tX))|_{t=0}$, for $f \in C^\infty(G)$, $X \in \underline{g}$.

Then $U(\underline{g}_C)$ is identified with the algebra of left-invariant differetial operators in the canonical way.

For $(z, H, x) \in \underline{k}\times\underline{a}\times\underline{n}$, we define differential operators δ_Z, δ_H, δ_X: $C^\infty(G) \to C^\infty(G)$, on G respectively by

$$(\delta_Z f)(kan) = \frac{d}{dt}f(k(\exp tz)an)|_{t=0},$$

$$(\delta_H f)(kan) = \frac{d}{dt}f(k(\exp tH)an)|_{t=0},$$

$$(\delta_X f)(kan) = \frac{d}{dt}f(ka(\exp tx)n)|_{t=0},$$

where $kan \in KAN$ is an Iwasawa decomposition of an element of G. These operators δ_Z, δ_H and δ_X are mutually commutative.

1.2. For $D \in U(\underline{g}_C)$, we define a differential operator $D^\dagger$ on G as follows. Extend an $f \in C^\infty(G)$ to $\tilde{f} \in C^\infty(G\times G)$ as

$$\tilde{f}(g, g_1) = f(kg_1an) \quad (g=kan),$$

and put

$$(D^{\dagger}f)(g) = (D_{(g_1)}\tilde{f})(g, g_1)|_{g_1=e} = (D_{(g_1)}\tilde{f})(g, e),$$

where $D_{(g_1)}$ means differentiation D with respect to the variable g_1, and e denotes the unit element of G.

Especially if $D \in \underline{g}$, then $D^{\dagger}$ is given as

$$(D^{\dagger}f)(kan) = \frac{d}{dt}f(k(\exp tD)an)|_{t=0}.$$

This operator $D^{\dagger}$ is left K-invariant and right AN-invariant for every $D \in U(\underline{g}_c)$. This trick $D \to D^{\dagger}$, applied to the Casimir operator, plays an important role in proof of our main result.

1.3. Let $\{H_i\}_{i=1}^{r}$, $\{u_i\}_{i=1}^{s}$ and $\{x_i\}_{i=1}^{t}$ be orthonormal bases of $\underline{a}$, $\underline{m}$ and $\underline{n}$ respectively. Set $z_i = (x_i+\theta x_i)/\sqrt{2}$, $y_i = (x_i-\theta x_i)/\sqrt{2}$, for $1 \leqq i \leqq t$. Then $z_i \in \underline{k}$, $y_i \in \underline{p}$. Let $\underline{m}^{\perp}$ (resp. $\underline{a}^{\perp}$) denote the orthogonal complement of $\underline{m}$ in $\underline{k}$ (resp. $\underline{a}$ in $\underline{p}$). Then $\{z_i\}_{i=1}^{t}$ (resp. $\{y_i\}_{i=1}^{t}$) is an orthonomal basis of $\underline{m}^{\perp}$ (resp. $\underline{a}^{\perp}$). So $\{H_i\}_{i=1}^{r} \cup \{y_i\}_{i=1}^{t}$ is an orthonormal basis of $\underline{p}$, and $\{u_i\}_{i=1}^{s} \cup \{z_i\}_{i=1}^{t}$ is an orthonomal basis of $\underline{k}$.

We choose an orthonomal basis $\{x_i\}_{i=1}^{t}$ of $\underline{n}$, consisting of root vectors, as follows. Write $\Sigma^{+}=\{\alpha_1,\dots,\alpha_q\}$, and for $1\leqq j\leqq q$, let $\{x_{i(j)};\ 1\leqq i\leqq m_{\alpha_j}\}$ be an orthonormal basis of $\mathfrak{g}_{\alpha_j}$. We put $\{x_i\}_{i=1}^{t}=\bigcup_{1\leqq j\leqq q}\{x_{i(j)};\ 1\leqq i\leqq m_{\alpha_j}\}$.

§2. A formula for the Casimir operator.

In this section we give a formula for the Casimir operator in Iwasawa coodinates.

2.1. First we recall the definition of Casimir operator. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbf{R}$, B the Killing form of $\mathfrak{g}$. Let $X_1,\cdots,X_n$ be a basis of $\mathfrak{g}$. Put $g_{ij}=B(X_i,X_j)$ and let (g^{ij}) be the inverse matrix of (g_{ij}). Then the differential operator

$$\Omega=\sum_{i,j} g^{ij}X_iX_j$$

is independent of a choice of the basis $\{X_i\}$, and is G-biinvariant. This Ω is called the Casimir operator on G.

In terms of the basis $\{H_i\}\cup\{y_i\}\cup\{z_i\}\cup\{u_i\}$ of $\mathfrak{g}$, the Casimir operator Ω is expressed as

$$(1)\qquad \Omega=\sum_{i=1}^{r}H_i^{\,2}+\sum_{i=1}^{t}y_i^{\,2}-\sum_{i=1}^{t}z_i^{\,2}-\sum_{i=1}^{s}u_i^{\,2}\,.$$

Computing the right hand side of (1), one obtains the following

LEMMA 2.1. The operator Ω has another expression as

$$\Omega = \sum_{i=1}^{r} (H_i{}^2 + 2\rho(H_i)H_i) + \sum_{i=1}^{t} (2x_i{}^2 - 2\sqrt{2}z_i x_i) - \sum_{i=1}^{s} u_i{}^2 .$$

PROOF. Since $y_i = \sqrt{2}x_i - z_i$, we get

$$
\begin{aligned}
(2) \qquad y_i{}^2 &= 2x_i{}^2 - \sqrt{2}x_i z_i - \sqrt{2}z_i x_i + z_i{}^2 \\
&= 2x_i{}^2 + z_i{}^2 - \sqrt{2}[x_i, z_i] - 2\sqrt{2}z_i x_i .
\end{aligned}
$$

The element $[x_i, z_i] = \frac{1}{\sqrt{2}} [x_i, \theta x_i]$ belongs to $\underline{a}$. So we put

$$[x_i, \theta x_i] = \sum_{k=1}^{r} c_k^{(i)} H_k \qquad \text{with } c_k^{(i)} \in \mathbf{R}.$$

Let $x_i = x_{\nu(\mu)} \in \underline{g}_{\alpha_\mu}$. Let us compute the coefficients $c_k^{(i)}$. First note that

$$B(H_k, [x_i, \theta x_i]) + B([x_i, H_k], \theta x_i) = 0,$$

which implies that

$$\sum_{j=1}^{r} c_j^{(i)} B(H_k, H_j) - \alpha_\mu(H_k) B(x_i, \theta x_i) = 0.$$

Since

$$B(H_k,H_j) = -B(H_k,\theta H_j) = \langle H_k,H_j\rangle = \delta_{kj},$$

$$-B(x_i,\theta x_i) = \langle x_i,x_i\rangle = 1,$$

we have $c_k^{(i)} = -\alpha_\mu(H_k)$. Therefore,

$$[x_i,z_i] = \frac{1}{\sqrt{2}}[x_i,\theta x_i] = -\frac{1}{\sqrt{2}}\sum_{k=1}^{r}\alpha_\mu(H_k)H_k .$$

So,

$$(3) \qquad \sqrt{2}\sum_{i=1}^{t}[x_i,z_i] = -\sum_{1\leqq i\leqq m_{\alpha_j}}\sum_{j=1}^{q}\sum_{k=1}^{r}\alpha_j(H_k)H_k = -2\sum_{k=1}^{r}\rho(H_k)H_k.$$

Then from (1)-(3) we obtain the desired expression. Q.E.D.

LEMMA 2.2. Let Ω be the Casimir operator on G. Then one has $\Omega = \Omega^{\dagger}$, where $D \to D^{\dagger}$ is the linear transform of differential operators on G, given in 1.2.

PROOF. Let $f \in C^{\infty}(G)$ and $kan \in KAN = G$. Since Ω is invariant under conjugation of elements of G, we obtain

$$\begin{aligned}(\Omega^{\dagger}f)(kan) &= (\Omega_{(g)}f)(kgan)|_{g=e} = (\Omega_{(g)}f)(k(angn^{-1}a^{-1})an)|_{g=e}\\ &= (\Omega_{(g)}f)(kang)|_{g=e} = (\Omega f)(kan).\end{aligned}$$

Thus $\Omega = \Omega^{\dagger}$. Q.E.D.

2.2. We now give, using Lemmas 2.1 and 2.2, an expression of Ω by means of the differential operators δ_{z_i}, δ_{u_i}, δ_{H_i} and δ_{x_i}, which is the main result of this paper.

THEOREM 2.3. The Casimir operator Ω is expressed as

$$\Omega = \sum_{1\leqq i\leqq r}(\delta_{H_i}^2 + 2\rho(H_i)\delta_{H_i}) - \sum_{1\leqq i\leqq s}\delta_{u_i}^2$$
$$+ \sum_{1\leqq j\leqq q}\ \sum_{1\leqq i\leqq m_{\alpha_j}}(2e^{-2\alpha_j}\delta_{x_{i(j)}}^2 - 2\sqrt{2}e^{-\alpha_j}\delta_{z_{i(j)}}\delta_{x_{i(j)}}),$$

where $e^{\alpha}(kan) = e^{\alpha(\log a)}$ for $\alpha \in \underline{a}^*$, $kan \in KAN = G$.

PROOF. By Lemma 2.2, $\Omega = \Omega^\dagger$. So we compute $\Omega^\dagger$. Thanks to Lemma 2.1, we get

$$(4) \qquad \Omega^\dagger = \sum_{i=1}^{r}((H_i^2)^\dagger + 2\rho(H_i)(H_i)^\dagger)$$
$$+ \sum_{i=1}^{t}(2(x_i^2)^\dagger - 2\sqrt{2}(z_i x_i)^\dagger) - \sum_{i=1}^{s}(u_i^2)^\dagger.$$

We compute each term of the right hand side of (4). Let $f \in C^\infty(G)$, $kan \in G$. As for $(H_i^2)^\dagger$, one obtains

$$(5)\qquad ((H_i{}^2)^{\dagger}f)(kan) = \frac{d}{ds}\frac{d}{dt} f(k(\exp tH_i)(\exp sH_i)an)\Big|_{t=s=0}$$
$$= ((\delta_{H_i})^2 f)(kan).$$

Similarly,

$$(6)\qquad (H_i{}^{\dagger}f)(kan) = (\delta_{H_i} f)(kan),$$

$$(7)\qquad ((u_i{}^2)^{\dagger}f)(kan) = ((\delta_{u_i})^2 f)(kan).$$

Let $x_i = x_{\nu(\mu)} \in \mathfrak{g}_{\alpha_\mu}$, then

$$(8)\qquad ((z_i x_i)^{\dagger}f)(kan) = \frac{d}{ds}\frac{d}{dt} f(k(\exp sz_i)(\exp tx_i)an)\Big|_{t=s=0}$$
$$= \frac{d}{ds}\frac{d}{dt} f(k(\exp sz_i)a\cdot\exp(tAd(a^{-1})x_i)\cdot n)\Big|_{t=s=0}$$
$$= \frac{d}{ds}\frac{d}{dt} f(k(\exp sz_i)a\cdot\exp(te^{-\alpha_\mu}(a)\ x_i)\cdot n)\Big|_{t=s=0}$$
$$= (e^{-\alpha_\mu}\ \delta_{z_i}\delta_{x_i} f)(kan),$$

$$(9)\qquad ((x_i{}^2)^{\dagger}f)(kan) = \frac{d}{ds}\frac{d}{dt} f(k(\exp sx_i)(\exp tx_i)an)\Big|_{t=s=0}$$
$$= \frac{d}{ds}\frac{d}{dt} f(ka\cdot\exp(sAd(a^{-1})x_i)\cdot\exp(tAd(a^{-1})x_i)\cdot n)\Big|_{t=s=0}$$
$$= \frac{d}{ds}\frac{d}{dt} f(ka\cdot\exp(se^{-\alpha_\mu}(a)x_i)\cdot\exp(te^{-\alpha_\mu}(a)x_i)\cdot n)\Big|_{t=s=0}$$
$$= (e^{-2\alpha_\mu}(\delta_{x_i})^2 f)(kan).$$

Finally (4)-(9) imply the formula in the theorem. Q.E.D

§3. Remarks on the formula in Lemma 2.1.

3.1. Through a discussion with Yamashita on the first version of this manuscript, we have realized that the formula of Ω in Lemma 2.1 is equivalent to the following well-known expression

$$\Omega = \sum_{i=1}^{r} (H_i^2 + 2\rho(H_i)H_i) - 2\sum_{i=1}^{t} \theta x_i x_i - \sum_{i=1}^{s} u_i^2 \tag{10}$$

by the relation $\sqrt{2}z_i - x_i = \theta x_i$. So, if one starts the process from this formula (10) instead of (1), then the proof of Lemma 2.1 can be cut out. In this way, one takes a futher shortcut to Theorem 2.3.

3.2. At last, we prove the formula (10) without using (1). Let $\underline{g}$ be a semisimple Lie algebra and $\{X_i\}_{i=1}^n$ be a basis of $\underline{g}$, and $\{Y_i\}_{i=1}^n$ be the dual basis of $\{X_i\}$ with respect to the Killing form B: $B(X_i, Y_j) = \delta_{ij}$. It follows immediately from the definition of Casimir operator that Ω is expressed as

$$\Omega = \sum_{i=1}^{n} Y_i X_i .$$

We take a basis $\{\theta x_i\} \cup \{u_i\} \cup \{H_i\} \cup \{x_i\}$ of $\underline{g}$. Then the dual bases of $\{\theta x_i\}$, $\{u_i\}$, $\{H_i\}$ and $\{x_i\}$ are $\{-x_i\}$, $\{-u_i\}$, $\{H_i\}$ and $\{-\theta x_i\}$ respectively. Therefore the operator Ω is expressed as

$$\Omega = \sum_{i=1}^{t}(-x_i)\theta x_i + \sum_{i=1}^{s}(-u_i)u_i + \sum_{i=1}^{r}H_i^2 + \sum_{i=1}^{t}(-\theta x_i)x_i$$

$$= \sum_{i=1}^{r}H_i^2 - \sum_{i=1}^{t}(x_i\theta x_i + \theta x_i x_i) - \sum_{i=1}^{s}u_i^2$$

$$= \sum_{i=1}^{r}H_i^2 - \sum_{i=1}^{t}([x_i,\theta x_i] + 2\theta x_i x_i) - \sum_{i=1}^{s}u_i^2.$$

As shown in Lemma 2.1, one gets $\sum_{i=1}^{t}[x_i,\theta x_i] = -\sum_{i=1}^{r}2\rho(H_i)H_i$. Thus we obtain (10) as desired.

References

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