

CHEVALLEY ALGEBRAS AND CHEVALLEY GROUPS

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1. Introduction In this note, we introduce and discuss Chevalley algebras over commutative rings R with identity, describe their arithmetic structure in the classical (i.e., non-Kac-Moody) cases, and relate that to the normal structure of Chevalley groups over R . In Section 3, we also describe recent work of Garland which leads to Chevalley algebras and groups associated with Kac-Moody Lie algebras. In several places, we discuss open questions and conjectures. The rest of this section is devoted to notational preliminaries.

Let L be a finite-dimensional simple Lie algebra over the complex field, H an m -dimensional Cartan subalgebra, Φ the set of roots of L relative to H , and $\Pi = \{r_1, r_2, \dots, r_m\}$ a simple system of roots. For $r \in \Phi$, let L_r be the corresponding root space. Chevalley [5] established the following basic fact.

1.1 Theorem There is a basis $B = \{\bar{e}_r \mid r \in \Phi\} \cup \{\bar{h}_1, \bar{h}_2, \dots, \bar{h}_m\}$, where $\bar{e}_r \in L_r$, $\bar{h}_i \in H$, such that

- (i) $[\bar{h}_i, \bar{h}_j] = 0$ for all i and j ,
- (ii) $[\bar{e}_r, \bar{e}_{-r}] = \bar{h}_r$, a certain [26, Lemma 1] integral linear combination of $\bar{h}_1, \bar{h}_2, \dots, \bar{h}_m$.
- (iii) If $r + s \neq 0$, then $[\bar{e}_r, \bar{e}_s] = \pm N_{rs} \bar{e}_{r+s}$, where N_{rs} is 0 if $r + s \notin \Phi$, and otherwise is $p+1$, where

p is the largest integer such that $s - pr \in \Phi$.

- (iv) $[\bar{h}_r, \bar{e}_s] = \frac{2(s,r)}{(r,r)} \bar{e}_s = s(\bar{h}_r) \bar{e}_s$, where $(\cdot, \cdot)$ is the Killing form on the dual H^* of H . We note that the Cartan integer $c(r,s) = \frac{2(s,r)}{(r,r)} = p - q$, where q is the largest integer such that $s + qr \in \Phi$.

Denote by L_Z the free abelian group on B . This is the Chevalley lattice of L corresponding to B and is closed under Lie products. Let R be a commutative ring with identity.

1.2 Definition The Chevalley algebra of L over R is

$$L_R = R \otimes_Z L_Z.$$

This is uniquely determined up to isomorphism by L [9, pp. 47-48].

Let H_Z be the free abelian group on $\{h_1, h_2, \dots, h_m\}$. Then we denote $R \otimes_Z H_Z$ by H_R . Similarly, if E_Z is the free abelian group on $\{\bar{e}_r \mid r \in \Phi\}$, then E_R stands for $R \otimes_Z E_Z$. Note that H_R is a subalgebra of L_R , but E_R is only an R -submodule.

2. Classical Chevalley algebras Results on the arithmetic structure of Lie algebras of Chevalley type tend to take the form of sandwich relations (cf. Equations (1) - (4) below). We consider in this section first the ideal structure of Chevalley algebras, and then the nature of orders in L when the underlying ground ring is an integral domain.

Even though L is simple over the complex field, L_R is not in general simple. For an ideal J of R for instance, we can from the projection homomorphism $f_J : R \rightarrow R/J$, produce a homomorphism from L_R

onto L_R/J with kernel JL_R , which we can identify with L_J . There are then ideals of L_R corresponding to ideals of the ring R . A natural question then arises.

Question 1 To what extent is the ideal structure of L_R determined by that of R ?

This question is answered in [10] and [27], to which the reader is referred for proofs of the first two results below. Assume that 2 and 3 are not zero divisors in R , and if L is of type A_m assume further that $m+1$ is not a multiple of the characteristic of R , or a 0-divisor.

2.1 Theorem Suppose that $I \not\subseteq H_R$. Then there is an ideal J of R and a positive integer n such that

$$(1) \quad n J L_R \subseteq I \subseteq J L_R.$$

Here, n is a product of divisors of $\det C$, where $C = (c_{ij}) = (c(r_i, r_j))$, and powers of $k = (\ell, \ell)/(s, s)$ where ℓ is a long root and s is a short root of L .

For fields of prime characteristic, Question 1 has also been answered by Hogewij [8], who determines all ideals of L_R even in case R is of characteristic 2 or 3 or in case its characteristic divides $m+1$ in type A_m . Using Theorem 2.1, one can obtain the following characterization of the circumstances under which all ideals of L_R arise from those of the ring R , again retaining the assumptions on 2, 3, and $m+1$.

2.2 Theorem Every ideal I of L_R has the form JL_R for J an ideal of R if and only if k and $\det C$ are invertible in R .

Question 2 What is the situation over a general commutative ring R

with identity in which 2 or 3 or $m+1$ may be a zero-divisor?

Chevalley algebras have also been used to study orders in split simple Lie algebras L over a field F which is the field of fractions of an integral domain D . Such algebras have a Chevalley basis over F , and we can identify L with $F \otimes_{\mathbb{Z}} L_{\mathbb{Z}}$. The results below generalize the principal theorems found in the Ph.D. dissertation of M. Harvey Hyman [16]. For a more complete discussion, consult [14]. We first give the basic definition.

2.3 Definition An order in L is a lattice (i.e., a finitely generated D -module whose F -span is L) X which is closed under multiplication.

We can then regard X as a Lie algebra over D . The Chevalley algebra L_D is, of course, a natural order to consider in L , and is called in this context the Chevalley order. In the remainder of this section, X stands for an arbitrary order in L .

2.4 Theorem If $X \supseteq L_D$, then there is an integer n as in Theorem 2.1 such that

$$(2) \quad n J L_D \subseteq X \subseteq L_D,$$

where J is the smallest D -submodule of F such that $J L_D \supseteq X$. If D is Noetherian, then J is a fractional ideal.

Observe that J is well-defined, since we have $X \subseteq J' L_D$ for the D -submodule J' of F generated by 1 and all coefficients of elements of X expressed as F -linear combinations of the Chevalley basis elements.

Let $\bar{D}$ denote the integral closure of D in F and $L_D' = E_D \oplus H_D'$,

where H_D' is the lattice of coroots,

$$H_D' = \{h \in H \mid r(h) \in D \text{ for all } r \in \Phi\}.$$

We have the following result.

2.5 Theorem (a) Suppose rank L is at least 2 and 2 has an

inverse in D in case L is of type B_m or C_m . Let D be a Noetherian domain. Then for any order $X \supseteq L_D$,

$$(3) \quad L_D \subseteq X \subseteq L_D'.$$

(b) If D is integrally closed and Noetherian, (e.g., a Dedekind domain), then for n as in Theorem 2.1,

$$(4) \quad n L_D' \subseteq X \subseteq L_D'$$

These results describe essentially the nature of orders which contain a certain fixed order L_D . It is perhaps worth noting that, even in the case of a Dedekind domain, infinite descending chains of orders are easily produced. If, for example, $a \in D$ is not invertible, then the chain

$$L_D \supseteq a L_D \supseteq a^2 L_D \supseteq a^3 L_D \supseteq \dots$$

is an infinite descending chain of orders. It seems to be appropriate then to study orders which contain a fixed order such as L_D . Such orders were referred to by Hyman as comprising the superstructure of the order L_D . One can ask the following question, whose answer one would expect to be related to the ideal structure of L_R' (cf. [11]).

Question 3 What is the superstructure of the order L_D' ?

3. Kac-Moody Lie algebras and Chevalley algebras We con-

tinue the notation of preceding sections. Garland [6] considers Kac-Moody Lie algebras $L_{\tilde{C}}$ associated with an $m+1$ -by- $m+1$ affine Cartan matrix $\tilde{C}$ obtained from a classical Cartan matrix C , and shows that over the

complex field such algebras have an integral basis closely related to the Chevalley basis for L . We discuss this from the more general viewpoint of Moody [20] first, and then specialize to the affine case to state Garland's theorem on Chevalley bases and pose two questions which arise naturally from his construction.

We begin with an n - by - n generalized Cartan matrix (GCM) $A = (a_{ij})$, that is, a matrix of integers such that for all i and j ,

$$a_{ij} \leq 0 \text{ if } i \neq j,$$

$$a_{ii} = 2 \text{ for all } i = 1, 2, \dots, n, \text{ and}$$

$$a_{ij} = 0 \text{ if and only if } a_{ji} = 0.$$

Let K be any field of characteristic zero. Let $L_1 = L_1(A)$ be the Lie algebra defined by a set $\{h_i, e_i, f_i\}_{i=1}^n$ of $3n$ generators with defining relations

$$(5) \quad \left\{ \begin{array}{l} [h_i, h_j] = 0, \text{ for all } i \text{ and } j, \\ [e_i, f_j] = \delta_{ij} h_i, \text{ for all } i \text{ and } j, \\ [h_i, e_j] = a_{ij} e_j, \text{ for all } i \text{ and } j, \\ [h_i, f_j] = -a_{ij} f_j, \text{ for all } i \text{ and } j, \\ (\text{ad } e_i)^{-a_{ij}+1}(e_j) = 0 = (\text{ad } f_i)^{-a_{ij}+1}(f_j) \text{ for } i \neq j, \end{array} \right.$$

$i, j = 1, 2, \dots, n$. Thus, L_1 is the quotient of the free Lie algebra on these $3n$ generators factored by the ideal generated by the elements obtained by rewriting each equation as an expression equated to zero.

For an n -tuple $(k_1, k_2, \dots, k_n)$ of integers, we define subspaces $L_1(k_1, k_2, \dots, k_n)$ as follows. $L_1(0, 0, \dots, 0) = H(A) =$ the abelian subalgebra of L_1 spanned by $\{h_1, h_2, \dots, h_n\}$. If $(k_1, k_2, \dots, k_n)$ consists of nonnegative (resp., nonpositive) integers, then $L_1(k_1, k_2, \dots, k_n)$ is the subspace of L_1 spanned by all products $[e_{i_1}, [e_{i_2}, \dots [e_{i_r-1},$

$e_{ir}]\dots]]$ (respectively, $[f_{i1}, [f_{i2}, \dots [f_{i,r-1}, f_{ir}]\dots]]$), where e_j (resp., f_j) occurs $|k_j|$ times. For all other n -tuples, $L_1(k_1, k_2, \dots, k_n)$ is defined to be 0. Each of these subspaces is seen to be finite dimensional, and L_1 is the sum of all the $L_1(k_1, k_2, \dots, k_n)$ over all members of Z^n . This gives us a Z^n -gradation of L_1 . There is a unique graded ideal R_1 maximal among all graded ideals which intersect the span of $\{h_i, e_i, f_i\}_{i=1}^n$ only in zero.

3.1 Definition The Kac-Moody Lie algebra L_A is L_1/R_1 .

Notice that if A is a classical Cartan matrix and K is the complex field, then $R_1 = 0$ and $L_A = L_C$ is a classical simple Lie algebra.

We denote the images of $h_i, e_i, f_i, H(A)$, and $L_1(k_1, k_2, \dots, k_n)$ by h_i, e_i, f_i, H_A , and $L(k_1, k_2, \dots, k_n)$ respectively. We define $D_i : L_A \rightarrow L_A$ for each $i = 1, 2, \dots, n$, to be multiplication by the scalar k_i on $L(k_1, k_2, \dots, k_n)$. This is then a derivation of L_A . Let D_0 be the n -dimensional subspace of commuting derivations spanned by $D_1, D_2, \dots, D_n$. Let D be a subspace of D_0 and form the semi-direct product algebra $L^e = D \times L_A$ with component-wise addition and multiplication by scalars, and Lie product $[d + \ell, d' + \ell'] = [d, d'] + (d(\ell') - d'(\ell) + [\ell, \ell'])$. Let $H_A^e = D \times H \subseteq L_A^e$, an abelian subalgebra which acts via scalar multiplication on $L(k_1, k_2, \dots, k_n)$. We further define $a_1, a_2, \dots, a_n \in (H^e)^*$ by

$$(6) \quad [h, e_i] = a_i(h) e_i, \quad \text{for } h \in H_A^e, \quad i = 1, 2, \dots, n.$$

Thus $a_j(h_i) = a_{ij}$, $i, j = 1, 2, \dots, n$. Henceforth we assume that D is so chosen that $\{a_1, a_2, \dots, a_n\}$ is a linearly independent set. This is possible since, for instance, $D = D_0$ will serve, although it is often convenient to use a smaller such D . Observe that $a_i(D_j) = \delta_{ij}$ for i, j

ranging between 1 and n . We can now define the roots of L_A .

3.2 Definition Let $a \in (H^e)^*$. Then $L^a = \{x \in L_A \mid [h, x] = a(h)x \text{ for all } h \in H_A^e\}$. A root of L_A relative to H_A^e is a member a of $(H^e)^*$ for which $L^a \neq 0$. The set of all roots is denoted by $\Delta = \Delta(A)$. The positive roots $\Delta_+ = \Delta_+(A)$ consist of all roots which are non-negative integral linear combinations of $a_1, a_2, \dots, a_n$. The negative roots $\Delta_- = \Delta_-(A)$ are defined to be the negatives of the positive roots.

Notice that $L_A^0 = H_A$ and $L = H_A \oplus \sum_{a \in \Delta_+} L^a \oplus \sum_{a \in \Delta_-} L^a$.

3.3 Definition The GCM A is symmetrizable if there exist positive rational numbers $q_1, q_2, \dots, q_n$ such that $\text{diag}(q_1, q_2, \dots, q_n)A$ is a symmetric matrix.

Henceforth, we assume that A is symmetrizable. Then we can define a symmetric bilinear form on the subspace of $(H^e)^*$ spanned by Δ by setting

$$(a_i, a_j) = q_i a_{ij},$$

for $i, j = 1, 2, \dots, n$. Then $q_i = (a_i, a_i)/2$ and we set

$$h_i' = \frac{1}{2} (a_i, a_i) h_i \in H,$$

for $i = 1, 2, \dots, n$. For $\phi = \sum_{i=1}^n x_i a_i$, we also define

$$h' = \sum_{i=1}^n x_i h_i',$$

and use this to transfer $(,)$ to H by defining $(h_i', h_j') = (a_i, a_j)$, for $i, j = 1, 2, \dots, n$, and then $(h_a', h_b') = (a, b)$ for any a and b in the span of Δ .

For $i = 1, 2, \dots, n$, we define the Weyl reflection $w_i : (H^e)^* \rightarrow (H^e)^*$ by

$$w_i(a) = a - a(h_i) a_i.$$

Thus, in particular, from (6) we see that $w_i(a_j) = a_j - a_{ij}a_i$ for $i, j = 1, 2, \dots, n$. The Weyl group W of L_A is the subgroup of $\text{Aut}(H^e)^*$ generated by all the w_i . We define the set $\Delta_R(B)$ of real roots to be $W(\{r_1, r_2, \dots, r_n\})$, and the set of imaginary roots $\Delta_I(B)$ to consist of all roots which are not real.

Now suppose that A is a classical m -by- m Cartan matrix C . We take $D = 0$, so that $H^e = H$, and $L_C^e = L_C$ is a classical Lie algebra over K . Our form $(,)$ on H_C^* is just the usual transferred Killing form from L . Using our notation Φ for the set of roots of L_C , the set Π of simple roots determines the positive roots $\Phi_+(C)$. Let $r_0 \in \Phi_+(C)$ be the highest root. We set $r_{m+1} = -r_0$, and form the affine Cartan matrix $\tilde{C}$ where $\tilde{c}_{ij} = 2(r_i, r_j)/(r_i, r_i)$, $i, j = 1, 2, \dots, m+1$. Then $\tilde{C}$ is a symmetrizable generalized Cartan matrix with associated Kac-Moody Lie algebra $L_{\tilde{C}}$.

Next let $K[t, t^{-1}]$ be the ring of Laurent polynomials over K . We define the infinite dimensional Laurent polynomial Lie algebra

$$\tilde{L} = K[t, t^{-1}] \otimes_K L_C,$$

with Lie product $[f \otimes x, g \otimes y] = fg \otimes [x, y]$ for $f, g \in K[t, t^{-1}]$ and $x, y \in L_C$. Note that from the decomposition of L_C into H_C and the sum of the root spaces L^r , we obtain

$$\tilde{L} = K[t, t^{-1}] \otimes_K H_C \oplus \sum_{r \in \Phi} L^r \oplus \sum_{n \in \mathbb{Z}^+} UZ^{-n} t^n \otimes_K L_C.$$

Now to avoid ambiguity, we write e_i^*, f_i^*, h_i^* for e_i, f_i, h_i in L_C , $i = 1, 2, \dots, n$ and h_r^* for h_r in H_C . For r_0 , choose $e_0^* \in L^{r_0}$ and $f_0^* \in L^{-r_0}$ so that $[e_0^*, f_0^*] = 2h_{r_0}^* / (r_0, r_0)$. The following theorem of Kac [17] and Moody [21] helps to describe the set of roots of

$L_{\tilde{C}}$. In our next result, we identify $1 \otimes x$ in $\tilde{L}$ with x in L_C .

3.4 Theorem There is a unique monomorphism $\tilde{\omega}: L_{\tilde{C}} \rightarrow \tilde{L}$ such that $\tilde{\omega}(e_i) = e_i^*$, $\tilde{\omega}(f_i) = f_i^*$, $\tilde{\omega}(h_i) = h_i^*$, $i = 1, 2, \dots, m$, $\tilde{\omega}(e_{m+1}) = t \otimes f_0^*$, $\tilde{\omega}(f_{m+1}) = t^{-1} \otimes e_0^*$, and $\tilde{\omega}(h_{m+1}) = 2h_{-r_0}^*/(r_0, r_0)$. The kernel of $\tilde{\omega}$ is the one-dimensional center of $L_{\tilde{C}}$ and is spanned by $h_1^* = \sum_{i=1}^m k_i h_i^* + h_{m+1}^*$, where $r_0 = \sum_{i=1}^m k_i r_i$.

We define $D_{m+1}: L_C \rightarrow L_C$ to be the $(m+1)$ -st degree derivation, and define D to be the one-dimensional subspace of D_0 spanned by D_{m+1} . It is easy to check that $\{a_1, a_2, \dots, a_{m+1}\}$ in the resulting $(H^e)^*$ is then a linearly independent set [6, p. 487]. Note that $\tilde{\omega}$ isomorphically maps

$$(7) \quad \sum_{a \in \Delta_+(\tilde{C})} L^a \rightarrow \sum_{r \in \Phi_+} L^r \oplus \sum_{n \in Z^+ \cup Z^-} t^n \otimes L_C$$

and similarly for $\sum_{a \in \Delta_-(C)} L^a$. We thus identify the two sides of (7). For $r \in \Phi$, $r = \sum_{i=1}^m n_i r_i$, $n_i \in Z^+$ or $n_i \in Z^-$ for all i , we define $a(r) \in (H^e)^*$ by the formula $a(r) = \sum_{i=1}^m n_i a_i$. We define the Lie algebra derivation $\bar{D}_0: L \rightarrow L$ by $\bar{D}_0(t^n \otimes x) = nt^n \otimes x$ for $n \in Z$ and $x \in L_C$. Then [6, p. 487] $\bar{D}_0 \circ \tilde{\omega} = \tilde{\omega} \circ D_{m+1}$. Setting

$$1 = \sum_{i=1}^m k_i a_i + a_{m+1} \in (H^e)^*,$$

it follows from Theorem 3.4 that

$$\Delta_+(C) = \{a(r)\}_{r \in \Phi_+} \cup \{a(r) + n_1\}_{\substack{r \in \Phi \\ n \in Z^+}} \cup \{n_1\}_{n \in Z^+}$$

3.5 Proposition (Kac [17, p. 287]) Let A be a GCM. Then the root $a \in \Delta_I(A)$ if and only if ja is a root for all integers $j \neq 0$.

It now follows that $\Delta_I(\tilde{C}) = \{n_1\}_{n \in \mathbb{Z} - \{0\}}$ and $\Delta_R(\tilde{C}) = \{a(r) + n_1\}_{n \in \mathbb{Z}, r \in \Phi}$. Using our identification (7) above, the root spaces L^a of L_C are therefore $L^a = t^n \otimes L^r$ (where $a = n_1 + a(r)$, $r \in \Phi$ and $n \in \mathbb{Z}$) and $L^a = t^n \otimes H_C$ (where $n \in \mathbb{Z} - \{0\}$ and $a = n_1$).

Next suppose that K is the complex field. We take $q_i = (r_i, r_i)/2$, $i = 1, 2, \dots, m+1$, so that $q_i > 0$ for each i . Then $\text{diag}(q_1, q_2, \dots, q_{m+1}) \tilde{C}$ is a symmetric matrix with ij -entry (r_i, r_j) for $i, j = 1, 2, \dots, m+1$. Notice then that with this choice of q_i , $(a_i, a_j) = (r_i, r_j)$, for $i, j = 1, 2, \dots, m+1$, and hence for $a \in \Delta(\tilde{C})$, we have $2(a, a_i)/(a_i, a_i) \in \mathbb{Z}$ for $i = 1, 2, \dots, m+1$. For each real root $a = a(r) + n_1$, we define $e_a \in L^a = t^n \otimes L^r$ by $e_a = t^n \otimes \bar{e}_r$, where $\bar{e}_r$ is as in Theorem 1.1. For each imaginary root n_1 , we define for $i = 1, 2, \dots, m$ and each nonzero integer n , $e_i(n) \in L^{n_1} = t^n \otimes H_A$ by $e_i(n) = t^n \otimes \bar{h}_i$. Note that $\{e_i(n)\}_{i=1}^m$ is a basis for $L^{n_1} = t^n \otimes H_C$ for each $n \in \mathbb{Z}$, and that $\{h_1, h_2, \dots, h_{m+1}\}$ is a basis for $H_{\tilde{C}}$.

3.6 Definition The set $\tilde{B} = \{h_i\}_{i=1}^{m+1} \cup \{e_a\}_{a \in \Delta_R(\tilde{C})} \cup \{e_i(n)\}_{i=1}^m, n \in \mathbb{Z}$ is called a Chevalley basis for $L_{\tilde{C}}$.

We now are in a position to state Theorem 4.12 of [6], which serves to explain and justify the terminology in the preceding definition.

3.7 Theorem $\tilde{B}$ is an integral basis for $L_{\tilde{C}}$. In fact, the equations of structure are

$$(i) [e_a, e_b] = \pm (p+1) e_{a+b} \quad \text{if } a+b \neq 0, \text{ where } a = a(r) + n_1, b = b(s) + j_1, r, s \in \Phi, n, j \in \mathbb{Z}, \text{ and } p \text{ as}$$

in Theorem 1.1 .

- (ii) $[e_a, e_{-a}] = h_a$, an integral linear combination of $h_1, h_2, \dots, h_{m+1}$ for all $a \in \Delta_R(\tilde{C})$.
- (iii) If $a = a(r) + n_1$, $b = a(-r) + j_1$, $r \in \Phi$, and $n + j = \ell \neq 0$, then $[e_a, e_b] = t^\ell \otimes h_r$ is an integral linear combination of $e_1(\ell), \dots, e_m(\ell)$.
- (iv) $[e_i(n), e_j(-n)] = n c_{ij} 2h_i^* / (r_j, r_j)$ is an integral linear combination of $h_1, h_2, \dots, h_{m+1}$
- (v) $[h_i, e_a] = (2(a, a_i)/(a_i, a_i)) e_a$ for $a \in \Delta_R(\tilde{C})$, $i = 1, 2, \dots, m+1$.
- (vi) $[e_i(n), e_a] = (2(r, r_i)/(r_i, r_i)) e_{a+n_1}$, where $i = 1, 2, \dots, m$, $n \neq 0$ is in $\mathbb{Z}$ and $a = a(r) + j_1$, $r \in \Phi$, $j \in \mathbb{Z}$.

All other products of elements in B are zero.

This result raises immediately the following question, in view of the results set forth in Section 2.

Question 4 If the free abelian group $(L_{\tilde{C}})_{\mathbb{Z}}$ is formed, and for a commutative ring R with identity 1 , the Kac-Moody Chevalley algebra $(L_{\tilde{C}})_R = R \otimes_{\mathbb{Z}} (L_{\tilde{C}})_{\mathbb{Z}}$ is constructed, then what is the ideal structure of $(L_{\tilde{C}})_R$, and in particular, how is it related to the ideal structure of the Chevalley algebra $L_R[t, t^{-1}]$?

This question is currently under investigation by the author and J. Morita. Theorem 3.4 has been generalized to a form which appears useful in considering Question 4.

Garland [7] goes on to construct groups of automorphisms analogous to Chevalley groups of classical simple Lie algebras. In [6], he already constructs (Theorem 5.8) a $\mathbb{Z}$ -form $U_{\mathbb{Z}}(\tilde{C})$ of the universal enveloping algebra $U(L_{\tilde{C}})$ of $L_{\tilde{C}}$ which is analogous to $U_{\mathbb{Z}}$ in [19]. He then proves (Theorem 11.3) the existence of a $V_{\mathbb{Z}}^{\lambda}$ which is invariant under $U_{\mathbb{Z}}(\tilde{C})$. He is able (Lemma 10.4) to exponentiate scalar multiples of e_a , $a \in \Delta_R(C)$, the Chevalley basis elements of Theorem 3.7 above, and obtains automorphisms of V^{λ} , and can thus define Chevalley groups for $L_{\tilde{C}}$ over a commutative ring R with identity. This brings up another question.

Question 5 What is the normal structure of these groups G and how does it relate to the ideal structure of $(L_{\tilde{C}})_R$ and of R ?

Garland [6, p. 495] remarks that the groups G have an infinite dimensional completion $\hat{G}$ which is a central extension of a Chevalley group with rational points in the field of formal Laurent series. Thus there may be some relation between Question 5 and recent work of Morita [22] on Chevalley groups over rings of Laurent polynomials. For a general idea of why one would expect some relationship between ideal structure of the Chevalley algebras $(L_{\tilde{C}})_R$ and the normal structure of the Chevalley groups G , refer to the next section.

4. Chevalley groups over rings We continue the notation of Sections 1 and 2, but remove the bars from the Chevalley basis elements in Theorem 1.1. Let U be the universal enveloping algebra of L . Let $U_{\mathbb{Z}}$ be the $\mathbb{Z}$ -algebra generated by all $e_r^m/m!$, $r \in \Phi$, $m \in \mathbb{Z}^+ \cup \{0\}$. Then [19, 26] under the adjoint representation, each generator of $U_{\mathbb{Z}}$ preserves $L_{\mathbb{Z}}$. If ρ is a faithful finite dimensional representation of L on a

complex vector space V , then there is [26, p. 17] a lattice M invariant under U_Z . Let $\bar{L}_Z$ be the part of L which preserves M . (If $\rho = \text{ad}$, then $\bar{L}_Z = L_Z$ of Section 1.) We can form the Chevalley algebra $L_R = R \otimes_Z \bar{L}_Z$ as before, and let $\exp(te_r)$, $t \in R$, $r \in \Phi$, act on $R \otimes_Z M$ in the natural [26, p. 21] way, and we label the resulting automorphism $x_r(t)$. The group $E_\rho(\Phi, R)$ generated by all such $x_r(t)$ is the elementary subgroup of the Chevalley group $G_\rho(\Phi, R)$ of L over R . The latter group consists of the points in R of a Chevalley-Demazure group scheme [4, §5] associated with L and ρ , which depends only on Φ and the lattice of weights of ρ . When this is the lattice of fundamental weights, $G_\rho(\Phi, \mathbb{C})$ is universal, and $G_\rho(\Phi, \mathbb{C})$ is simply connected [26, p. 89] over the complex field $\mathbb{C}$. In this case, $G_\rho(\Phi, R) = E_\rho(\Phi, R)$ when R is a field, local or even semi-local ring, or a Euclidean ring.

While $G_\rho(\Phi, K)$ is simple over a field K in almost all cases when $\rho \neq \text{ad}$, $G_\rho(\Phi, R)$ has normal subgroups arising from the ideal structure of R in a way that is reminiscent of the ideals of L_R . For notational simplicity, let us fix ρ and Φ and delete them from our notation for the groups. Let $f_J : G(R) \rightarrow G(R/J)$ be the natural epimorphism induced by reduction of the ring R modulo the ideal J . Then $G(R, J) = \text{Ker } f_J$ is of course a normal subgroup of $G(R)$, as is $f_J^{-1}(\text{Center } G(R/J)) = G^*(R, J)$.

4.1 Definition A congruence subgroup of $G(R)$ is a subgroup N such that

$$(8) \quad G(R, J) \subseteq N \subseteq G^*(R, J) .$$

Note the resemblance between (8) and (1). Also note that if N is any congruence subgroup of $G(R)$, then N is necessarily a normal subgroup. For

letting (X, Y) stand for the group generated by all commutators $(x, y) = xyx^{-1}y^{-1}$ for $x \in X$ and $y \in Y$, we have $(N, G) \subseteq (N, G^*(R, J)) \subseteq G(R, J) \subseteq N$ for any congruence subgroup N . Hence in particular, we have $G(R, J) \supseteq (G^*(R, J), G^*(R, J))$, which finishes the proof of the following basic result.

4.2 Corollary Every congruence subgroup is normal, and the factor group $G^*(R, J)/G(R, J)$ is an abelian group.

The study of normal subgroups of $G(R)$ has focused on congruence subgroups. We begin our description of the present status of this study by stating the following result of E. Abe [1].

4.3 Theorem Suppose $G(\Phi, \mathbf{C})$ is simple and simply connected as a Lie group. Let R be a local ring such that $R/M \neq \text{GF}(3)$ and $\text{char } R/M \neq 2$ if L is of type A_1, B_n, C_n, F_4 . Suppose also that $R/M \neq \text{GF}(2)$ or $\text{GF}(3)$ in types B_2 or G_2 . (Here M is the maximal ideal of R .) Then $G(R) = E(R)$ and the only normal subgroups of $G(R)$ are congruence subgroups.

4.4 Theorem [12, 13] Let R be any commutative ring with identity, with 2 and 3 not zero divisors in R and $n+1$ not a zero divisor if L is of type A_n . Let $\rho = \text{ad}$. Then corresponding to an ideal I of L_R there is a normal subgroup G_I of $E(R)$ generated by $x_r(t)$ such that $te_r \in I$ and by all iterated conjugates of $x_r(t)$ by elements of the form $x_{-r}(u_1), x_r(u_2), x_{-r}(u_3)$, etc. This G_I is the normal closure in $E(R)$ of the subgroup generated by all $x_r(t)$ such that $te_r \in I$. If L is not of type C_n , then $G_I = G_I'$ if and only if $I \cap E_R = I' \cap E_R$.

The next theorem actually holds more generally [3], but for simplicity of statement, we restrict ourselves to the following version.

4.5 Theorem (Abe - Suzuki) Let G be simple and simply connected as a complex Lie group, and have rank at least two. Let R be a Noetherian ring or a direct product of fields. Let $\text{Spm}(R) = \{M \mid M \text{ is a maximal ideal of } R\}$. If ϕ is of type B_2 or G_2 , assume for all $M \in \text{Spm}(R)$ that $R/M \neq GF(2)$. If ϕ is of type B_n , C_n , or F_4 , suppose for all $M \in \text{Spm}(R)$ that $\text{char } R/M \neq 2$, and if ϕ is of type G_2 , suppose that $\text{char } R/M \neq 3$. Let $G_0(R)$ be the subgroup of $G(R)$ generated by all $x_r(t)$ for $t \in R$, $r \in \phi$ and by all $h(\chi) = \text{diag}(\chi(\lambda_1), \dots, \chi(\lambda_n))$ for a certain [1, pp. 475-476] set $\{\lambda_1, \dots, \lambda_n\}$ which generates the additive abelian group P generated by the weights of ρ , and arbitrary $\chi \in \text{Hom}(P, \mathbb{C}^*)$. Then any subgroup of $G_0(R)$ which is normalized by $E(R)$ (in particular, any normal subgroup of $E(R)$) satisfies for a unique ideal J

$$E(R, J) \subseteq N \subseteq G^*(R, J),$$

where $E(R, J)$ is the normal closure in $E(R)$ of all $x_r(t)$, $t \in R$, $r \in \phi$. That is, $E(R, J) = G_I$, where $I = JL_R$.

4.6 Theorem [13] Suppose ϕ has a single root length and rank at least two, with $\rho = \text{ad}$. Then $x_r(t)$ has normal closure G_I , where $I = JL_R$ for J the principal ideal in R generated by t . The same result holds if $\phi = B_n$, $n \geq 3$, or F_4 if r is a long root.

Comparing the preceding result with [18, Satz 3], the following question is suggested.

Question 6 Is $E(R, J) = \text{Ker } f_J|_{E(R)}$, at least in the single root length cases?

E. Abe has been able to answer Question 6 affirmatively in case the exact sequence $0 \rightarrow J \rightarrow R \rightarrow R/J \rightarrow 0$ splits. It would be interesting to find other conditions on R which provide a positive answer. Swan [29] states without proof that Question 6 has a positive answer in the case of the stable group $E(R)$ for any commutative ring with identity. Silvester [24] makes a similar statement in the nonstable case. (In both these claims, $\Phi = A_n$.)

4.7 Theorem [13] Under the hypotheses of Theorem 4.6 (first part), the normal closure of a product $x_r(t_1)x_s(t_2)$ where $r \neq s$ in $E(R)$ is G_I where $I = J L_R$ for J the ideal in R generated by t_1 and t_2 .

Theorem 4.7 also holds for products of three root elements, but in general no such fact is known. This leads to the following question.

Question 7 Under what hypotheses on R and Φ can Theorem 4.7 be extended?

Suslin [28] has shown that $E(R) \trianglelefteq G(R)$ for $\Phi = A_n$, $n \geq 2$, and in fact has shown that $E(R, J) \trianglelefteq G(R)$ in that case. He even showed normality in $GL_n(R)$, a still larger group. This raises another natural question.

Question 8 Under what hypotheses on R and Φ is $E(R) \trianglelefteq G(R)$ and $E(R, J) \trianglelefteq G(R)$?

This question relates directly to the K_1 -functor on Chevalley groups of Stein [25]. Let $\text{rank } \Phi \geq 2$. $St(\Phi, R)$ stands for the group generated by $x_r(t)$, $t \in R$, $r \in \Phi$, subject to the relations

$$(R1) \quad x_r(t)x_s(u) = x_r(t+u)$$

$$(R2) \quad (x_r(t), x_s(u)) = \prod_{ir+js \in \Phi} x_{ir+js}(c_{ijrs} t^i u^j).$$

where $r+s \in \Phi$ and the product is taken in some fixed order, with $c_{ijrs} \in \mathbb{Z}$ for all $ir+js \in \Phi$.

Since the relations (R1) and (R2) hold, we have a mapping $\pi: \text{St}(R) \rightarrow G(R)$ whose image is the elementary subgroup $E(R)$. By definition, the group $K_2(\Phi, R)$ is the kernel of the map π , and $K(\Phi, R) = \text{Cok } \pi = G(R)/E(R)$ as a homogeneous space. Question 8 can therefore be rephrased in the language of algebraic K-theory as follows.

Question 8' Under what conditions on Φ and R is $K_1(\Phi, R)$ a group?

In [25], Stein gives the following partial answer to this form of the question.

4.8 Theorem Let Φ be one of the types A_n, B_n, C_n, D_n , or G_2 . Let R be a ring whose maximal ideal space is Noetherian of finite dimension d . Suppose also that if Φ is of type A_n , then $n \geq d+1$; if Φ is of type B_n , then $n \geq d+2$; if Φ is of type C_n , then $n \geq (d+2)/2$; if Φ is of type D_n , then $n \geq d+2$; and if Φ is of type G_2 , then $d \leq 1$. Then $E(R, J) \leq G(\Phi, R)$.

Finally, Silvester [23] considered the ring $R = K\langle X \rangle$ freely generated over K by a set X of noncommuting indeterminates. For $\Phi = A_n$, he considered $GE(R)$, the subgroup of $GL_n(R)$ generated by all $x_r(t)$ and all $h_i(z) = w_{r_i}(z)w_{r_i}(-1)$, $i = 1, 2, \dots, n$, where $w_r(u) = x_r(u)x_{-r}(-u^{-1})x_r(u)$. In addition to this group, he also considered $GEU(R)$, the group generated by symbols $x_r(t)$ and $h_i(z)$ with the usual

relations [23, p. 37] in $GE(R)$. He showed that the natural homomorphism $f : GEU(R) \rightarrow GE(R)$ is an isomorphism, in which circumstance R is said to be universal for f . Consideration of the analogous notions for general Φ leads naturally to the following question which is currently under joint investigation by the author and E. Abe.

Question 9 Under what conditions on R and Φ is f an isomorphism for more general root systems? In particular, is it an isomorphism in the case of $R = K[X]$, the free commutative K -algebra generated by a set X of indeterminates, at least in the single root length systems?

Silvester's results for the case $R = K\langle X \rangle$ was a major tool in showing that $K_2(A_n, K\langle X \rangle) = K_2(A_n, K)$. Thus the answer to Question 9 bears directly upon the question of computing $K_2(\Phi, K[X])$ which is raised by E. Abe [2] in his article in these Proceedings.

In a similar vein, if R is a commutative ring with identity, then for $\Phi = A_n$, $\text{Center } E_n(R) = \Omega_n$, the group of n -th roots of 1 in R [15]. Passing to the stable group, $\text{Center } E(R) = 1$. For more general Φ , we can pose the following question.

Question 10 What is the center of the group $E(\Phi_n, R)$? What is the center of $E_\rho(R)$, the direct limit of $E_\rho(\Phi_n, R)$ for the classical Φ_n ?

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