

ON THE EXISTENCE OF CONDENSER-TYPE MEASURES

WITH RESPECT TO FUNCTION-KERNELS

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§1. Introduction.

Let X be a locally compact Hausdorff space with a countable base, G be a continuous function-kernel on X and N be a lower semicontinuous function-kernel which is locally bounded outside the diagonal set. Further, assume that each non-empty open set is not G -negligible and not N -negligible. On this note we shall ask necessary and sufficient conditions in order that for any pair $\langle K, F \rangle$ of disjoint compact sets there exists a (G, N) -condenser-type measure α of each $\lambda \in M_K^+$ of which $N\lambda$ is locally bounded on K , i.e., there exists a measure $\alpha = \mu_0 - \mu_1$ such that

$$\begin{aligned} \text{supp}(\mu_0) &\subset K, & \text{supp}(\mu_1) &\subset F, \\ G\alpha &= N\lambda \quad G\text{-n.e. on } K, & G\alpha &= 0 \quad G\text{-n.e. on } F, \\ 0 &\leq G\alpha \leq N\lambda & \text{on } X. \end{aligned}$$

In case $N = 1$ (the constant kernel) this problem has been solved in [6], i.e., G satisfies the condenser principle if and only if, G is non-degenerate and it satisfies the domination and the classical maximum principles.

In §3 we shall show that for any pair $\langle K, F \rangle$ of a non- G -negligible compact set K and a compact set F with $A(K) \cap F = \emptyset$, there exists a (G, N) -condenser-type measure of each $\lambda \in M_K^+$ of which $N\lambda$ is bounded on K if and only if, G satisfies the domination principle and the relative domination principle with respect to N .

It is well-known that in Dirichlet spaces there exists a condenser measure for any pair $\langle \bar{U}, \bar{V} \rangle$ of a relatively compact open set U and an open set V with $\bar{U} \cap \bar{V} = \emptyset$. Further, C. Berg has proved in [1] the existence of the condenser measure for any above pair $\langle \bar{U}, \bar{V} \rangle$ with respect to the kernel $\kappa = \int \mu_t dt$, where $(\mu_t)_{t>0}$ is a transient convolution semigroup on a locally compact abelian group.

In §4 we shall ask similar problems for any pair $\langle K, F \rangle$ of a non-G-negligible compact set K and a closed set F with $A(K) \cap F = \emptyset$.

§2. Various domination principles.

Let X be a locally compact Hausdorff space with a countable base. A lower semicontinuous function-kernel on X is a non-negative lower semicontinuous function defined on $X \times X$ which is strictly positive on the diagonal set Δ of $X \times X$. A continuous function-kernel is a lower semicontinuous kernel which is continuous in the extended sense in $X \times X$ and finite-valued outside Δ . We use G and N for denoting lower semicontinuous function-kernels and introduce various domination principles.

Definition 1. We say that G satisfies the relative domination principle with respect to N , if for $\mu, \nu \in M_K^+$ with $\int G d\mu < +\infty$

$$G\mu \leq N\nu \quad \text{on } \text{supp}(\mu) \quad \text{implies} \quad G\mu \leq N\nu \quad \text{in } X.$$

We use the symbol $G \prec N$ for this principle.

" $G \prec G$ " is said to be simply the domination principle.

" $G \prec 1$ " is said to be the classical maximum principle.

Further, we say that G satisfies the elementary relative domination principle with respect to N if for $\mu \in M_K^+$ with $\int G d\mu < +\infty$ and for $x_0 \in C\text{supp}(\mu)$

$$G\mu \leq N\epsilon_{x_0} \quad \text{on } \text{supp}(\mu) \quad \text{implies} \quad G\mu \leq N\epsilon_{x_0} \quad \text{on } X.$$

Definition 2. We say that G satisfies the transitive domination principle with respect to N , if for $\mu, \nu \in M_K^+$ with $\int G d\mu < +\infty$

$$G\mu \leq G\nu \text{ on } \text{supp}(\mu) \text{ implies } N\mu \leq N\nu \text{ in } X.$$

We use the symbol $G \ll N$ for this principle.

Definition 3. We say that G satisfies the relative balayage principle with respect to N , if for every compact set K and every $\mu \in M_K^+$ such that $N\mu \neq +\infty$ on K there exists $\nu \in M_K^+$ such that

$$\text{supp}(\nu) \subset K, \quad G\nu = N\mu \text{ G-n.e. on } K, \quad G\nu \leq N\mu \text{ in } X.$$

Here " $G\nu = N\mu$ G-n.e. on K " means that the set $\{x \in K; G\nu(x) \neq N\mu(x)\}$ is a G -negligible set and a Borel measurable set A of X is called G -negligible if any measure $\lambda \in M_K^+$ with $\text{supp}(\lambda) \subset A$ and $\int G\lambda d\lambda < +\infty$ is equal to 0. Such a measure ν is called a relatively balayaged measure of μ onto K with respect to (G, N) . If G satisfies the relative balayage principle with respect to G , we say that G satisfies the balayage principle and a relatively balayaged measure with respect to (G, G) is called a G -balayaged measure, simply a balayaged measure. If G satisfies the relative balayage principle with respect to the constant kernel 1, we say that G satisfies the equilibrium principle.

Definition 4. We say that G satisfies the continuity principle if for $\mu \in M_K^+$ $G\mu$ is finite and continuous everywhere whenever it is finite and continuous on $\text{supp}(\mu)$.

We denote by $\mathcal{L}(G)$ (resp. $\mathcal{F}(G)$) the set of measures $\mu \in M_K^+$ such that $G\mu$ are locally bounded (resp. $G\mu$ are finite and continuous).

Hereafter we shall assume that G is a continuous function-kernel on X and N is a lower semicontinuous function-kernel on X which is locally bounded outside the diagonal set Δ of $X \times X$.

Proposition 1. Assume that each non-empty open set is not G -negligible and not N -negligible. If $G \ll N$, then $\mu \in \mathcal{L}(G)$ implies $\mu \in \mathcal{L}(N)$.

Proof. Let μ be a measure in $\mathcal{L}(G)$. Obviously the potential $N\mu$ is locally bounded outside $\text{supp}(\mu)$. Let x_0 be an arbitrary point in $\text{supp}(\mu)$. Since $G(x_0, x_0) > 0$, we can find a compact neighborhood K of x_0 and a constant c such that $cG(x, y) \geq 1$ on $K \times K$. Denote by μ_0 the restriction of μ to K . Since $G\mu_0$ is also locally bounded, there are a neighborhood K_1 of x_0 and a constant b such that $K_1 \subset K$ and $G\mu_0 \leq b$ on K_1 . First, assume that x_0 is an isolated point. Then $\{x_0\}$ is not G -negligible and not N -negligible by the assumption and we have $G(x_0, x_0) < +\infty$ and $N(x_0, x_0) < +\infty$. Denote by μ_1 the restriction of μ_0 to K_1 . Then $G\mu_1 \leq bcG\epsilon_{x_0}$ on K_1 and hence $N\mu_1 \leq bcN\epsilon_{x_0}$ on X by $G \ll N$. Since $N\epsilon_{x_0}$ and $N(\mu - \mu_1)$ are bounded on K_1 , $N\mu$ is also bounded on K_1 . Secondly, we consider the case where x_0 is not an isolated point. Take $x_1 \in K_1 \setminus \{x_0\}$ and find a compact neighborhood K_2 of x_0 satisfying $K_2 \subset K_1$ and $x_1 \notin K_2$. If we denote by μ_2 the restriction of μ_0 to K_2 , we have $G\mu_2 \leq bcG\epsilon_{x_1}$ on K_2 and hence $N\mu_2 \leq bcN\epsilon_{x_1}$ on X . Since $N\epsilon_{x_1}$ and $N(\mu - \mu_2)$ are bounded on K_2 , $N\mu$ is bounded on K_2 . Thus we obtain that $N\mu$ is locally bounded.

Proposition 2. Assume that each non-empty open set is not G -negligible and not N -negligible. If G satisfies the continuity principle and $G \ll N$, then $\check{G}$ satisfies the relative balayage principle with respect to $\check{N}$. Here $\check{G}$ (resp. $\check{N}$) is the adjoint kernel of G (resp. N).

Proof. Let K be a non- G -negligible compact set and μ be a measure in M_K^+ with $\check{N}\mu \neq +\infty$ on K . Put

$$S := \{u = G\sigma - G\tau; \sigma \in M_K^+, \tau \in \mathcal{F}(G), \text{supp}(\tau) \subset K\}$$

and define for each $f \in C(K)$

$$p(f) := \inf \left\{ \int N\sigma d\mu - \int N\tau d\mu; u = G\sigma - G\tau \in S, u \geq f \text{ on } K \right\}.$$

We remark that $N\tau$ is locally bounded by Proposition 1 and $\int N\tau d\mu < +\infty$. By the assumption we can find, for each $f \in C(K)$, $\sigma_0 \in \mathcal{F}(G)$ satisfying $|f| \leq G\sigma_0$ on K . Then $p(f) \leq \int N\sigma_0 d\mu < +\infty$. Take $u = G\sigma - G\tau \in S$ with $u \geq f$ on K and $\text{supp}(\tau) \subset K$. Then $G\tau \leq G\sigma + G\sigma_0$ on K and $G \ll N$, we have $N\tau \leq N\sigma + N\sigma_0$. Consequently $p(f) \geq -\int N\sigma_0 d\mu > -\infty$. Since the mapping $f \rightarrow p(f)$ is sublinear on $C(K)$, there exists a linear functional ν on $C(K)$ such that

$$\nu(f) \leq p(f) \quad \text{for all } f \in C(K).$$

If $f \leq 0$, it follows that $\nu(f) \leq p(f) \leq 0$. Hence ν is a positive measure on K . For each $\sigma \in M_K^+$, we have

$$\begin{aligned} \int G\sigma d\nu &= \sup \left\{ \int f d\nu; 0 \leq f \leq G\sigma \text{ on } K, f \in C(K) \right\} \\ &\leq \sup \{p(f); 0 \leq f \leq G\sigma \text{ on } K, f \in C(K)\} \leq \int N\sigma d\mu. \end{aligned}$$

Especially, $\int G\varepsilon_x d\nu \leq \int N\varepsilon_x d\mu$ and hence $\check{G}\nu(x) \leq \check{N}\mu(x)$ for each $x \in X$.

For each $\tau \in \mathcal{F}(G)$ with $\text{supp}(\tau) \subset K$, we have

$$\int -G\tau d\nu \leq p(-G\tau) \leq \int -N\tau d\mu.$$

Thus $\int \check{G}\nu d\tau = \int \check{N}\mu d\tau$ for each $\tau \in \mathcal{F}(G)$ with $\text{supp}(\tau) \subset K$. Since G satisfies the continuity principle by the assumption, it follows that $G\nu = N\mu$ G -n.e. on K .

§3. Condenser-type theorems for disjoint compact sets.

Hereafter we assume that G is a continuous function-kernel and N is a lower semicontinuous function-kernel which is locally bounded outside Δ . Further, assume that each non-empty open set is not G -negligible and not N -negligible.

For each $x \in X$ and for a closed set K , we define

$$A(x) := \{y \in X; \quad b > 0, \quad G(z, y) = bG(z, x) \text{ for all } z \in X\},$$

$$A(K); = \bigcup_{x \in K} A(x).$$

We shall consider the necessary and sufficient conditions in order that (G, N) has the following property:

(b) Let K be a non- G -negligible compact set, F be a compact set with $A(K) \cap F = \emptyset$ and λ be a measure in M_K^+ of which $N\lambda$ is bounded on K . Then there exists a (G, N) -condenser-type measure $\alpha = \mu_0 - \mu_1$ of λ onto $\langle K, F \rangle$, i.e., there exist $\mu_0 \in \mathcal{L}(G)$ and $\mu_1 \in \mathcal{L}(G)$ such that

$$b_1) \quad \text{supp}(\mu_0) \subset K, \quad \text{supp}(\mu_1) \subset F,$$

$$b_2) \quad G\alpha = N\lambda \quad G\text{-n.e. on } K, \quad G\alpha = 0 \quad G\text{-n.e. on } F,$$

$$b_3) \quad 0 \leq G\alpha \leq N\lambda \quad \text{on } X.$$

Theorem 1. Let N and G be continuous function-kernels. Then the following assertions (1), (2) are equivalent:

(1) (G, N) has the property (b),

(2) $G \prec G$ and $G \prec N$.

Proof. (1) $\rightarrow$ (2): Let K be a non- G -negligible compact set and λ be a measure in M_K^+ such that $N\lambda$ is bounded on K . Then there exists a (G, N) -condenser-type measure $\mu_0 - \mu_1$ of λ onto $\langle K, \emptyset \rangle$ by the assumption. Since $\mu_1 = 0$, μ_0 is a relatively balayaged measure of λ onto K with respect to (G, N) . Using this, we can show that G satisfies the elementary relative domination principle (cf. Proof of Lemma 2 in [3]). By Theorem 1 in [3], we have $G \prec N$. Next, we shall show that $G \prec G$. Assume that $\check{G}\mu \leq \check{G}_{\varepsilon_X}$ on $\text{supp}(\mu)$, where $\mu \in M_K^+$ with $\int G_\mu d\mu < +\infty$ and $x \in C(\text{supp}(\mu))$. If there is a point $x_1 \in X \setminus \text{supp}(\mu)$ such that $\check{G}\mu(x_1) > \check{G}_{\varepsilon_X}(x_1)$, we can find a compact neighborhood K of x_1 such that $\check{G}\mu(y) > \check{G}_{\varepsilon_X}(y)$ for all $y \in K$. Remark that $A(K) \cap \text{supp}(\mu)$

$= \emptyset$. By the assumption there exists a (G, N) -condenser-type measure $\mu_0 - \mu_1$ of ε_z for $z \in \text{supp}(\mu)$ onto $\langle K, \text{supp}(\mu) \rangle$. It is obvious that $\mu_0 \neq 0$. We obtain

$$\begin{aligned} 0 &= \int (G\mu_0 - G\mu_1) d\mu = \int \check{G} \mu d\mu_0 - \int \check{G} \mu d\mu_1 > \int \check{G} \varepsilon_x d\mu_0 - \int \check{G} \varepsilon_x d\mu_1 \\ &= G\mu_0(x) - G\mu_1(x) \geq 0. \end{aligned}$$

This is a contradiction. Thus we have $\check{G} \mu \leq \check{G} \varepsilon_x$ on X . Since $G \prec N$ implies that G satisfies the continuity principle (cf. [3]).

Consequently $\check{G}$ satisfies the domination principle (cf. [5, Theorem II.3]) and G does (cf. [2]).

(2) $\rightarrow$ (1): Since $G \prec G$, G satisfies the continuity principle. Let K be a non- G -negligible compact set, F be a compact set with $A(K) \cap F = \emptyset$ and λ be a measure in M_K^+ such that $N\lambda$ is bounded on K . The assumption $G \prec N$ implies $\check{G} \ll \check{N}$. By Proposition 2, G satisfies the relative balayage principle with respect to N . If ν_0 is a relatively balayaged measure of λ onto K with respect to (G, N) , $G\nu_0$ is bounded on K . Since there is $\alpha_0 \in \mathcal{F}(G)$ satisfying $G\nu_0 \leq G\alpha_0$ on X , $G\nu_0$ is locally bounded. Further we can choose successively a sequence $\{\nu_n\}$ of measures in $\mathcal{L}(G)$ with the following properties:

- (i) $\text{supp}(\nu_{2m}) \subset K, \quad \text{supp}(\nu_{2m+1}) \subset F,$
- (ii) ν_{2m+1} is a G -balayaged measure of ν_{2m} onto F ,
- (iii) ν_{2m+2} is a G -balayaged measure of ν_{2m+1} onto K .

Then $\{G\nu_n\}$ is decreasing and $\lim_{m \rightarrow \infty} G\nu_{2m} = \lim_{m \rightarrow \infty} G\nu_{2m+1}$. Since $G \prec G$, we have $\check{G} \prec \check{G}$. Let β be an arbitrary measure in $\mathcal{F}(\check{G})$. Then we can find $\sigma \in \mathcal{L}(\check{G})$ and $\tau \in \mathcal{L}(\check{G})$ satisfying

$$\begin{aligned} \check{G}\sigma - \check{G}\tau &= \check{G}\beta \quad G\text{-n.e. on } K, \quad \check{G}\sigma - \check{G}\tau = 0 \quad G\text{-n.e. on } F, \\ 0 &\leq \check{G}\sigma - \check{G}\tau \leq \check{G}\beta \quad G\text{-n.e. on } X \quad (\text{cf. [8]}). \end{aligned}$$

Remarking that $\text{supp}(\nu_{2m}) \subset K$ and $\text{supp}(\nu_{2m+1}) \subset F$, we have

$$\begin{aligned}
\int \lim_{m \rightarrow \infty} Gv_{2m} d\beta &= \lim_{m \rightarrow \infty} \int Gv_{2m} d\beta = \lim_{m \rightarrow \infty} \int \check{G}\beta dv_{2m} = \lim_{m \rightarrow \infty} \int (\check{G}\sigma - \check{G}\tau) dv_{2m} \\
&= \int (\lim_{m \rightarrow \infty} Gv_{2m}) d\sigma - \int (\lim_{m \rightarrow \infty} Gv_{2m}) d\tau \\
&= \int (\lim_{m \rightarrow \infty} Gv_{2m+1}) d\sigma - \int (\lim_{m \rightarrow \infty} Gv_{2m+1}) d\tau \\
&= \lim_{m \rightarrow \infty} \int (\check{G}\sigma - \check{G}\tau) dv_{2m+1} = 0.
\end{aligned}$$

Consequently $\lim_{m \rightarrow \infty} Gv_{2m} = 0$ G-n.e. on X and hence $\lim_{n \rightarrow \infty} Gv_n = 0$ G-n.e.

on X. Thus we see that the alternative series $\sum_{n=0}^{\infty} (-1)^n Gv_n$

converges G-n.e. on X. Put

$$g = \sum_{m=0}^{\infty} (Gv_{2m} - Gv_{2m+1}).$$

Since $Gv_{2m} - Gv_{2m+1} = 0$ G-n.e. on F and $Gv_{2m+1} - Gv_{2m+2} = 0$ G-n.e. on K, we have

$$\begin{aligned}
(3.1) \quad g &= Gv_0 = N\lambda \text{ G-n.e. on K, } g = 0 \text{ G-n.e. on F,} \\
0 &\leq g \leq Gv_0 \leq N\lambda \text{ on X}
\end{aligned}$$

Further we can choose $\gamma, \delta \in \mathcal{L}(\check{G})$ satisfying

$$\begin{aligned}
\check{G}\gamma - \check{G}\delta &\geq 1 \text{ G-n.e. on K, } \check{G}\gamma - \check{G}\delta = 0 \text{ G-n.e. on F,} \\
0 &\leq \check{G}\gamma - \check{G}\delta \text{ G-n.e. on X.}
\end{aligned}$$

For any natural number p, it follows that

$$\begin{aligned}
\sum_{m=0}^p \int dv_{2m} &\leq \sum_{m=0}^p \int (\check{G}\gamma - \check{G}\delta) dv_{2m} - \sum_{m=0}^p \int (\check{G}\gamma - \check{G}\delta) dv_{2m+1} \\
&= \sum_{m=0}^p \int (Gv_{2m} - Gv_{2m+1}) d\gamma - \sum_{m=0}^p \int (Gv_{2m} - Gv_{2m+1}) d\delta \\
&\leq \int g d\gamma = \int Gv_0 d\gamma < +\infty
\end{aligned}$$

and hence $\sum_{m=0}^{\infty} \int dv_{2m} < +\infty$. Put $\mu_0 = \sum_{m=0}^{\infty} v_{2m}$. Then μ_0 is a positive

measure supported by K. If we can choose $\beta_0 \in \mathcal{L}(\check{G})$ satisfying $\check{G}\beta_0 \geq 1$ on F, we have

$$\sum_{m=0}^{\infty} \int dv_{2m+1} \leq \sum_{m=0}^{\infty} \int \check{G}\beta_0 dv_{2m+1} = \sum_{m=0}^{\infty} \int Gv_{2m+1} d\beta_0$$

$$\leq \sum_{m=0}^{\infty} \int G v_{2m} d\beta_0 = \sum_{m=0}^{\infty} \int G \beta_0 dv_{2m} < +\infty.$$

Consequently $\mu_1 := \sum_{m=0}^{\infty} v_{2m+1}$ is a positive measure supported by F .

Obviously it follows that $\sum_{m=0}^{\infty} G v_{2m} = G\mu_0$, $\sum_{m=0}^{\infty} G v_{2m+1} = G\mu_1$ and $g = G\mu_0 - G\mu_1$ on X . Since $G\mu_0 \leq G\mu_1 + Gv_0$ and $G\mu_1$ is a continuous on K , $G\mu_0$ is bounded on K and hence it is locally bounded on X . Using $G\mu_1 \leq G\mu_0$, $G\mu_1$ is also locally bounded on X . Therefore $\mu_0 - \mu_1$ is a (G, N) -condenser-type measure of λ onto $\langle K, F \rangle$. Thus Theorem 1 has been proved.

Putting $N = G$ in Theorem 1, we obtain the following corollary.

Corollary 1. G satisfies the domination principle if and only if (G, G) has the property (b).

Definition 5. We say that G satisfies the condenser principle if for each non- G -negligible compact set K and each compact set F with $K \cap F = \emptyset$, there exists a measure $\alpha = \mu_0 - \mu_1$ ($\mu_0 \in \mathcal{I}(G)$, $\mu_1 \in \mathcal{L}(G)$) such that

$$\begin{aligned} \text{supp}(\mu_0) &\subset K \quad \text{and} \quad \text{supp}(\mu_1) \subset F, \\ G\alpha &= 1 \quad G\text{-n.e. on } K, \quad G\alpha = 0 \quad G\text{-n.e. on } F, \\ 0 &\leq G\alpha \leq 1 \quad \text{on } X. \end{aligned}$$

If $A(x) = \{x\}$ for any $x \in X$, we say that G is non-degenerate. Putting $N = 1$ (the constant kernel) in Theorem 1, we have easily the following well-known result (cf. [6]).

Corollary 2. Let G be non-degenerate. Then G satisfies the condenser principle if and only if $G \prec G$ and $G \prec 1$.

Corollary 3. If $G \prec G$, $G \prec N$ and G is non-degenerate, then for any pair of a non- G -negligible compact set K and :

F with $K \cap F = \emptyset$, there exists uniquely a measure $\alpha = \mu_0 - \mu_1$ ($\mu_0, \mu_1 \in \mathcal{L}(G)$) satisfying $b_1), b_2), b_3)$ in Theorem 1.

Proof. Let K be a non- G -negligible compact set and F be a compact set with $K \cap F = \emptyset$. Since G is non-degenerate, we have $A(K) \cap F = \emptyset$. By Theorem 1 there exists a measure $\alpha = \mu_0 - \mu_1$ ($\mu_0, \mu_1 \in \mathcal{L}(G)$) satisfying $b_1), b_2), b_3)$. Let $\beta = \nu_0 - \nu_1$ ($\nu_0, \nu_1 \in \mathcal{L}(G)$) be another measure satisfying $b_1), b_2), b_3)$. Since $G \prec G$ and G is non-degenerate, $\check{G} \prec \check{G}$ and $\check{G}$ is also non-degenerate. By Corollary 1 there exists, for each $\gamma \in \mathcal{L}(\check{G})$, a measure $\delta = \sigma - \tau$ ($\sigma, \tau \in \mathcal{L}(\check{G})$) such that

$$\begin{aligned} \check{G}\delta &= \check{G}\gamma \quad G\text{-n.e. on } K, \quad \check{G}\delta = 0 \quad G\text{-n.e. on } F, \\ 0 &\leq \check{G}\delta \leq \check{G}\gamma \quad \text{on } X. \end{aligned}$$

Then we have

$$\begin{aligned} \int \check{G}\gamma d\mu_0 &= \int (\check{G}\sigma - \check{G}\tau) d\mu_0 - \int (\check{G}\sigma - \check{G}\tau) d\mu_1 \\ &= \int (G\mu_0 - G\mu_1) d\sigma - \int (G\mu_0 - G\mu_1) d\tau \\ &= \int (G\nu_0 - G\nu_1) d\sigma - \int (G\nu_0 - G\nu_1) d\tau \\ &= \int (\check{G}\sigma - \check{G}\tau) d\nu_0 = \int \check{G}\gamma d\nu_0. \end{aligned}$$

Let f be a non-negative continuous function on X with compact support. Since G is non-degenerate, we can find sequences $\{\sigma_n\}, \{\tau_n\}$ in $\mathcal{L}(\check{G})$ satisfying

$$\begin{aligned} 0 &\leq \check{G}\sigma_n - \check{G}\tau_n \leq \check{G}\lambda \quad \text{on } X \text{ for some } \lambda \in \mathcal{F}(\check{G}), \\ \lim_{n \rightarrow \infty} (\check{G}\sigma_n - \check{G}\tau_n) &= f \quad G\text{-n.e. on } X. \end{aligned}$$

Consequently we have $\mu_0 = \nu_0$. If F is non- G -negligible, we have $\mu_1 = \nu_1$ analogously. If F is G -negligible, it follows that $\mu_1 = \nu_1 = 0$. Thus we have $\mu_0 = \nu_0$ and $\mu_1 = \nu_1$.

§4. (G, N)-condenser-type theorem for a pair $\langle K, F \rangle$ of a compact set K and a closed set F.

We say that (G, N) has the property (b') if the following property (b') is satisfied:

(b') Let K be a non-G-negligible compact set and F be a closed set with $A(K) \cap F = \emptyset$. Then for each $\lambda \in \mathcal{L}(N)$ there exists a measure $\alpha = \mu_0 - \mu_1$ satisfying

- $b'_1)$ $\text{supp}(\mu_0) \subset K, \quad \text{supp}(\mu_1) \subset F,$
- $b'_2)$ $\int G\mu_1 d\beta < +\infty$ for each $\beta \in \mathcal{L}(G),$
- $b'_3)$ $\mu_0(B) = \mu_1(B) = 0$ for each G-negligible set B,
- $b'_4)$ $G\alpha = N\lambda$ G-n.e. on K, $G\alpha = 0$ G-n.e. on F,
- $b'_5)$ $0 \leq G\alpha \leq N\lambda$ on X if F is compact and $0 \leq G\alpha \leq N\lambda$ G-n.e. on X if F is not compact.

In this section we ask a necessary and sufficient condition in order that (G, N) has the property (b') .

Theorem 2. Assume that $G \prec G$ and $\mathcal{L}(N) \neq \{0\}$. Then (G, N) has the property (b') if and only if (G, N) has the following three properties (c), (d), (e):

(c) Let K be a non-G-negligible compact set. Then for every $\mu \in \mathcal{L}(N)$ there exists a measure $\nu \in M_K^+$ such that

$$\text{supp}(\nu) \subset K, \quad G\nu = N\mu \text{ G-n.e. on } K, \quad G\nu \leq N\mu \text{ on } X.$$

(d) Any $\mu \in \mathcal{L}(G)$ is balayable onto any closed set F, i.e., there exists a positive measure ν such that

$$G\nu = G\mu \text{ G-n.e. on } F, \quad G\nu \leq G\mu \text{ on } X.$$

(e) Let K be a non-G-negligible compact set and F be a closed set with $A(K) \cap F = \emptyset$. Then there exist sequences $\{\sigma_n\}, \{\tau_n\}$ (σ_n, τ_n

$\in \mathcal{L}(\check{G})$) satisfying

$$e_1) \quad \lim_{n \rightarrow \infty} (\check{G}\sigma_n - \check{G}\tau_n) = 0 \quad G\text{-n.e. on } F,$$

$$\lim_{n \rightarrow \infty} (\check{G}\sigma_n - \check{G}\tau_n) \geq 1 \quad G\text{-n.e. on } K,$$

$$e_2) \quad 0 \leq \lim_{n \rightarrow \infty} (\check{G}\sigma_n - \check{G}\tau_n) \leq \check{G}\eta \quad G\text{-n.e. on } X \text{ for some } \eta \in \mathcal{L}(\check{G}),$$

$$e_3) \quad \check{G}\sigma_n \leq \check{G}\sigma_{n+1} \text{ and } \check{G}\tau_n \leq \check{G}\tau_{n+1} \text{ for each natural number } n,$$

$$e_4) \quad \liminf_{n \rightarrow \infty} \int \check{N}\sigma_n d\beta < +\infty \text{ and } \liminf_{n \rightarrow \infty} \int \check{N}\tau_n d\beta < +\infty \text{ for each } \beta \in \mathcal{L}(\check{N}).$$

Proof. Suppose that (G, N) has the property (b'). First, we shall show that (G, N) has the property (c). Let K be a non- G -negligible set and μ be a measure in $\mathcal{L}(N)$. Then there exists a (G, N) -condenser-type measure $\alpha = \mu_0 - \mu_1$ of μ onto $\langle K, \emptyset \rangle$. Since $\mu_1 = 0$, we have

$$G\mu_0 = N\mu \quad G\text{-n.e. on } K, \quad G\mu_0 \leq N\mu \quad \text{on } X.$$

Secondly, we shall show that any measure in $\mathcal{L}(G)$ is balayable onto any closed set F . For this purpose it is sufficient to prove that for each closed set F

$$(4.1) \quad \lim_{n \rightarrow \infty} (\check{G}\sigma_n - \check{G}\tau_n) = 0 \quad G\text{-n.e. on } F \text{ implies } \lim_{n \rightarrow \infty} (\check{G}\sigma_n - \check{G}\tau_n) = 0 \\ G\text{-n.e. on } X, \text{ where } \{\sigma_n\}, \{\tau_n\} \in \mathcal{L}(\check{G}) \text{ satisfying} \\ \text{supp}(\sigma_n) \subset F, \quad \text{supp}(\tau_n) \subset F, \\ 0 \leq \check{G}\sigma_n - \check{G}\tau_n \leq \check{G}\gamma \quad G\text{-n.e. on } F \text{ for some } \gamma \in \mathcal{L}(\check{G}), \\ \check{G}\sigma_{n+1} - \check{G}\tau_{n+1} \leq \check{G}\sigma_n - \check{G}\tau_n \quad G\text{-n.e. on } F \quad (\text{cf. [8, Theorem 1]}).$$

To show (4.1), we suppose that it does not hold that $\lim_{n \rightarrow \infty} (\check{G}\sigma_n - \check{G}\tau_n) = 0$ G -n.e. on X . Since $0 \leq \check{G}\sigma_n - \check{G}\tau_n$ on X , there exists a non- G -negligible set contained in $\{x \in CF; \lim_{n \rightarrow \infty} (\check{G}\sigma_n(x) - \check{G}\tau_n(x)) > 0\}$.

We remark that $A(K) \cap F = \emptyset$. By the assumption there exists, for $\lambda \in \mathcal{L}(N)$ ($\lambda \neq 0$), a measure $\mu_0 - \mu_1$ satisfying $b'_1) \sim b'_5)$. Since $\lambda \neq 0$,

we have $\mu_0 \neq 0$. Consequently

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \left\{ \int (G\mu_0 - G\mu_1) d\sigma_n - \int (G\mu_0 - G\mu_1) d\tau_n \right\} \\ &= \int \lim_{n \rightarrow \infty} (\check{G}\sigma_n - \check{G}\tau_n) d\mu_0 - \int \lim_{n \rightarrow \infty} (\check{G}\sigma_n - \check{G}\tau_n) d\mu_1 \\ &= \int \lim_{n \rightarrow \infty} (\check{G}\sigma_n - \check{G}\tau_n) d\mu_0 > 0. \end{aligned}$$

This is a contradiction. Thus $\lim_{n \rightarrow \infty} (\check{G}\sigma_n - \check{G}\tau_n) = 0$ G-n.e. on F

implies $\lim_{n \rightarrow \infty} (\check{G}\sigma_n - \check{G}\tau_n) = 0$ G-n.e. on X . Finally, we shall show that

(G, N) has the property (e). Let $\{K_n\}$ be an increasing sequence of compact sets with $\bigcup_{n=1}^{\infty} K_n = X$. Suppose that K is a non-G-negligible

compact set and F be a closed set with $A(K) \cap F = \emptyset$. Put $F_n = F \cap K_n$.

We can find $\lambda_0 \in \mathcal{L}(\check{G})$ satisfying $\check{G}\lambda_0 \geq 1$ on K . Since $\check{G} \prec \check{G}$, there exists, by Corollary 1, $\delta_n = \alpha_n - \beta_n$ such that

$$\begin{aligned} \text{supp}(\alpha_n) &\subset K, \quad \text{supp}(\beta_n) \subset F_n, \\ \check{G}\delta_n &= \check{G}\lambda_0 \quad \text{G-n.e. on } K, \quad \check{G}\delta_n = 0 \quad \text{G-n.e. on } F_n, \\ 0 &\leq \check{G}\delta_n \leq \check{G}\lambda_0 \quad \text{on } X. \end{aligned}$$

Since $\check{G}\alpha_n - \check{G}\beta_n \leq \check{G}\alpha_{n-1} - \check{G}\beta_{n-1}$ on $K \cup F_n$, it follows that $\check{G}\alpha_n - \check{G}\beta_n$

$\leq \check{G}\alpha_{n-1} - \check{G}\beta_{n-1}$ on X . We remark that $\check{G}\alpha_n \leq \check{G}\alpha_{n+1}$ on X . In fact,

there exists, for each $\eta \in \mathcal{L}(G)$, a measure $\sigma_n - \tau_n$ ($\sigma_n, \tau_n \in \mathcal{L}(G)$) satisfying

$$\begin{aligned} \text{supp}(\sigma_n) &\subset K, \quad \text{supp}(\tau_n) \subset F_n, \\ G\sigma_n - G\tau_n &= G\eta \quad \text{G-n.e. on } K, \quad G\sigma_n - G\tau_n = 0 \quad \text{G-n.e. on } F_n, \\ 0 &\leq G\sigma_n - G\tau_n \leq G\eta. \end{aligned}$$

Then we have

$$\begin{aligned} \int \check{G}\alpha_n d\eta &= \int G\eta d\alpha_n \\ &= \int (G\sigma_n - G\tau_n) d\alpha_n - \int (G\sigma_n - G\tau_n) d\beta_n \\ &= \int (\check{G}\alpha_n - \check{G}\beta_n) d\sigma_n - \int (\check{G}\alpha_n - \check{G}\beta_n) d\tau_n \\ &\leq \int (\check{G}\alpha_{n+1} - \check{G}\beta_{n+1}) d\sigma_n - \int (\check{G}\alpha_{n+1} - \check{G}\beta_{n+1}) d\tau_n \\ &= \int (G\sigma_n - G\tau_n) d\alpha_{n+1} - \int (G\sigma_n - G\tau_n) d\beta_{n+1} \end{aligned}$$

$$\leq \int G \eta d\alpha_{n+1} = \int \check{G}\alpha_{n+1} d\eta.$$

Consequently $\check{G}\alpha_n \leq \check{G}\alpha_{n+1}$ G-n.e. on X and hence $\check{G}\alpha_n \leq \check{G}\alpha_{n+1}$.

Similarly we have $\check{G}\beta_n \leq \check{G}\beta_{n+1}$ on X. Let λ be an arbitrary measure in $\mathcal{L}(N)$. By the assumption we can find $\alpha = \mu_0 - \mu_1$ satisfying $b'_1) \sim b'_5)$. Then we have

$$\begin{aligned} +\infty &> \int \check{G}\lambda_0 d\mu_0 \geq \int \lim_{n \rightarrow \infty} (\check{G}\alpha_n - \check{G}\beta_n) d\mu_0 - \int \lim_{n \rightarrow \infty} (\check{G}\alpha_n - \check{G}\beta_n) d\mu_1 \\ &= \lim_{n \rightarrow \infty} \left\{ \int (G\mu_0 - G\mu_1) d\alpha_n - \int (G\mu_0 - G\mu_1) d\beta_n \right\} \\ &= \lim_{n \rightarrow \infty} \int N \lambda d\alpha_n = \lim_{n \rightarrow \infty} \int \check{N}\alpha_n d\lambda. \end{aligned}$$

Since $\check{G}\beta_n \leq \check{G}\alpha_n$ on X and $\alpha_n, \beta_n \in \mathcal{L}(\check{G})$, we have, using (c),

$$\int \check{N}\beta_n d\lambda \leq \int \check{N}\alpha_n d\lambda \quad \text{for all } \lambda \in \mathcal{L}(N).$$

Consequently $\liminf_{n \rightarrow \infty} \int \check{N}\beta_n d\lambda \leq \lim_{n \rightarrow \infty} \int \check{N}\alpha_n d\lambda < +\infty$.

Conversely, suppose that (G, N) has the properties (c), (d), (e).

Let K be a non-G-negligible compact set and F be a closed set with $A(K) \cap F = \emptyset$ and λ be a measure in $\mathcal{L}(N)$. We denote by v_0 a relatively balayaged measure of λ onto K with respect to (G, N) . Since G has the property (d), we can choose a sequence $\{v_n\}$ of positive measures satisfying (i) ~ (iii) in Theorem 1. Obviously $\{Gv_n\}$ is decreasing. We shall show that $\lim_{n \rightarrow \infty} Gv_n = 0$ G-n.e. on X. Let β be an arbitrary measure in $\mathcal{L}(\check{G})$ and b be a positive real number satisfying $b\check{G}\beta \leq 1$ on K. Using the property (e), for $u = \lim_{n \rightarrow \infty} \check{G}\sigma_n$, $v = \lim_{n \rightarrow \infty} \check{G}\tau_n$ in (e) we have

$$\begin{aligned} \int u dv_n &= \lim_{p \rightarrow \infty} \int \check{G}\sigma_p dv_n = \lim_{p \rightarrow \infty} \int Gv_n d\sigma_p \\ &\leq \liminf_{p \rightarrow \infty} \int N \lambda d\sigma_p = \liminf_{p \rightarrow \infty} \int \check{N}\sigma_p d\lambda < +\infty. \end{aligned}$$

Since the sequence $\{\int u dv_n\}$ is decreasing, it follows that

$$\lim_{m \rightarrow \infty} \int u dv_{2m} = \lim_{m \rightarrow \infty} \int u dv_{2m+1} < +\infty.$$

Analogously

$$\lim_{m \rightarrow \infty} \int v dv_{2m} = \lim_{m \rightarrow \infty} \int v dv_{2m+1} < +\infty.$$

Using these relations, we obtain

$$\begin{aligned} b \int \lim_{m \rightarrow \infty} G v_{2m} d\beta &= \lim_{m \rightarrow \infty} \int b G v_{2m} d\beta = \lim_{m \rightarrow \infty} \int b \check{G} \beta dv_{2m} \\ &\leq \lim_{m \rightarrow \infty} \int \lim_{p \rightarrow \infty} (\check{G} \sigma_p - \check{G} \tau_p) dv_{2m} \\ &= \lim_{m \rightarrow \infty} \int u dv_{2m} - \lim_{m \rightarrow \infty} \int v dv_{2m} \\ &= \lim_{m \rightarrow \infty} \int u dv_{2m+1} - \lim_{m \rightarrow \infty} \int v dv_{2m+1} \\ &= \lim_{m \rightarrow \infty} \int \lim_{p \rightarrow \infty} (\check{G} \sigma_p - \check{G} \tau_p) dv_{2m+1} = 0. \end{aligned}$$

Consequently $\int \lim_{m \rightarrow \infty} G v_{2m} d\beta = 0$ for all $\beta \in \mathcal{L}(\check{G})$. Thus we have

$$\lim_{m \rightarrow \infty} G v_{2m} = 0 \quad G\text{-n.e. on } X \text{ and hence } \lim_{n \rightarrow \infty} G v_n = 0 \quad G\text{-n.e. on } X.$$

Thus we see that the alternative series $\sum_{n=0}^{\infty} (-1)^n G v_n$ converges

$G\text{-n.e. on } X$. Put

$$g := \sum_{m=0}^{\infty} (G v_{2m} - G v_{2m+1}).$$

Then g has the property (3.1). Next we shall show that $\sum_{m=0}^{\infty} v_{2m}$ is

a positive measure. For any natural number q we have

$$\begin{aligned} \sum_{m=0}^q \int dv_{2m} &\leq \sum_{m=0}^q \int \lim_{p \rightarrow \infty} (\check{G} \sigma_p - \check{G} \tau_p) dv_{2m} - \sum_{m=0}^q \int \lim_{p \rightarrow \infty} (\check{G} \sigma_p - \check{G} \tau_p) dv_{2m+1} \\ &= \lim_{p \rightarrow \infty} \sum_{m=0}^q \int (G v_{2m} - G v_{2m+1}) d\sigma_p \\ &\quad - \lim_{p \rightarrow \infty} \sum_{m=0}^q \int (G v_{2m} - G v_{2m+1}) d\tau_p \\ &\leq \liminf_{p \rightarrow \infty} \int g d\sigma_p \leq \liminf_{p \rightarrow \infty} \int N \lambda d\sigma_p < +\infty. \end{aligned}$$

Consequently $\sum_{m=0}^{\infty} \int dv_{2m} \leq \int \liminf_{p \rightarrow \infty} \int N \lambda d\sigma_p < +\infty$ and hence $\mu_0 :=$

$\sum_{m=0}^{\infty} v_{2m}$ is a positive measure supported by K . Let f be a non-negative continuous function on X with compact support and η be a measure in $\mathcal{H}(\check{G})$ with $\check{G}\eta \geq f$ on X . Then

$$\int f dv_{2m+1} \leq \int \check{G}\eta dv_{2m+1} = \int Gv_{2m+1} d\eta \leq \int Gv_{2m} d\eta = \int \check{G}\eta dv_{2m}.$$

Hence

$$\sum_{m=0}^{\infty} \int f dv_{2m+1} \leq \sum_{m=0}^{\infty} \int \check{G}\eta dv_{2m} < +\infty.$$

This implies that the positive linear functional: $f \mapsto \sum_{m=0}^{\infty} \int f dv_{2m+1}$

on the space of continuous functions on X with compact support

is a positive measure μ_1 supported by F . Obviously $\sum_{m=0}^{\infty} Gv_{2m} = G\mu_0$

and $\sum_{m=0}^{\infty} Gv_{2m+1} = G\mu_1$. For any $\beta \in \mathcal{L}(\check{G})$ we obtain

$$\int G\mu_1 d\beta = \sum_{m=0}^{\infty} \int Gv_{2m+1} d\beta \leq \sum_{m=0}^{\infty} \int Gv_{2m} d\beta \leq \sum_{m=0}^{\infty} \int \check{G}\beta dv_{2m} < +\infty.$$

Thus both $G\mu_1$ and $G\mu_0$ are finite on G -n.e. on X and it follows that

$g = G\mu_0 - G\mu_1$ G -n.e. on X . We remark that $g = G\mu_0 - G\mu_1$ on X if F

is compact. Therefore $\mu_0 - \mu_1$ is a (G, N) -condenser-type measure of

λ onto $\langle K, F \rangle$. Thus Theorem 2 has been proved.

We define

$$S(\check{G}) := \{u = \lim_{n \rightarrow \infty} \check{G}\sigma_n ; \sigma_n \in \mathcal{L}(\check{G}), \check{G}\sigma_n \leq \check{G}\sigma_{n+1}, \int u d\beta < +\infty \\ \text{for all } \beta \in \mathcal{H}(G)\}.$$

Putting $G = N$, we have easily the following corollary.

Corollary 4. Assume that $G \prec G$. Then (G, G) has the property

(b') if and only if G has the following properties (d), (e'):

(d) Any $\mu \in \mathcal{L}(G)$ is balayable onto any closed set F ,

(e') Let K be a non- G -negligible compact set and F be a closed set with $A(K) \cap F = \emptyset$. Then there exist $u, v \in S(\check{G})$ such that

$$u - v \geq 1 \quad G\text{-n.e.}$$

$$0 \leq u - v \leq \check{G}\eta \quad G\text{-I}$$

Proposition 3. Assume that G is non-degenerate and $G \prec G$. Further assume that G has the following property (f):

(f) For any $\lambda \in \mathcal{F}(\check{G})$, for any $\varepsilon > 0$ and for any compact set K there exist $u, v \in S(\check{G})$ and a compact set K' such that

$$(4.2) \quad \begin{aligned} v &= \lim_{n \rightarrow \infty} \check{G}\tau_n, \quad \text{supp}(\tau_n) \subset K', \\ u - v &\leq \varepsilon \quad G\text{-n.e. on } K, \quad u - v \geq \check{G}\lambda \quad G\text{-n.e. on } CK', \\ 0 &\leq u - v \quad G\text{-n.e. on } X. \end{aligned}$$

Then (G, G) has the property (b').

Proof. First we remark that we can choose $u, v \in S(\check{G})$ in (f) satisfying (4.2) and

$$u - v = \check{G}\lambda \quad G\text{-n.e. on } CK', \quad 0 \leq u - v \leq \check{G}\lambda \quad G\text{-n.e. on } X.$$

In fact, since $(u - v) \wedge \check{G}\lambda = u \wedge (\check{G}\lambda + v) - v$ and $u \wedge (\check{G}\lambda + v) \in S(\check{G})$, we can take $(u - v) \wedge \check{G}\lambda$ instead of $u - v$. By Proposition 4 in [8] any $\mu \in \mathcal{L}(G)$ is balayable to any closed set. Further, let K be a non- G -negligible compact set and F be a closed set with $K \cap F = \emptyset$. Choose $\lambda_0 \in \mathcal{F}(\check{G})$ satisfying $\check{G}\lambda_0 \geq 2$ on K . By the assumption (f) there exist $u, v \in S(\check{G})$ and a compact set K' with $K \subset K'$ such that

$$\begin{aligned} u - v &= \check{G}\lambda_0 \quad G\text{-n.e. on } CK', \quad u - v \leq 1 \quad G\text{-n.e. on } K, \\ 0 &\leq u - v \leq \check{G}\lambda_0 \quad G\text{-n.e. on } X. \end{aligned}$$

Then we obtain, putting $w_1 := \check{G}\lambda_0 + v - u$,

$$\begin{aligned} w_1 &\geq 1 \quad G\text{-n.e. on } K, \quad w_1 = 0 \quad G\text{-n.e. on } CK', \\ 0 &\leq w_1 \leq \check{G}\lambda_0 \quad G\text{-n.e. on } X. \end{aligned}$$

Since $(F \cap K') \cap K = \emptyset$ and $\check{G} \prec \check{G}$, it follows from Corollar there exists $w_2 = \check{G}\alpha - \check{G}\beta$ ($\alpha, \beta \in \mathcal{L}(\check{G})$) such that

$$w_2 = \frac{1}{2} \check{G}\lambda_0 \geq 1 \quad G\text{-n.e. on } K, \quad w_2 = 0 \quad G\text{-n.e. on } F \cap K',$$

$$0 \leq w_2 \leq \frac{1}{2} \check{G}\lambda_0 \quad G\text{-n.e. on } X.$$

We can write $w_1 \wedge w_2 = u_1 - u_2$ ($u_1, u_2 \in S(\check{G})$) G -n.e. on X and obtain

$$u_1 - u_2 \geq 1 \quad G\text{-n.e. on } K, \quad u_1 - u_2 = 0 \quad G\text{-n.e. on } F,$$

$$0 \leq u_1 - u_2 \leq \frac{1}{2} \check{G}\lambda_0 \quad G\text{-n.e. on } X.$$

Thus (G, G) has the property (b') by Corollary 4.

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