

Razumikhin type theorems for
differential equations with infinite delay

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Our concern is on the stability problem for functional
differential equations with infinite delay

$$(1) \quad \dot{x}(t) = f(t, x_t).$$

For functional differential equations with infinite
delay, there are several ways to specify the phase space.
A typical one is the Hale's space $\mathcal{B}$ (see [1]) consisting of
functions defined on $(-\infty, 0]$, which is provided a norm $|\cdot|_{\mathcal{B}}$
and the conditions;

(i) if $x(t)$ is defined on $(-\infty, a)$, $a > 0$, continuous
on $[0, a)$ and $x_0 \in \mathcal{B}$, then for $t \in [0, a)$, $x_t \in \mathcal{B}$
and it is continuous in t , where

$$x_t(s) = x(t + s) \quad \text{for } s \in (-\infty, 0];$$

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(ii) there exist two positive constants c, d such that

$$|\phi|_{\mathcal{B}} \leq c \sup_{-\beta \leq s \leq 0} |\phi(s)| + d |\phi|_{\beta}$$

for any $\beta \geq 0$, where

$$|\phi|_{\beta} = \inf \{ |\psi|_{\mathcal{B}}; \psi \in \mathcal{B}, \psi(s) = \phi(s) \text{ on } (-\infty, -\beta] \}$$

together with other conditions.

In our case, the space $\mathcal{B}$ is assumed to satisfy the properties

$$|\phi(0)| \leq M |\phi|_{\mathcal{B}}, \quad |\phi|_{\beta} \leq M(\beta) |\phi_{-\beta}|_{\mathcal{B}} \text{ if } \phi_{-\beta} \in \mathcal{B},$$

in addition to the conditions (i) and (ii), though c and d in (ii) may continuously depend on β . In particular, if $x(t)$ is defined on $(-\infty, a)$ and continuous on $[\tau, a)$, $\tau < a$, and if $x_{\tau} \in \mathcal{B}$ then we have

$$(2) \quad |x_t|_{\mathcal{B}} \leq c(t-\tau) \sup_{\tau \leq s \leq t} |x(s)| + d(t-\tau)M(t-\tau)|x_{\tau}|_{\mathcal{B}}.$$

It is assumed for the equations (1) to have the trivial solution, where $f(t, \phi)$ in (1) is defined and continuous on

$R \times \mathcal{B}$.

The following definition will be made:

Definition. The trivial solution of (1) is said to be
(I) *stable* if for any $\epsilon > 0$ and any $\tau \geq 0$ there exists a
 $\delta > 0$ such that

$$|x_\tau|_{\mathcal{B}} < \delta \text{ implies } |x(t)| < \epsilon \text{ for all } t \geq \tau;$$

(II) *asymptotically stable* if in addition to the stability
for any $\tau \geq 0$ there exists a $\delta_0 > 0$ and for any $\epsilon > 0$
there is a T such that

$$|x_\tau|_{\mathcal{B}} < \delta_0 \text{ and } t \geq \tau + T \text{ imply } |x(t)| < \epsilon;$$

where $x(t)$ denotes any solution of (1). Here, δ , δ_0 , T may
depend on τ but not on each solution. If these numbers are
independent of τ , then the stabilities are called *uniform*.

The following theorem is a simple version of the Liapunov-
Krasovskii's theorem (see [2] and also [3]).

Theorem A. Suppose that there exists a continuous
function $V(t, \phi)$ defined on $R \times \mathcal{B}$ such that $V(t, 0) = 0$,

$$(3) \quad a(|\phi(0)|) \leq V(t, \phi)$$

for a continuous, increasing, positive-definite function $a(r)$ and that for a continuous function $c(t, r) \geq 0$, which is non-decreasing in r ,

$$(4) \quad \dot{V}(t, x_t) \leq -c(t, V(t, x_t))$$

along any solution $x(t)$ of (1), where

$$\dot{V}(t, x_t) = \overline{\lim}_{h \rightarrow +0} \frac{1}{h} \{V(t+h, x_{t+h}) - V(t, x_t)\}.$$

Then the trivial solution of (1) is asymptotically stable if for any $r > 0$

$$(5) \quad \int_t^{t+T} c(s, r) ds \rightarrow \infty \quad \text{as } T \rightarrow \infty;$$

and uniformly asymptotically stable if the divergence in (5) is uniformly in t and if we have

$$(6) \quad V(t, \phi) \leq b(|\phi|)$$

for a continuous function $b(r)$ with $b(0) = 0$.

Since the solutions may belong to the more restrictive class as the time elapses, the following theorem is expected to be more effective. Such a theorem has been given by

Barnea[4] for the uniform stability of an autonomous system with finite delay (also refer. [5]).

Theorem B. In Theorem A, it is sufficient for $V(t, \phi)$ to satisfy (4) under the case (*) $x(s)$ is a solution of (1) at least on the interval $[p(t, V(t, x_t)), t]$, where the continuous function $p(t, r) \leq t$ is increasing in $t \geq 0$ and in $r > 0$ and satisfies $p(t, r) \rightarrow \infty$ as $t \rightarrow \infty$, $p(t, r) \rightarrow \infty$ as $r \rightarrow 0$. For the uniform stability we assume

$$(7) \quad p(t, r) = t - q(r).$$

Here, also we assume that the trivial solution of (1) is unique for the stability and that $f(t, \phi)$ in (1) satisfies

$$(8) \quad |f(t, \phi)| \leq L|\phi|$$

for the uniform stability.

Proof. Let $\epsilon > 0$ be given. Suppose that $V(\tau, x_\tau) < \frac{a(\epsilon)}{2}$ but $V(t, x_t) > a(\epsilon)$ for a $t > \tau$. Then there exists

$$t_1 = \inf \{t > \tau; V(t, x_t) \geq a(\epsilon)\}.$$

Set $t_2 = \max \{t < t_1; V(t, x_t) \leq \frac{a(\epsilon)}{2}\}.$

Since we have

$$|x_t|_{\mathcal{B}} \leq c(t-\tau) \sup_{\tau \leq s \leq t} |x(s)| + d(t-\tau)M(t-\tau)|x_\tau|_{\mathcal{B}}$$

for $t \geq \tau$ by (2) and since the uniqueness of the trivial solution implies

$$(9) \quad \sup_{\tau \leq s \leq t} |x(s)| \leq K(t, \tau, |x_\tau|_{\mathcal{B}})$$

with $K(t, \tau, r) \rightarrow 0$ as $r \rightarrow 0$, we shall have

$$t \in [t_2, t_1] \text{ and } |x_\tau|_{\mathcal{B}} < \delta \text{ imply } \tau < p(t, V(t, x_t)).$$

For this purpose, it is enough to choose δ so that $\delta < \frac{a(\epsilon)}{2}$ and

$$|\phi|_{\mathcal{B}} < A(p_t^{-1}(\tau, \frac{a(\epsilon)}{2}), \tau, \delta) \text{ implies } V(t, \phi) < \frac{a(\epsilon)}{2}$$

if $\tau \leq t \leq p_t^{-1}(\tau, \frac{a(\epsilon)}{2})$, where $A(t, \tau, r) = c(t-\tau)K(t, \tau, r) + d(t-\tau)M(t-\tau)r$. Thus, by the assumptions $V(t, x_t)$ is non-increasing on $[t_2, t_1]$, which contradicts $V(t_1, x_{t_1}) = a(\epsilon)$.

If f in (1) satisfies (8), we may choose K in (9) so that

$$K(t, \tau, r) = K(t-\tau)r$$

for a continuous function $K(t)$. Hence, in this case A is

a function of $t - \tau$ and r , and under the condition $p(t, r) = t - q(r) \delta$ can be chosen independent of τ so that

$$r \leq A(\tau + q(\frac{a(\varepsilon)}{2}), \tau, \delta) \text{ implies } b(r) < \frac{a(\varepsilon)}{2}.$$

In the second step, we should note that

$$(10) \quad \dot{V}(t, x_t) \leq -c(t, V(t, x_t)) \text{ as long as } V(t, x_t) \geq p_r^{-1}(t, \tau)$$

and that $p_r^{-1}(t, \tau)$ tends to 0 as $t \rightarrow \infty$.

Let δ_0 and T_1 be such that $\delta_0(\tau) = \delta(\tau, 1)$ and

$$\int_{\sigma}^{\sigma+T_1} c(s, \varepsilon) ds > \eta(\sigma, \tau) - \varepsilon,$$

where $\sigma = p_t^{-1}(\tau, \varepsilon)$ and

$$\eta(\sigma, \tau) \geq \sup \{V(\sigma, \phi); |\phi|_{\mathcal{B}} \leq b(\sigma - \tau) + c(\sigma - \tau)M(\sigma - \tau)\delta_0(\tau)\}.$$

Suppose that for a $t_1 > T + \tau$, $T = T_1 + \sigma - \tau$, we have

$V(t_1, x_{t_1}) \geq \varepsilon$. Clearly,

$$V(t_1, x_{t_1}) > p_r^{-1}(t_1, \tau).$$

Let $t_2 = \max \{\sup \{t < t_1; V(t, x_t) = p_r^{-1}(t, \tau)\}, \tau\}$. Then, by

(10), $V(t, x_t)$ is non-increasing on $[t_2, t_1]$. Hence, we have

$$p_r^{-1}(t_2, \tau) \geq V(t_2, x_{t_2}) \geq V(t_1, x_{t_1}) \geq \epsilon,$$

which implies

$$\tau \geq p(t_2, \epsilon).$$

Therefore, $\sigma \stackrel{\text{def}}{=} p_t^{-1}(\tau, \epsilon) \geq t_2$, that is,

$$\dot{V}(t, x_t) \leq -c(t, V(t, x_t)) \quad \text{and} \quad V(t, x_t) \geq \epsilon \quad \text{for} \quad t \in [\sigma, t_1],$$

and hence we have

$$\begin{aligned} \epsilon \leq V(t_1, x_{t_1}) &\leq V(\sigma, x_\sigma) - \int_{\sigma}^{t_1} c(s, V(s, x_s)) ds \\ &\leq V(\sigma, x_\sigma) - \int_{\sigma}^{t_1} c(s, \epsilon) ds, \end{aligned}$$

which implies

$$\int_{\sigma}^{t_1} c(s, \epsilon) ds \leq \eta(\sigma, \tau) - \epsilon.$$

This contradicts $t_1 > \tau + T(\tau, \epsilon)$.

When $p(t, r) = t - q(r)$, $\sigma = \tau + q(\epsilon)$. Therefore, if

the divergence in (5) is uniformly in t , then we can choose T independent of τ .

Remark 1. It is sufficient that in the Theorem B for each τ there exists a Liapunov function $V(t, \phi; \tau)$ which is defined on $\{(t, x_t); t \geq \tau, x(t) \text{ is continuous on } [\tau, \infty), x_\tau \in \mathcal{B}\}$ and satisfies the conditions (3), (4) with a, c independent of τ , and corresponding to (6) we assume

$$V(t, x_t; \tau) \leq b \left(\sup_{\tau \leq s \leq t} |x_s| \right),$$

because to estimate solutions we can choose different Liapunov function for each solution.

Now, we try to construct a Razumikhin type theorem for the equations (1). Such theorems have been given in [3], [6], [7]. Here, we shall state the following theorem by extending the ideas in [5], [8].

Theorem C. In Theorem B, suppose that $p(t, r)$ is of the form (7).

Then, we can restrict $x(s)$ in (*) within a solution of (1) satisfying

$$(11) \quad V(s, x_s) \leq F(V(t, x_t)) \quad \text{for } s \in [p(t, V(t, x_t)), t],$$

where $F(r)$ is a continuous function such that $F(r) > r$ and $F(r)/r$ is non-decreasing for $r > 0$.

To prove Theorem C, by Remark 1 it is sufficient to construct a Liapunov function for each τ , which satisfies the conditions in Theorem B on $[\tau, \infty)$. The existence of such a Liapunov function follows from the following lemma.

Lemma. Let F be as in Theorem C, and let p be as in Theorem B with $q(t, r) = t - p(t, r)$ which is non-decreasing in t .

If a Liapunov function $V(t, \phi)$ satisfies (3), (4) under the condition (11) and

$$V(t, \phi) \leq b(t, |\phi|_B),$$

then for each τ there exists a Liapunov function $W(t, x_t; \tau)$ which satisfies

$$(12) \quad a(|x(t)|) \leq W(t, x_t; \tau) \leq b^*(t, \tau, \sup_{\tau \leq s \leq t} |x_s|) \quad \text{B}$$

and

$$(13) \quad \dot{W}(t, x_t; \tau) \leq -c^*(t, W(t, x_t; \tau)),$$

if $x(s)$ is a solution of (1) on $[p(t, W(t, x_t; \tau)), t]$, where

$$b^*(t, \tau, r) = \sup_{\tau \leq s \leq t} b(s, r),$$

$$c^*(t, r) = \min \{c(t, r), r\alpha(t, r)\},$$

a, b, c, p, q for V , and

$$\alpha(t, r) = \frac{1}{q(p_t^{-1}(t, F^{-1}(\frac{r}{2})), F^{-1}(\frac{r}{2}))} \log \frac{r}{F^{-1}(r)}.$$

Proof. Define

$$W(t, x_t; \tau) = \sup_{\tau \leq s \leq t} V(s, x_s) e^{\alpha(s, V(s, x_s))(s - t)},$$

and for a fixed $x(s)$ set

$$W(t) = W(t, x_t; \tau), \quad V(t) = V(t, x_t),$$

$$P(s, t) = V(s) e^{\alpha(s, V(s))(s - t)}.$$

Since $\alpha(t, r) > 0$ ($r > 0$), obviously we have (12).

To prove (13), we choose $s(t) \in [\tau, t]$ so that

$$W(t) = P(s(t), t).$$

For small $h > 0$ we may assume that $s(t+h) \rightarrow s(t)$ as $h \rightarrow 0$.

Case 1. $s(t+h) \leq t$ for small $h > 0$. In this case, since $W(t) \geq P(s(t+h), t)$, we have

$$\begin{aligned} \frac{W(t+h) - W(t)}{h} &\leq \frac{P(s(t+h), t+h) - P(s(t+h), t)}{h} \\ &\leq W(t+h) \frac{1}{h} \{1 - e^{\alpha(s(t+h), V(s(t+h)))h}\} \\ &\leq -W(t) \alpha(s(t), V(s(t))) + o(1) \\ &\leq -W(t) \alpha(t, W(t)) + o(1). \end{aligned}$$

Here, we note that $\alpha(t, r)$ is non-decreasing in r , non-increasing in t and that $V(s(t)) \geq W(t)$.

Case 2. $t \leq s(t+h) \leq t+h$ for some arbitrarily small $h > 0$. Then, clearly $s(t) = t$. Therefore,

$$V(t) = W(t) \geq P(s, t) \quad \text{for any } s \leq t.$$

Hence,

$$(14) \quad V(t) \geq V(s) e^{-\alpha(s, V(s))q(t, V(t))} \quad \text{for any } s \in [p(t, V(t)), t].$$

Assume that $x(s)$ is a solution of (1) at least on $[p(t, W(t)), t]$ and, in particular, $\tau \leq p(t, W(t))$.

If we can prove that

$$(15) \quad V(t) \geq F^{-1}\left(\frac{V(s)}{2}\right),$$

immediately we have

$$t \leq p_t^{-1}\left(s, F^{-1}\left(\frac{V(s)}{2}\right)\right) \quad \text{if } s \geq p(t, V(t)),$$

and hence by the definition of $\alpha(t, r)$

$$\alpha(s, V(s))q(t, V(t)) \leq \log \frac{V(s)}{F^{-1}(V(s))},$$

which implies $V(t) \geq F^{-1}(V(s))$, that is,

$$F(V(t)) \geq V(s) \quad \text{for } s \in [p(t, V(t)), t] \quad \text{with (15).}$$

This fact also proves (15) for all $s \in [p(t, V(t)), t]$, and hence we have

$$(16) \quad F(V(t)) \geq V(s) \quad \text{for all } s \in [p(t, V(t)), t].$$

Since $s(t) = t$, we have

$$\frac{W(t+h) - W(t)}{h} = V(s(t+h)) \frac{1}{h} \{e^{\alpha(s(t+h), V(s(t+h)))(s(t+h)-t-h)} - 1\}$$

$$\begin{aligned}
& + \frac{V(s(t+h)) - V(t)}{h} \\
& = V(t)\alpha(t, V(t))\left\{\frac{s(t+h) - t}{h} - 1\right\} + V(t)\frac{s(t+h) - t}{h} \\
& \quad + o(1) \\
& \leq -W(t)\alpha(t, W(t))\left\{1 - \frac{s(t+h) - t}{h}\right\} \\
& \quad - c(t, W(t))\frac{s(t+h) - t}{h} + o(1) \\
& \leq -c^*(t, W(t)) + o(1),
\end{aligned}$$

because V satisfies (4) under (16) and $\frac{s(t+h) - t}{h} \in [0, 1]$.

To complete the proof of Theorem C, it is sufficient to note that if q is independent of t , then so is α and that the property (5) for $c(t, r)$ implies the same property for $c^*(t, r)$.

Remark 2. As is clear from the lemma, for the stability it is sufficient that the property (5) holds for $c^*(t, r)$. In addition to the case given in Theorem C, this is satisfied if c is independent of t and

$$\int_t^{t+T} \frac{ds}{q(p_t^{-1}(s, r), r)} \rightarrow \infty \quad \text{as } T \rightarrow \infty.$$

The asymptotic stability of

$$\dot{x}(t) = -ax(t) + b(t)x(p(t)),$$

$$|b(t)| \leq \beta < a, \quad p(t) = \varepsilon t, \quad 0 < \varepsilon < 1,$$

can be proved as the case.

However, unfortunately the case where

$$p(t) = \frac{\sqrt{1+4t} - 1}{2}$$

is not covered by our result, though the asymptotic stability can be proved by the method in [3].

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