

Burgers equation vs. large N limit in $T\bar{T}$ -deformed $O(N)$ vector model

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Abstract

We study a $T\bar{T}$ -deformed $O(N)$ vector model, which is classically equivalent to the Nambu-Goto action with static gauge. The thermal free energy density can be computed exactly by using the Burgers equation as a special property of $T\bar{T}$ -deformation. The resulting expression is valid for an arbitrary value of N . One may consider a large N limit while preserving this expression. We try to derive this result in the field-theoretical approach directly by employing the large N limit. As a result, the leading contribution coincides with the exact one. That is, the $1/N$ corrections are canceled out through a non-trivial mechanism.

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1. Introduction

Let us consider the theory space of 2D quantum field theories (QFTs), where the existence of the Lagrangian description is assumed. One may consider a flow parameterized by a real constant parameter α , along which the Lagrangian is denoted as $\mathcal{L}^{(\alpha)}$.

In the following, we will discuss a special flow caused by an irrelevant operator

$$\det \left(T_{\mu\nu}^{(\alpha)} \right) \equiv T_{00}^{(\alpha)} T_{11}^{(\alpha)} - (T_{01}^{(\alpha)})^2 \quad (\mu, \nu = 0, 1). \quad (1.1)$$

This is just the determinant of the energy-momentum tensor derived from $\mathcal{L}^{(\alpha)}$ and often called the $T\bar{T}$ -operator. The dimension of this operator is $(\text{length})^{-4}$ in natural units and α has dimension $(\text{length})^2$. Although this is a composite operator, it is well-defined as a local operator as shown by Sasha Zamolodchikov [1]. Furthermore, the expectation value of this operator sandwiched between the excited states is factorized without taking some limits or relying on supersymmetry [1].

The flow caused by the $T\bar{T}$ -operator is called a $T\bar{T}$ -flow [2,3]. The defining relation of the flow is given by

$$\mathcal{L}^{(\alpha+\delta\alpha)} = \mathcal{L}^{(\alpha)} + \delta\alpha \det \left(T_{\mu\nu}^{(\alpha)} \right). \quad (1.2)$$

This relation describes an infinitesimal deviation from $\mathcal{L}^{(\alpha)}$ (a deformed theory), rather than $\mathcal{L}^{(0)}$ (the undeformed theory). It can be rewritten as a differential equation:

$$\frac{\partial}{\partial\alpha} \mathcal{L}^{(\alpha)} = \det \left(T_{\mu\nu}^{(\alpha)} \right). \quad (1.3)$$

This is called the $T\bar{T}$ -flow equation, and the deformation governed by it is called the $T\bar{T}$ -deformation.

As an application of the $T\bar{T}$ -flow equation at classical level, we can derive a deformed Lagrangian $\mathcal{L}^{(\alpha)}$ by solving it directly. For example, if we consider the $T\bar{T}$ -deformation of a free massless scalar field theory, the deformed theory leads to the Nambu-Goto (NG) action (with static gauge) where the deformation parameter α is identified with α' in the string tension $T = 1/(2\pi\alpha')$. Namely, the $T\bar{T}$ -deformation is closely related to string theory and provides a strong motive to study it.

A significant feature of the $T\bar{T}$ -deformation is that the energy spectrum of the deformed theory may be obtained explicitly. If the system is defined on a infinitely-long cylinder, the deformation of the energy spectrum is governed by the inviscid Burgers equation [2,3]:

$$\frac{\partial E_n^{(\alpha)}}{\partial \alpha} = E_n^{(\alpha)} \frac{\partial E_n^{(\alpha)}}{\partial R} + R^{-1} (P_n)^2, \quad (1.4)$$

where $E_n^{(\alpha)}$ and P_n are the energy and momentum of the n -th excited state in the $T\bar{T}$ -deformed theory $\mathcal{L}^{(\alpha)}$ respectively, and R is the circumference of the cylinder. If we know the energy spectrum of the undeformed theory $\mathcal{L}^{(0)}$ such as conformal field theory (CFT) and integrable quantum field theory (IQFT), we can get that of the deformed theory $\mathcal{L}^{(\alpha)}$ by solving the Burgers equation (1.4).

In a massless $O(N)$ vector model, for example, the deformed thermal free energy can be computed as

$$f = \frac{1}{2\alpha} \left(-1 + \sqrt{1 - \frac{2\pi N}{3} \frac{\alpha}{\beta^2}} \right), \quad (1.5)$$

where $\beta := T^{-1}$ is the inverse temperature. This expression is valid for an arbitrary value of N . Note here that one may take a large N limit [4,5]

$$\alpha N = \tilde{\alpha} : \text{fixed}, \quad N \rightarrow \infty, \quad (1.6)$$

preserving the expression of (1.5). It is significant to notice that this result is exact while it may be reproduced as well in the field-theoretical approach directly by employing the large N limit. Our aim here is to try to carry out it.

As another motive to consider the large N limit, it is instructive to make some comments on our previous work [5]. It has been shown there that a $T\bar{T}$ -deformed free $O(N)$ vector model has negative-norm states inevitably (at least) in the large N limit. Hence it would be interesting to see whether this negative norm state would be relevant to the Burgers result or not.

Based on the above motives, we compute the thermal free energy density in the field-theoretical approach by employing the large N limit and compare the result to the exact one obtained by solving the Burgers equation.¹ As a result, we find that the leading contribution in the large N limit coincides with the exact result. Our result indicates that the $1/N$ corrections are canceled out through a non-trivial mechanism.

This note is organized as follows. Section 2 is a brief review of the $T\bar{T}$ -deformed massless $O(N)$ vector model and the derivation of its thermal free energy density through (1.4). In section 3, we compute it in the large N limit. We summarize and discuss the result in Section 4. Appendix A is the derivation of the Hamiltonian for the $T\bar{T}$ -deformed model. Appendix B is devoted to the detailed computation of the effective potential. In Appendix C, we revisit the derivation of the Burgers equation in the operator formalism under some conditions.

2. Review of the $T\bar{T}$ -deformed model

In this section, we shall introduce a free massless $O(N)$ vector model and its $T\bar{T}$ -deformation.

Our starting point is the classical Lagrangian in 2D Euclidean space with the 2D coordinates (x^1, x^2) . The undeformed Lagrangian is given by

¹ This large N limit has also been discussed in [6], where $c = N$ is the total central charge. In this limit, correlation functions in $T\bar{T}$ -deformed CFTs may be computed exactly. For other aspects of correlation functions in $T\bar{T}$ -deformed theories, see [7–9].

$$\mathcal{L} = \frac{1}{2} \delta^{\mu\nu} \partial_\mu \vec{\phi} \cdot \partial_\nu \vec{\phi}, \quad (2.1)$$

where each of ϕ^m ($m = 1, \dots, N$) is a real scalar field and $\delta^{\mu\nu} = \text{diag}(1, 1)$ is the (inversed) 2D Euclidean metric. By solving the flow equation (1.3), the deformed Lagrangian [3,10] is explicitly obtained as

$$\mathcal{L} = \frac{1}{2\alpha} \left(-1 + \sqrt{1 + 2\alpha \left[\left(\partial_1 \vec{\phi} \right)^2 + \left(\partial_2 \vec{\phi} \right)^2 \right] + 4\alpha^2 \left[\left(\partial_1 \vec{\phi} \right)^2 \left(\partial_2 \vec{\phi} \right)^2 - \left(\partial_1 \vec{\phi} \cdot \partial_2 \vec{\phi} \right)^2 \right]} \right), \quad (2.2)$$

where α is a deformation parameter with dimension (length)². This Lagrangian can be identified with the Nambu-Goto (NG) Lagrangian

$$\mathcal{L}_{\text{NG}} = \sqrt{\det(h_{\mu\nu})}, \quad (2.3)$$

with static gauge (up to a constant term). Here $h_{\mu\nu}$ is induced from the target space of the $(N+2)$ -D Euclidean spacetime with the metric δ_{mn} and is defined as

$$h_{\mu\nu} := \delta_{mn} \partial_\mu \phi^m \partial_\nu \phi^n \quad (m, n = 1, \dots, N+2). \quad (2.4)$$

The static gauge is realized as

$$\phi^{N+1} = \frac{1}{\sqrt{2\alpha}} x^1, \quad \phi^{N+2} = \frac{1}{\sqrt{2\alpha}} x^2 \quad (2.5)$$

and then $h_{\mu\nu}$ is evaluated as

$$h_{\mu\nu} = \frac{1}{2\alpha} \delta_{\mu\nu} + \partial_\mu \vec{\phi} \cdot \partial_\nu \vec{\phi}. \quad (2.6)$$

In this expression, the fields $\vec{\phi}$ describe classical fluctuations around 2D flat subspace in $(N+2)$ -D target space.

The derivation of the thermal free energy density with the Burgers equation is the following. There are two points to note. First, with the inverse temperature $\beta = T^{-1}$, the thermal free energy density $f(\beta)$ on flat space is computed from the ground-state energy $E_0(R)$ on the cylinder with the circumference R through the relation [11]

$$f(\beta) = \frac{E_0(R)}{R} \Big|_{R=\beta}. \quad (2.7)$$

Second, the $T\bar{T}$ -deformed energy spectrum on the cylinder can be computed by solving the Burgers equation (1.4).

For the undeformed theory (2.1) on the cylinder, the ground-state energy $E_0^{(\text{free})}$ is just the Casimir energy and is given by

$$\frac{E_0^{(\text{free})}(R)}{R} = -\frac{\pi N}{6R^2}. \quad (2.8)$$

By taking it as the initial value to solve the Burgers equation (1.4), we obtain the ground-state energy of the $T\bar{T}$ -deformed theory [2,3]:

$$\frac{E_0^{(\alpha)}(R)}{R} = \frac{1}{2\alpha} \left(-1 + \sqrt{1 - \frac{2\pi N}{3} \frac{\alpha}{R^2}} \right). \quad (2.9)$$

Here we have used the fact that the total momentum of the ground state vanishes for any finite value of α . Thus, the formula (2.7) leads to the thermal free energy density:

$$f = \frac{N}{2\tilde{\alpha}} \left(-1 + \sqrt{1 - \frac{2\pi}{3} \frac{\tilde{\alpha}}{\beta^2}} \right). \quad (2.10)$$

In the last equality, we have introduced a new quantity

$$\tilde{\alpha} \equiv N\alpha \quad (2.11)$$

for later convenience in considering a large N limit.

3. Free energy density in large N

In this section, let us derive the thermal free energy density in the field-theoretical approach by employing a certain large N limit.

3.1. Derivation

We start from the following partition function²

$$Z = \int D\vec{\phi} D\vec{\pi} \exp \left(\int_0^\beta dx^2 \int_{\mathbb{R}} dx^1 \left\{ i\vec{\pi} \cdot \partial_2 \vec{\phi} - \mathcal{H} \right\} \right). \quad (3.1)$$

As usual, the fields have periodicity of β in the imaginary-time direction x^2 , and the model is essentially defined on the cylinder. The Hamiltonian is derived as³

$$\mathcal{H} = \frac{1}{2\alpha} \left[-1 + \sqrt{1 + 2\alpha \left(\partial_1 \vec{\phi} \right)^2 + 2\alpha \vec{\pi}^2 + 4\alpha^2 \left(\vec{\pi} \cdot \partial_1 \vec{\phi} \right)^2} \right], \quad (3.2)$$

where $\vec{\pi}$ is the conjugate momentum of $\vec{\phi}$.

To evaluate the partition function, we consider here a large N limit [4,5]:

$$\alpha N = \tilde{\alpha} : \text{fixed}, \quad N \rightarrow \infty. \quad (3.3)$$

As shown later, $\vec{\phi}$ and $\vec{\pi}$ can formally be integrated out by introducing auxiliary fields. Then, in the large N limit (3.3), the integrals of the auxiliary fields can also be evaluated perturbatively with respect to $1/N$.

Let us now calculate the thermal partition function. In the following, we will consider the $\alpha > 0$ case based on the physical ground. By rescaling the fields as

$$\sqrt{2\alpha} \vec{\pi} \rightarrow \vec{\pi}, \quad \sqrt{2\alpha} \vec{\phi} \rightarrow \vec{\phi}, \quad (3.4)$$

the partition function is rewritten as

² Note that we cannot start naively from the Lagrangian because the Lagrangian (2.2) includes higher-order terms of $\partial_0 \vec{\phi}$ in the Lorentzian formulation. This point is also mentioned in a recent work [12].

³ For the derivation of this Hamiltonian, see Appendix A.

$$Z = \int D\vec{\phi} D\vec{\pi} \exp\left(\frac{N}{2\tilde{\alpha}} \int d^2x \left[i\vec{\pi} \cdot \partial_2 \vec{\phi} + 1 - \sqrt{1 + \left(\partial_1 \vec{\phi}\right)^2 + \vec{\pi}^2 + \left(\vec{\pi} \cdot \partial_1 \vec{\phi}\right)^2} \right]\right), \quad (3.5)$$

where we have introduced a shorthand notation $\int d^2x$ for $\int_0^\beta dx^2 \int_{\mathbb{R}} dx^1$. Here one might worry if there may be a subtlety concerned with the rescaling (3.4) because α is sent to zero so as to keep αN finite. Then it may seem that the scaling of the field should be taken into account in the path integral in some manner. However, the range of the path integral is not changed and hence the field configurations with all multiplicative scales has already been taken into account. In addition, the extra constant Jacobian has no physical contribution. Therefore, the rescaling (3.4) has no problem.

Then, let us introduce auxiliary fields λ and ρ . By inserting the identity

$$\begin{aligned} 1 &= \int D\rho \prod_x \delta\left(\left(\partial_1 \vec{\phi}(x)\right)^2 + \vec{\pi}(x)^2 + \left(\vec{\pi}(x) \cdot \partial_1 \vec{\phi}(x)\right)^2 - \rho(x)\right) \\ &= \int D\lambda D\rho \exp\left(\frac{iN}{2\tilde{\alpha}} \int d^2x \lambda \left[(\partial_1 \vec{\phi})^2 + \vec{\pi}^2 + (\vec{\pi} \cdot \partial_1 \vec{\phi})^2 - \rho\right]\right) \end{aligned} \quad (3.6)$$

to the partition function (3.5), we obtain the following expression:

$$\begin{aligned} Z &= \int D\vec{\phi} D\vec{\pi} D\lambda D\rho \exp\left(\frac{N}{2\tilde{\alpha}} \int d^2x \left[i\vec{\pi} \cdot \partial_2 \vec{\phi} + 1 - \sqrt{1 + \rho} \right. \right. \\ &\quad \left. \left. + i\lambda \left[(\partial_1 \vec{\phi})^2 + \vec{\pi}^2 + (\vec{\pi} \cdot \partial_1 \vec{\phi})^2 - \rho\right] \right]\right) \\ &= \int D\vec{\phi} D\vec{\pi} D\lambda D\rho D\sigma \exp\left(\frac{N}{2\tilde{\alpha}} \int d^2x \left[i\vec{\pi} \cdot \partial_2 \vec{\phi} + 1 - \sqrt{1 + \rho} \right. \right. \\ &\quad \left. \left. + i\lambda \left[(\partial_1 \vec{\phi})^2 + \vec{\pi}^2 + 2\sigma \vec{\pi} \cdot \partial_1 \vec{\phi} - \sigma^2 - \rho\right] \right] + \frac{1}{2} \text{Tr} \log \lambda \right). \end{aligned} \quad (3.7)$$

In the first equality, we have replaced $(\partial_1 \vec{\phi})^2 + \vec{\pi}^2 + (\vec{\pi} \cdot \partial_1 \vec{\phi})^2$ with ρ in the square root by using the delta function. In the second equality, we have used the Gaussian integral formula

$$\begin{aligned} &\exp\left(\frac{iN}{2\tilde{\alpha}} \int d^2x \lambda (\vec{\pi} \cdot \partial_1 \vec{\phi})^2\right) \\ &\propto \int D\sigma \exp\left(\frac{1}{2} \text{Tr} \log \lambda + \frac{iN}{2\tilde{\alpha}} \int d^2x \lambda (-\sigma^2 + 2\sigma \vec{\pi} \cdot \partial_1 \vec{\phi})\right) \end{aligned} \quad (3.8)$$

to rewrite the four-point interaction of $(\vec{\pi} \cdot \partial_1 \vec{\phi})^2$. Note here that the symbol “Tr” denotes the trace over the functional space. Since the argument of the exponential in (3.7) is a quadratic form of $\vec{\pi}, \vec{\phi}$, we can integrate out them. The resulting form is given by

$$Z = \int D\lambda D\rho D\sigma \exp(-S_{\text{eff}}) \quad (3.9)$$

with the effective action S_{eff} for λ, ρ and σ of the following form:

$$\begin{aligned} \frac{S_{\text{eff}}}{N} &:= \frac{1}{2} \text{Tr} \log \left[\lambda \partial_1 \lambda \partial_1 - \frac{1}{4} (\partial_2 + 2\partial_1 \lambda \sigma) (\partial_2 + 2\lambda \sigma \partial_1) \right] \\ &\quad - \frac{1}{2\tilde{\alpha}} \int d^2x \left(1 - \sqrt{1 + \rho} - i\lambda (\sigma^2 + \rho) \right) - \frac{i}{2N} \text{Tr} \log \lambda. \end{aligned} \quad (3.10)$$

Note here that this expression is valid for finite values of N .

Then, let us consider the large N limit (3.3) and evaluate the leading-order contribution. In this limit, the stationary point of S_{eff} dominates in the integrals over the auxiliary fields. The stationary conditions are

$$\frac{\delta S_{\text{eff}}}{\delta \lambda(x)} = \frac{\delta S_{\text{eff}}}{\delta \rho(x)} = \frac{\delta S_{\text{eff}}}{\delta \sigma(x)} = 0. \quad (3.11)$$

Because λ , ρ and σ are constant due to translational invariance of the system, it suffices to focus on the effective potential:

$$\frac{V_{\text{eff}}}{N} = -\frac{\pi}{6\beta^2} \frac{1}{2|\lambda||1-\sigma^2|} - \frac{1}{2\tilde{\alpha}} \left(1 - \sqrt{1+\rho} - i\lambda(\sigma^2 + \rho)\right). \quad (3.12)$$

For the detailed derivation of (3.12), see Appendix B. Note that the third term in the right-hand side of (3.10) vanishes at the leading in large N .

The stationary conditions for constant λ , ρ and σ

$$\frac{\partial V_{\text{eff}}}{\partial \lambda} = \frac{\partial V_{\text{eff}}}{\partial \rho} = \frac{\partial V_{\text{eff}}}{\partial \sigma} = 0 \quad (3.13)$$

are solved as

$$i\lambda = -\frac{1}{2} \left(1 - \frac{2\pi}{3} \frac{\tilde{\alpha}}{\beta^2}\right)^{\frac{1}{2}}, \quad \sigma = 0, \quad \rho = \left(1 - \left(\frac{2\pi}{3} \frac{\tilde{\alpha}}{\beta^2}\right)^{-1}\right)^{-1}. \quad (3.14)$$

The value of the effective action at the stationary point is evaluated as

$$\begin{aligned} \left. \frac{S_{\text{eff}}}{N} \right|_{\text{stat pt, large } N} &= \frac{1}{N} \int d^2x V_{\text{eff}} \Big|_{\text{stat pt, large } N} \\ &= \int d^2x \frac{1}{2\tilde{\alpha}} \left(-1 + \sqrt{1 - \frac{2\pi}{3} \frac{\tilde{\alpha}}{\beta^2}}\right). \end{aligned} \quad (3.15)$$

Then the leading contribution to the free energy density is given by

$$f = \frac{N}{2\tilde{\alpha}} \left(-1 + \sqrt{1 - \frac{2\pi}{3} \frac{\tilde{\alpha}}{\beta^2}}\right) \quad (3.16)$$

and *does* coincide with the exact result (2.10).

3.2. Miscellaneous comments

Let us make some comments on our result as follows.

Negative α

So far, we have supposed that the deformation parameter α is positive. But even for the negative α , one can derive the same results by following almost the same manner in the previous subsection.

When α is negative, we should rescale the fields as

$$\sqrt{-2\alpha}\vec{\phi} \rightarrow \vec{\phi}, \quad \sqrt{-2\alpha}\vec{\pi} \rightarrow \vec{\pi}, \quad (3.17)$$

instead of Eq. (3.4). By using the following identity

$$1 = \int D\lambda D\rho \exp\left(\frac{iN}{2\tilde{\alpha}} \int d^2x \lambda [-(\partial_1 \vec{\phi})^2 - \vec{\pi}^2 + (\vec{\pi} \cdot \partial_1 \vec{\phi})^2 - \rho]\right), \quad (3.18)$$

the resulting effective action is given by

$$\begin{aligned} \frac{S_{\text{eff}}}{N} := & \frac{1}{2} \text{Tr} \log \left[\lambda \partial_1 \lambda \partial_1 - \frac{1}{4} (\partial_2 - 2\partial_1 \lambda \sigma) (\partial_2 - 2\lambda \sigma \partial_1) \right] \\ & - \frac{1}{2\tilde{\alpha}} \int d^2x (1 - \sqrt{1 + \rho} - i\lambda(\sigma^2 + \rho)) - \frac{i}{2N} \text{Tr} \log \lambda. \end{aligned} \quad (3.19)$$

The difference from Eq. (3.10) is just the sign of ∂_1 , which can be absorbed into the redefinition of σ . As a result, we get the same effective potential as in (3.12). Hence the leading contribution is valid even for the negative α as well.

Microcanonical entropy

The microcanonical entropy density is calculated by the Legendre transformation of the free energy density with respect to the temperature. It is given by

$$s := \min_T \left(\frac{e - f}{T} \right) = \sqrt{\frac{2\pi N}{3}} e \left(1 + \frac{\tilde{\alpha}}{N} e \right) = \sqrt{\frac{2\pi N}{3}} e (1 + \alpha e), \quad (3.20)$$

where e is the internal energy density. In the high energy limit ($e \rightarrow \infty$), s behaves as

$$s \sim \sqrt{\frac{2\pi N}{3}} \alpha e = \beta_H e, \quad \beta_H = (T_H)^{-1} := \left(\frac{2\pi N \alpha}{3} \right)^{-\frac{1}{2}}, \quad (3.21)$$

where T_H is the Hagedorn temperature [4]. This result can also be obtained from the Cardy formula [13].

Since the microcanonical entropy density counts the number of energy eigenstates with energy e , some information on the negative-norm states [5] may be encoded into it. However, it seems likely that their contributions do not appear in (3.20). This issue is a puzzle to be resolved in the future work.

Derivation of Burgers equation

As a possibility, one might suspect that the Burgers equation is not exact. In the original derivation [1], a physical interpretation of the expectation value of T_{11} with excited states is used and might make a loophole. In Appendix C, we have proven that the Burgers equation is valid without using the physical interpretation by supposing that 1) the $T\bar{T}$ operator is well-defined as a local operator, 2) the expectation value of the $T\bar{T}$ operator is factorized and 3) the Hamiltonian consists only of mutually commutative operators. The third condition is very strong and the situation becomes restrictive. But the $T\bar{T}$ -deformed massless $O(N)$ vector model is included in this analysis and the argument there is a bit general. Hence, this possibility has been excluded in this class of theories.

UV divergences

It is also helpful to make a comment on the ultraviolet divergences. Recall that we have not explicitly encountered them in the calculation in Section 3. This is just because we have implicitly tuned the cosmological constant so that the free energy should be zero at the zero-temperature limit ($\beta \rightarrow \infty$).

4. Conclusion and discussion

In this note, we have discussed the thermal free energy density in a $T\bar{T}$ -deformed $O(N)$ vector model. It can be computed by using the Burgers equation and the resulting expression is valid for an arbitrary value of N . Since we may take a large N limit so as to keep this expression, we have tried to derive this result in the field-theoretical approach directly by employing the large N limit. As a result, the leading contribution coincides with the exact one.

What would have happened about the sub-leading contributions to the large N result (3.16)? This result indicates that the large N calculation gives rise to the exact result and there should be a non-trivial cancellation mechanism in the sub-leading corrections. The cancellation may come from the integrability of the $O(N)$ vector model preserved under the $T\bar{T}$ -deformation. As another possibility, beyond the integrability, it may be related to the one-loop exactness discussed by Dubovsky et al. [14] in the context of the Jackiw-Teitelboim (JT) gravity [15,16]. It is a significant future problem to reveal this cancellation mechanism explicitly.

Although the massless $O(N)$ vector model has been studied so far, one may consider the massive case. Then, the undeformed theory is a free massive theory and there is no conformal symmetry (though the $T\bar{T}$ -deformed theory is still integrable). Hence the resulting entropy is not governed by the Cardy formula and it would enable us to see more apparently the relation between the thermodynamic property and integrability.

One may also consider a generalization to higher-dimensional cases with a higher-dimensional version of $T\bar{T}$ -operator [10,17–19]. It is curious to study whether the non-trivial cancellation would occur even in higher dimensions or not. It is also nice to try to interpret our result from the bulk point of view in the context of AdS/CFT.

We hope that our result would shed light on the mysterious property of the $T\bar{T}$ -deformation and find an interesting application in String Theory.

CRediT authorship contribution statement

Junichi Haruna: Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **Katsuta Sakai:** Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **Kentaroh Yoshida:** Conceptualization, Formal analysis, Supervision, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Derivation of the Hamiltonian

In Section 3.1, we had to start from the Hamiltonian (3.2) in the path-integral formulation. But in the main text of the present note, we have entirely worked in the Euclidean signature, while in deriving the Hamiltonian (3.2) we need to begin in the Lorentzian formulation so as to follow the standard manner. Hence our purpose here is to derive the Hamiltonian (3.2) by starting in the Lorentzian formulation with the coordinates $x^\mu = (x^0, x^1)$ and the metric $\eta_{\mu\nu} = \text{diag}(-1, +1)$.

The deformed Lagrangian is given by

$$\mathcal{L} = \frac{1}{2\alpha} \left(1 - \sqrt{1 - 2\alpha \left[(\partial_0 \vec{\phi})^2 - (\partial_1 \vec{\phi})^2 \right] - 4\alpha^2 \left[(\partial_0 \vec{\phi})^2 (\partial_1 \vec{\phi})^2 - (\partial_0 \vec{\phi} \cdot \partial_1 \vec{\phi})^2 \right]} \right). \quad (\text{A.1})$$

Then, the conjugate momentum for $\vec{\phi}$ is

$$\begin{aligned} \vec{\pi} &:= \frac{\partial \mathcal{L}}{\partial (\partial_0 \vec{\phi})} \\ &= \frac{\partial_0 \vec{\phi} + 2\alpha \left[(\partial_1 \vec{\phi})^2 \partial_0 \vec{\phi} - (\partial_0 \vec{\phi} \cdot \partial_1 \vec{\phi}) \partial_1 \vec{\phi} \right]}{\sqrt{1 - 2\alpha \left[(\partial_0 \vec{\phi})^2 - (\partial_1 \vec{\phi})^2 \right] - 4\alpha^2 \left[(\partial_0 \vec{\phi})^2 (\partial_1 \vec{\phi})^2 - (\partial_0 \vec{\phi} \cdot \partial_1 \vec{\phi})^2 \right]}}. \end{aligned} \quad (\text{A.2})$$

By solving it with respect to $\partial_0 \vec{\phi}$, we obtain

$$\partial_0 \vec{\phi} = \frac{\vec{\pi} + 2\alpha (\vec{\pi} \cdot \partial_1 \vec{\phi}) \partial_1 \vec{\phi}}{\sqrt{1 + 2\alpha (\partial_1 \vec{\phi})^2 + 2\alpha \vec{\pi}^2 + 4\alpha^2 (\vec{\pi} \cdot \partial_1 \vec{\phi})^2}} \quad (\text{A.3})$$

and the Hamiltonian (3.2) is expressed as

$$\begin{aligned} \mathcal{H} &:= \vec{\pi} \cdot \partial_0 \vec{\phi} - \mathcal{L} \Big|_{\partial_0 \vec{\phi} = \partial_0 \vec{\phi}(\vec{\phi}, \vec{\pi})} \\ &= \frac{1}{2\alpha} \left[-1 + \sqrt{1 + 2\alpha (\partial_1 \vec{\phi})^2 + 2\alpha \vec{\pi}^2 + 4\alpha^2 (\vec{\pi} \cdot \partial_1 \vec{\phi})^2} \right]. \end{aligned}$$

Let us discuss the relation between the signature of α and stability of the system. When α is positive, $\mathcal{H}$ is real for any $\partial_1 \vec{\phi}$ and $\vec{\pi}$, and is bounded from below. On the other hand, when α is negative, $\mathcal{H}$ may be complex because the inside of the square root becomes negative for some configurations. Even for the configurations for which $\mathcal{H}$ is real, $\mathcal{H}$ has no lower bound and the system becomes unstable. Therefore, α should be positive in order to obtain a consistent quantum theory.

So far, we have discussed at classical level. Let us make a comment on the quantization of the Hamiltonian in the operator formalism. Since the Hamiltonian includes $\partial_1 \vec{\phi}$ and $\vec{\pi}$, one may worry about the ordering problem. However, these commute with each other at the identical point because we can show the following relation:

$$\begin{aligned} \left[\partial_1 \phi_i(x^0, x^1), \pi_j(x^0, y^1) \right] &= \lim_{y^1 \rightarrow x^1} \left[\partial_1 \phi_i(x^0, x^1), \pi_j(x^0, y^1) \right] \\ &= \lim_{y^1 \rightarrow x^1} \delta_{ij} \partial_1 \delta(x^1 - y^1). \end{aligned} \quad (\text{A.4})$$

Since $\partial_1 \delta(x^1 - y^1)$ is an odd distribution, the limit must vanish. Therefore there is no ordering problem. Note that this is a special property in the *massless* case. If we consider the massive

case, $\vec{\phi}$ appears in the Hamiltonian and it causes the ordering problem. It is also obvious that the Hamiltonian is hermite and hence one may use the path-integral formalism as usual.

Appendix B. Calculation of the effective potential

In this appendix, we compute the effective potential on an infinitely-long cylinder with the circumference β .

For later use, we first evaluate the following quantity:

$$\text{tr} \log(-\partial_a c^{ab} \partial_b), \quad (\text{B.1})$$

where c^{ab} is a constant symmetric tensor which satisfies $\det c^{ab} > 0$. Let us employ a combined symbol for the integral and sum over the momentum space:

$$\begin{aligned} \frac{1}{\text{Vol}} \text{tr} f &= \frac{1}{\text{Vol}} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{1}{\beta} \langle k_1, k_2 | f | k_1, k_2 \rangle \\ &\equiv \oint_k f(k_1, k_2), \end{aligned} \quad (\text{B.2})$$

where k_2 denotes $2\pi n/\beta$ and “Vol” is the coordinate space volume. The quantity (B.1) is then written as

$$\begin{aligned} \frac{1}{\text{Vol}} \text{tr} \log(-\partial_a c^{ab} \partial_b) &= \oint_k \log(c^{ab} k_a k_b) \\ &= \oint_k \log(c^{11} k_1^2 + 2c^{12} k_1 k_2 + c^{22} k_2^2) \\ &= \oint_k \log(c^{11}) + \oint_k \log\left(k_1^2 + 2\frac{c^{12}}{c^{11}} k_1 k_2 + \frac{c^{22}}{c^{11}} k_2^2\right) \end{aligned} \quad (\text{B.3})$$

The first term is a constant and is to be subtracted by choosing an appropriate measure of the path integral as in the ordinary field theories. On the other hand, the second term in the last expression of (B.3) can be evaluated just as the Casimir energy on a cylinder. By introducing a new quantity

$$R := |c^{11}| \beta / \sqrt{\det c^{ab}}, \quad (\text{B.4})$$

it can be evaluated as

$$\begin{aligned} \oint_k \log(k_1^2 + 2\frac{c^{12}}{c^{11}} k_1 k_2 + \frac{c^{22}}{c^{11}} k_2^2) &= \oint_k \log\left[\left(k_1 + \frac{c^{12}}{c^{11}} k_2\right)^2 + \frac{\det c^{ab}}{(c^{11})^2} k_2^2\right] \\ &= \frac{|c^{11}|}{\sqrt{\det c^{ab}}} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \frac{1}{R} \sum_{n \in \mathbb{Z}} \log\left[k_1^2 + \left(\frac{2\pi}{R} n\right)^2\right] \\ &\sim \frac{|c^{11}|}{\sqrt{\det c^{ab}}} \left(-\frac{\pi}{3R^2}\right) \\ &= -\frac{\sqrt{\det c^{ab}}}{|c^{11}|} \frac{\pi}{3\beta^2}. \end{aligned} \quad (\text{B.5})$$

We have again subtracted a divergent constant, which also corresponds to choosing the measure of the path integral or zeta-function regularization.

In summary, we have obtained the following formula:

$$\text{tr log}(-\partial_a c^{ab} \partial_b) = \text{Vol} \times \left(-\frac{\sqrt{\det c^{ab}}}{|c^{11}|} \frac{\pi}{3\beta^2} \right). \quad (\text{B.6})$$

Using the above formula (B.6), we can readily evaluate Eq. (3.10). When the auxiliary fields λ , σ and ρ are constant, the first term in (3.10) can be computed as

$$\begin{aligned} & \frac{1}{2} \text{tr log} \left[\lambda \partial_1 \lambda \partial_1 - \frac{1}{4} (\partial_2 + 2\partial_1 \lambda \sigma) (\partial_2 + 2\lambda \sigma \partial_1) \right] \\ &= \frac{1}{2} \int d^2 x \sum_k \log \left[-\lambda^2 (1 - \sigma^2) k_1^2 + \lambda \sigma k_2 k_1 + \frac{1}{4} k_2^2 \right] \\ &= \int d^2 x \left(-\frac{\pi}{6\beta^2} \frac{1}{2|\lambda| |1 - \sigma^2|} \right). \end{aligned} \quad (\text{B.7})$$

In the above computation, we have identified the components of c^{ab} as

$$c^{11} = -\lambda^2 (1 - \sigma^2), \quad c^{21} = c^{12} = \lambda \sigma / 2, \quad c^{22} = 1/4. \quad (\text{B.8})$$

Thus we have derived the effective potential (3.12).

Appendix C. Revisiting the derivation of the Burgers equation

In this Appendix, we will revisit the derivation of the Burgers equation in the operator formalism under some conditions.

C.1. Response under parameter variation

Here, we will discuss a general system defined on a $(d - 1)$ dimensional space with the metric

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + h_{ij}(x) dx^i dx^j \quad (\text{C.1})$$

and with some parameter γ . Suppose here that γ is irrelevant to the space volume $\int d^{d-1}x$.

We will show that the following relation holds:

$$\frac{d}{d\gamma} H = - \frac{d}{d\gamma} L \Big|_{\partial_0 \vec{\phi} = \partial_0 \vec{\phi}(\vec{\phi}, \vec{\Pi})}, \quad (\text{C.2})$$

$$L = \int d^{d-1}x \sqrt{h} \mathcal{L}^{(\gamma, h)}. \quad (\text{C.3})$$

Here $\vec{\Pi}$ denotes the conjugate momentum for $\vec{\phi}$ that is defined as a scalar density:

$$\vec{\Pi} := \frac{\partial}{\partial(\partial_0 \vec{\phi})} (\sqrt{h} \mathcal{L}^{(\gamma, h)}), \quad (\text{C.4})$$

and $\partial_0 \vec{\phi}(\vec{\phi}, \vec{\Pi})$ is the solution to Eq. (C.4) as a function of $\vec{\phi}$ and $\vec{\Pi}$. The latter implicitly depends on γ and h_{ij} . The associated classical Poisson structure and Hamiltonian are defined respectively as

$$\{\phi_n(t, x), \Pi_m(t, y)\} = \delta_{nm} \delta^{d-1}(x - y), \quad (\text{C.5})$$

$$H := \int d^{d-1}x \left(\vec{\Pi} \cdot \partial_0 \vec{\phi} - \sqrt{h} \mathcal{L}^{(\gamma, h)} \right) \Big|_{\partial_0 \vec{\phi} = \partial_0 \vec{\phi}(\vec{\phi}, \vec{\Pi})}. \quad (\text{C.6})$$

Note that Eq. (C.5) is independent of $h_{ij}(x)$.

Then we can easily prove the relation (C.2) like

$$\begin{aligned} \frac{d}{d\gamma} H &= \int d^{d-1}x \frac{d}{d\gamma} \left[\vec{\Pi} \cdot \partial_0 \vec{\phi} - \sqrt{h} \mathcal{L}^{(\gamma, h)} \right] \Big|_{\partial_0 \vec{\phi} = \partial_0 \vec{\phi}(\vec{\phi}, \vec{\Pi})} \\ &= \int d^{d-1}x \left(\vec{\Pi} \cdot \frac{d(\partial_0 \vec{\phi})}{d\gamma} - \left[\frac{d(\partial_0 \vec{\phi})}{d\gamma} \cdot \frac{\partial}{(\partial_0 \vec{\phi})} + \frac{\partial}{\partial \gamma} \right] \sqrt{h} \mathcal{L}^{(\gamma, h)} \right) \Big|_{\partial_0 \vec{\phi} = \partial_0 \vec{\phi}(\vec{\phi}, \vec{\Pi})} \\ &= \int d^{d-1}x \left(\vec{\Pi} \cdot \frac{d(\partial_0 \vec{\phi})}{d\gamma} - \frac{d(\partial_0 \vec{\phi})}{d\gamma} \cdot \vec{\Pi} - \frac{\partial}{\partial \gamma} (\sqrt{h} \mathcal{L}^{(\gamma, h)}) \right) \Big|_{\partial_0 \vec{\phi} = \partial_0 \vec{\phi}(\vec{\phi}, \vec{\Pi})} \\ &= - \int d^{d-1}x \frac{\partial}{\partial \gamma} (\sqrt{h} \mathcal{L}^{(\gamma, h)}) \Big|_{\partial_0 \vec{\phi} = \partial_0 \vec{\phi}(\vec{\phi}, \vec{\Pi})} \\ &= - \frac{d}{d\gamma} L \Big|_{\partial_0 \vec{\phi} = \partial_0 \vec{\phi}(\vec{\phi}, \vec{\Pi})}. \end{aligned} \quad (\text{C.7})$$

Therefore, in the context of $T\bar{T}$ -deformation, we find that

$$\frac{dH}{d\alpha} = - \frac{dL}{d\alpha} \Big|_{\partial_0 \vec{\phi} = \partial_0 \vec{\phi}(\vec{\phi}, \vec{\Pi})} \quad (\text{C.8})$$

at classical level.

It is remarkable that one may also consider a variation with $h_{ij}(x)$ as above. It results in the following relation:

$$\begin{aligned} \delta H &= -\delta L \Big|_{\partial_0 \vec{\phi} = \partial_0 \vec{\phi}(\vec{\phi}, \vec{\Pi})} \\ &= -\frac{1}{2} \int d^{d-1}x \sqrt{h} T^{ij} \delta h_{ij}, \end{aligned} \quad (\text{C.9})$$

where $T^{ij}(x)$ is the energy-momentum tensor.

Finally, note that the two relations (C.2) and (C.9) can easily be promoted to the quantum level in some special cases where the Hamiltonian and Lagrangian consist of commutative operators $\vec{\Pi}$ and $\partial_1 \vec{\phi}$, including our present case, a $T\bar{T}$ -deformed massless $O(N)$ vector model. Then one can formally take derivatives with respect to them (in particular, one can use the chain rule in the second term in the second line in Eq. (C.7)).

C.2. Derivation of the Burgers equation

Let us next derive the Burgers equation for a $T\bar{T}$ -deformed theory on a cylinder with circumference R .

The coordinate for the circumference direction is represented by x^1 with the normalized period:

$$x^1 \sim x^1 + 1. \quad (\text{C.10})$$

In turn, the metric $g_{\mu\nu}$ includes the information of the circumference:

$$\begin{aligned} g_{\mu\nu} &= \text{diag}(-1, R^2), \\ h_{11} &= R^2. \end{aligned} \quad (\text{C.11})$$

Note that in this coordinate system, the $T\bar{T}$ -flow equation is written as

$$\frac{\partial}{\partial\alpha} \mathcal{L}^{(\alpha)} = R^{-2} \det(T_{\mu\nu}^{(\alpha)}). \quad (\text{C.12})$$

In the following, we will suppose the conditions that 1) the $T\bar{T}$ operator is well-defined as a local operator, 2) the expectation value of the $T\bar{T}$ operator is factorized and 3) the Hamiltonian consists only of mutually commutative operators. Then we will discuss at quantum level.

The quantum version of (C.8) is given by

$$\frac{d\hat{H}^{(\alpha)}}{d\alpha} = - \int dx R^{-1} \det(\hat{T}_{\mu\nu}^{(\alpha)}). \quad (\text{C.13})$$

By taking the expectation value of both sides of (C.13) in the n -th excited state $|n\rangle$. Now notice that the following relation holds:

$$\left\langle \frac{d\hat{H}^{(\alpha)}}{d\alpha} \right\rangle_n = R^{-1} \left\langle \det(\hat{T}_{\mu\nu}^{(\alpha)}) \right\rangle_n = -R^{-1} \left(\left\langle \hat{T}_{00}^{(\alpha)} \right\rangle_n \left\langle \hat{T}_{11}^{(\alpha)} \right\rangle_n - \left\langle \hat{T}_{01}^{(\alpha)} \right\rangle_n^2 \right), \quad (\text{C.14})$$

where we have defined $\langle \mathcal{O} \rangle_n$ by $\langle n | \mathcal{O} | n \rangle$. We have used the translational symmetry of the eigenstates and the factorization property of $\det \hat{T}_{\mu\nu}^{(\alpha)}$ [1].

Let us evaluate the right-hand side of (C.14). Here, each state is an eigenstate simultaneously for the Hamiltonian and the total momentum:

$$\hat{H}^{(\alpha)} |n\rangle = E_n^{(\alpha)} |n\rangle \quad \left(\hat{H}^{(\alpha)} := \int dx R \hat{T}_{00}^{(\alpha)} \right), \quad (\text{C.15})$$

$$\hat{P}^{(\alpha)} |n\rangle = P_n |n\rangle \quad \left(\hat{P}^{(\alpha)} := \int dx \hat{T}_{01}^{(\alpha)} \right), \quad (\text{C.16})$$

and hence we obtain

$$\left\langle \hat{T}_{00}^{(\alpha)} \right\rangle_n = \frac{E_n^{(\alpha)}}{R}, \quad \left\langle \hat{T}_{01}^{(\alpha)} \right\rangle_n = \left\langle \hat{T}_{10}^{(\alpha)} \right\rangle_n = P_n. \quad (\text{C.17})$$

Note here that P_n is not changed under the $T\bar{T}$ -deformation because it is determined by the winding number. As for $\left\langle \hat{T}_{11}^{(\alpha)} \right\rangle_n$, the differentiation of the Hamiltonian by R yields to

$$\begin{aligned} \frac{dH^{(\alpha)}}{dR} &= \frac{d(R^2)}{dR} \frac{d\hat{H}^{(\alpha)}}{d(R^2)} \\ &= 2R \frac{d\hat{H}^{(\alpha)}}{dh_{11}} \\ &= -2R \int_0^1 dx^1 \frac{R}{2} \hat{T}^{(\alpha)11} \end{aligned}$$

$$= -R^{-2} \int_0^1 dx^1 \hat{T}_{11}^{(\alpha)}. \quad (\text{C.18})$$

By taking the expectation value of both sides, we get⁴

$$\langle \hat{T}_{11}^{(\alpha)} \rangle = -R^2 \frac{dE_n^{(\alpha)}}{dR}. \quad (\text{C.19})$$

On the other hand, the left-hand side of (C.14) can be obviously evaluated as

$$\left\langle \frac{d\hat{H}^{(\alpha)}}{d\alpha} \right\rangle_n = \frac{\partial E_n^{(\alpha)}}{\partial \alpha}. \quad (\text{C.20})$$

By substituting (C.17), (C.19) and (C.20) into (C.14), we obtain the Burgers equation:

$$\frac{\partial E_n^{(\alpha)}}{\partial \alpha} = E_n^{(\alpha)} \frac{\partial E_n^{(\alpha)}}{\partial R} + R^{-1} (P_n)^2. \quad (\text{C.21})$$

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⁴ We have utilized the Hellmann-Feynman theorem.