

On the Strong Novikov Conjecture of Locally Compact Groups for Low Degree Cohomology Classes

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Abstract

The main result of this paper is non-vanishing of the image of the index map from the G -equivariant K -homology of a proper G -compact G -manifold X to the K -theory of the C^* -algebra of the group G . Under the assumption that the Kronecker pairing of a K -homology class with a low-dimensional cohomology class is non-zero, we prove that the image of this class under the index map is non-zero. Neither discreteness of the locally compact group G nor freeness of the action of G on X are required. The case of free actions of discrete groups was considered earlier by B. Hanke and T. Schick.

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1 Introduction

The main result of this paper is the non-vanishing of the image of the index map, assuming that the non-vanishing of the pairing with low dimensional cohomology. In this paper, actions of a locally compact group G on X will be assumed proper and co-compact, but neither discreteness of G nor the freeness of the action will be needed. First of all, let us recall the Strong Novikov Conjecture and a result in the case of discrete groups.

Conjecture 1.1 (Strong Novikov Conjecture) *Let Γ be a discrete group. The Baum-Connes assembly map $\mu: K_0(B\Gamma) \rightarrow K_0(C_{\text{Max}}^*(\Gamma))$ is injective after tensoring $\mathbb{Q}$.*

Theorem 1.2 [HS, Theorem 1.2] *Let Γ be a discrete group. If a K -homology class $h \in K_0(B\Gamma)$ satisfies the following condition; there exists a cohomology class $c \in H^*(B\Gamma; \mathbb{Q})$ generated by classes of degree at most 2 such that $\langle c, \text{ch}(h) \rangle \neq 0$. Then $\mu(h) \neq 0$.*

Next, the statement of Baum-Connes conjecture for the proper actions is the following;

Conjecture 1.3 [BCH, (3.15)] *Let G be a locally compact Hausdorff second countable group and let $\mathcal{E}G$ be the universal example for proper actions of G defined in definition 2.9. Then*

$$\mu: \mathcal{R}K_0^G(\mathcal{E}G) \rightarrow K_0(C^*(G))$$

is an isomorphism.

This conjecture is true for such groups as, for instance, hyperbolic groups and Lie groups with finitely many connected components.

In this paper, few such assumptions on groups will be required. For example, we allow the semi-direct product of a compact group with a countable discrete group. Instead, we will set some assumptions on the domain of the Baum Connes map like [Ma03] and [HS], and we focus on the rational injectivity of this map.

Now let us fix our settings. Let X be a complete Riemannian manifold with $H_1(X; \mathbb{R}) = 0$ and let G be a second countable locally compact Hausdorff unimodular group which act on X properly and isometrically. Assume that X is G -compact.

Let $\mu_G: KK^G(C_0(X), \mathbb{C}) \rightarrow K_0(C^*(G))$ be the index map, and $E = L^1 \oplus \dots \oplus L^N$ be a finite direct sum of smooth Hermitian G -line bundles each of which has vanishing first Chern class; $c_1(L^k) = 0 \in H^2(X; \mathbb{R})$.

Theorem A *Let A be a properly supported G -invariant elliptic operator of order 0. If $[A] \in KK^G(C_0(X), \mathbb{C})$ satisfies that the Kronecker pairing $\langle [E], [A] \rangle_G$ is not equal to $0 \in \mathbb{R}$ for some E , then $\mu_G([A]) \neq 0 \in K_0(C^*(G))$.*

The Kronecker pairing $\langle [E], [A] \rangle_G$ will be defined later in the definition 5.20. Instead of taking the pairing on the level of cohomology $H^*(X)$ and homology $H_*(X)$ as in [Ma03] and [HS], we will consider it on the level of equivariant K -theory. In the case of free actions of discrete groups, they coincide.

Remarks 1.4

- (1) In this paper $C^*(G)$ is either reduced or maximal C^* -algebra. So even if G is discrete, theorem A is not covered by [HS, Theorem 1.2] since in [HS], only the maximal group C^* -algebras are considered to use the universal property of them.

Since we have a natural projection map $C_{\text{Max}}^*(G) \rightarrow C_{\text{red}}^*(G)$, it is sufficient to consider the reduced one and prove that $\mu_G([A]) \neq 0 \in C_{\text{red}}^*(G)$.

- (2) The assumption that $c_1(L) = 0 \in H^2(X; \mathbb{R})$ implies that such line bundle L over X is topologically trivial, however, remark that it does not follow that L is trivial as a G -line bundle.

One of the classical examples is as follows; let us consider the case of free action of a discrete group, namely, let M be a closed Riemannian manifold, G be its fundamental group $\pi_1(M)$, and X be its universal cover $\widetilde{M}$. The projection will be denoted by $\pi: X \rightarrow M$. Moreover, let $f: M \rightarrow BG$ be the classifying map, and consider a line bundle f^*L_0 obtained by the pull back by f of any Hermitian line bundle L_0 over BG . Then, let L be its lift $\pi^*f^*L_0$. Since f is lifted to $\widetilde{f}: X \rightarrow EG$ and EG is contractible, we obtain $c_1(L) = 0$.

Similar argument also works in the case of proper actions. Let G be a locally compact group acting properly on a complete Riemannian manifold X . Let $\mathcal{E}G$ be the universal example of proper action of G , $f: X \rightarrow \mathcal{E}G$ be a G equivariant map, and L_0 be any Hermitian G -line bundle over $\mathcal{E}G$. Then, the pull back $L := f^*L_0$ is a Hermitian G -line bundle over X , whose first Chern class vanishes since $\mathcal{E}G$ is contractible as a topological space.

Remark 1.5 In [HS], the problem on $B\Gamma$ is reduced to the case of manifolds, using the following lemma;

Lemma 1.6 [BHS, Theorem 6.2] *For any $h \in K_0(B\Gamma)$, there exists a geometric K -cycle (M, E, ϕ) realizing h . Precisely there exists (M, E, ϕ) , where M is a spin^c -manifold, E is a smooth Hermitian vector bundle, and $\phi: M \rightarrow B\Gamma$ is a continuous map, such that*

$$\phi_*([E] \widehat{\otimes} [M]) = h.$$

Unfortunately we do not have the corresponding result for G -equivariant K -homology of proper G -spaces, so the general case of proper G -topological space including $\mathcal{E}G$ could not be reduced to the case of G -manifolds.

As a corollary of Theorem A, we obtain the following Gromov-Lawson-Rosenberg conjecture for low dimensional cohomology classes.

Corollary B *Let X and G be as above and additionally we assume X is a spin manifold and the scalar curvature of X is positive. Then for any Hermitian G -vectorbundle E satisfying the conditions above, the following higher G - $\widehat{A}$ -genus vanishes.*

$$\widehat{A}_G(X; E) := \int_X c(x) \widehat{A}(TX) \wedge \text{ch}(E) = 0,$$

where $c \in C_c(X)$ denotes an arbitrary cut-off function.

The higher G - $\widehat{\mathcal{A}}$ -genus is defined later in Definition 5.23 using the L^2 -index formula; Theorem 5.22 [Wa, Proposition 6.11]. In the case of a free action a discrete group Γ , our higher G - $\widehat{\mathcal{A}}$ -genus of X coincides with the higher $\widehat{\mathcal{A}}$ -genus of X/Γ by Γ -index theorem. This gives a generalization of [Ma99, Corollary 1] to non-discrete groups.

Remark that the following Novikov conjecture for low dimensional cohomology classes does not follow unless the acting group G is discrete and the action on X is free. This is because we do not know, unfortunately, the homotopy equivalence of the $(C^*(G))$ -index of the signature operator, $\mu_G([d^X + \delta^X]) \in K_0(C^*(G))$ in general cases.

Conjecture 1.7 *Let G be a second countable locally compact Hausdorff group. Let X be a G -compact proper complete G -Riemannian manifold. Let X' be another complete Riemannian G -compact proper G -manifold and assume that there exists a G -equivariant orientation preserving smooth homotopy equivalence $h: X' \rightarrow X$. Let $f: X \rightarrow \mathcal{E}G$ be a G -equivariant map. Then for any smooth Hermitian G -vector bundle E over X given by $L = f^*E_0$ for some G -vector bundle E_0 over $\mathcal{E}G$, the higher signature for E_0 is homotopy invariant, that is,*

$$\int_X c(x) \mathcal{L}(X) \wedge \text{ch}(E) = \int_{X'} c'(x') \mathcal{L}(X') \wedge \text{ch}(h^*E),$$

where $c \in C_c(X)$ denotes an arbitrary cut-off function and $\mathcal{L}(X)$ denotes the L -class of X .

2 Proper Actions

2.1 Proper Actions

Definition 2.1 Let G be a second countable locally compact Hausdorff group. Let X be a complete Riemannian manifold.

- (1) X is called a G -Riemannian manifold if G acts on X isometrically.
- (2) The action of G on X is said to be proper or X is called a proper G -space if the following continuous map is proper;

$$G \times X \rightarrow X \times X, \quad (\gamma, x) \mapsto (\gamma x, x).$$

- (3) The action of G on X is said to be co-compact or X is called G -compact space if the quotient space X/G is compact.

Definition 2.2 A compactly supported smooth function $c \in C_c^\infty(X)$ is called a cut-off function if $c \geq 0$ is non-negative everywhere and it satisfies

$$\int_G c(\gamma^{-1}x) d\gamma = 1$$

for any $x \in X$.

Lemma 2.3 *Any G -compact manifold with proper action admits a cut-off function.*

Proof. Let $c' \in C_c^\infty(X)$ be an arbitrary non-negative function whose support intersects every orbit, namely, $\text{supp}(c') \cap G(\{x\}) \neq \emptyset$. There actually exists such a function because X is G -compact. Then consider the function c by normalizing c' on each orbit;

$$c(x) := \frac{c'(x)}{\int_G c'(\gamma^{-1}x) d\gamma}.$$

This c provides a cut-off function.

Note that since the action of G is proper, $\{\gamma \in G \mid \gamma^{-1}x \in \text{supp}(c')\} \subset G$ is compact and hence the value $\int_G c'(\gamma^{-1}x)d\gamma$ is always finite for any $x \in X$. $\square$

Definition 2.4 The action of G on X induces actions on TX and T^*X given by

$$\begin{aligned} \gamma: T_x X &\rightarrow T_{\gamma x} X & \text{and} & & \gamma: T_x^* X &\rightarrow T_{\gamma x}^* X \\ v &\mapsto \gamma(v) := \gamma_* v & & & \xi &\mapsto \gamma(\xi) := (\gamma^{-1})^* \xi. \end{aligned}$$

The action on $\mathfrak{X}(X)$ and $\Omega^*(X)$ is given by

$$\gamma[V] := \gamma_* V \quad \text{and} \quad \gamma[\omega] := (\gamma^{-1})^* \omega$$

for $\gamma \in G$, $V \in \mathfrak{X}(X)$ and $\omega \in \Omega^*(X)$. Obviously, $\gamma[\omega \wedge \eta] = \gamma[\omega] \wedge \gamma[\eta]$ and $d(\gamma[\omega]) = \gamma[d\omega]$.

Lemma 2.5 Let $\omega \in \Omega^n(X)$ and $\eta \in \Omega^{n-1}(X)$ be G -invariant differential forms, where $n = \dim(X)$ and $c \in C_c^\infty(X)$ be a cut-off function. Then,

- (1) $\int_X c(x)\omega$ does not depend on the choice of a cut-off function.
- (2) $\int_X c(x)d\eta = 0$.

Proof. Firstly, note that for any $\gamma \in G$, $\gamma: X \rightarrow X$ is a diffeomorphism, so

$$\int_X \zeta = \int_X \gamma[\zeta]$$

holds for any compactly supported differential form $\zeta \in \Omega_c^n(X)$.

Let $c' \in C_c^\infty(X)$ be another cut-off function and set $a := \int_X c(x)\omega - \int_X c'(x)\omega$. Then,

$$\begin{aligned} a &= \int_X \gamma[c(x)\omega] - \int_X \gamma[c'(x)\omega] \\ &= \int_X c(\gamma^{-1}x)\gamma[\omega] - \int_X c'(\gamma^{-1}x)\gamma[\omega] \\ &= \int_X c(\gamma^{-1}x)\omega - \int_X c'(\gamma^{-1}x)\omega \\ &= \int_X (c(\gamma^{-1}x) - c'(\gamma^{-1}x))\omega, \\ \text{hence, } \int_G a \, d\gamma &= \int_X \left(\int_G c(\gamma^{-1}x) \, d\gamma - \int_G c'(\gamma^{-1}x) \, d\gamma \right) \omega \\ &= \int_X (1 - 1) \omega = 0. \end{aligned}$$

Therefore $a = 0$. As for the second part, set $b := \int_X c(x)d\eta$. Then,

$$\begin{aligned} b &= - \int_X dc(x) \wedge \eta = - \int_X \gamma[dc(x) \wedge \eta] = - \int_X dc(\gamma^{-1}x) \wedge \gamma[\eta] = - \int_X dc(\gamma^{-1}x) \wedge \eta, \\ \text{hence, } - \int_G b \, d\gamma &= \int_X d \left(\int_G c(\gamma^{-1}x) \, d\gamma \right) \wedge \eta = \int_X d(1) \wedge \eta = 0. \end{aligned}$$

Therefore $b = 0$. $\square$

Remark 2.6 In the case when $G = \Gamma$ is a discrete group acting on X freely, if $\omega \in \Omega^n(X)$ is a Γ -invariant form on X , then the value $\int_X c(x)\omega$ is equal to $\int_F c(x)\omega$ for an arbitrary fundamental domain $F \subset X$. Therefore if $\omega' \in \Omega(X/\Gamma)$ is a differential form covered by the Γ -invariant form ω , we have $\int_X c(x)\omega = \int_{X/\Gamma} \omega'$.

Proposition 2.7 (Slice theorem) *Let G be a second countable locally compact Hausdorff group and act properly and isometrically on X . Then for any neighborhood O of any point $x \in X$ there exists a compact subgroup $K \subset G$ including the stabilizer at x , $K \supset G_x := \{\gamma \in G \mid \gamma x = x\}$ and there exists a K -slice $\{x\} \subset S \subset O$.*

Here $S \subset X$ is called K -slice if the followings are satisfied;

- S is K -invariant; $K(S) = S$,
- the tubular subset $G(S) \subset X$ is open,
- there exists a G -equivariant map $\psi: G(S) \rightarrow G/K$ satisfying $\psi^{-1}([e]) = S$, called a slice map.

We will give a proof later only in the case when G is a Lie group, in which case K can be taken as G_x itself. The proof in the general case is found in [Ab, Theorem 3.3.] and [An, Theorem 3.6.].

Corollary 2.8 *We additionally assume that X is G -compact. Then for any open covering $X = \bigcup_{x \in X} O_x$, there exists a sub-family of finitely many open subsets $\{O_{x_1}, \dots, O_{x_N}\}$ such that*

$$\bigcup_{\gamma \in G} \bigcup_{i=1}^N \gamma(O_{x_i}) = X.$$

In particular, X is of bounded geometry, namely, the injective radius is bounded below and the norm of Riemannian curvature is bounded.

Proof. Due to the slice theorem we have compact subgroups K_x including the stabilizers G_x and K_x -slice $\{x\} \subset S_x \subset O_x$. Since X/G is compact, and $\{[G(S_x)]\}_{x \in X}$ is an open covering of X/G , one can take finite sub-cover $X/G = \bigcup_{i=1}^N [G(S_{x_i})]$ and hence obtain a finite G -invariant open cover $X = \bigcup_{i=1}^N G(S_{x_i})$. On the other hand, obviously $G(O_{x_i}) \supset G(S_{x_i})$ so it follows that $X = \bigcup_{\gamma \in G} \bigcup_{i=1}^N \gamma(O_{x_i})$.

In order to prove the bounded geometry of X , take an arbitrary relatively compact neighborhood $\{x\} \subset O_x \subset X$ for each $x \in X$ and find $\{O_{x_1}, \dots, O_{x_N}\}$ as above. Noting that each of $\gamma(O_{x_i})$ is isometric to O_{x_i} for every $\gamma \in G$, the lower bound of injective radius and the norm of Riemannian curvature on whole X is determined only on $O_{x_1} \cup \dots \cup O_{x_N}$, which is relatively compact. Therefore they are bounded. $\square$

Proof of the Proposition 2.7. Let G be a Lie group. Set $K = G_x$. Due to the properness of the action, G_x is a compact sub-group. Besides every orbit set $G\{x\} \subset X$ is a closed subset of X since the orbit map $G \rightarrow X; \gamma \mapsto \gamma x$ is proper. At the same time, $G/G_x \rightarrow X; [\gamma] \mapsto \gamma x$ is an injective immersion. In particular, the orbit set $G\{x\} \subset X$ is a sub-manifold in X diffeomorphic to G/G_x .

Let $N(G\{x\})$ denote the normal bundle of $G\{x\} \subset X$, namely $N_y(G\{x\})$ is the orthogonal complement of $T_y(G\{x\})$ in $T_y X$. Since the action of G on X is isometrically, the action on $N(G\{x\})$ preserves the metric of the fibers.

Let $N_x^r G\{x\} := \{v \in N(G\{x\}) \mid \|v\| < r\}$ for positive $r \in \mathbb{R}$. Choose a small $\delta > 0$ so that the following restricted exponential map

$$\exp_x: N_x^{4\delta}(G\{x\}) \rightarrow X$$

is diffeomorphism onto its image and the image does not intersect with $G\{x\}$ at any other points than $x \in G\{x\}$. Then

$$\exp: N^\delta(G\{x\}) \rightarrow X; (y, v) \mapsto \exp_y(v)$$

is a diffeomorphism onto its image.

Now let $S := \exp_x(N_x^\delta G\{x\})$, with δ replaced by a smaller value so that S is contained in O if needed. This S is obviously G_x -invariant. Moreover the tubular subset $G(S)$ is equal to the image of the exponential map

$$G(S) = \bigcup_{y \in G\{x\}} \exp_y(N_y^\delta G\{x\}) = \exp(N^\delta(G\{x\}))$$

which is diffeomorphic to a disk bundle $N^\delta(G\{x\})$ and hence open. One can take a slicing map $\psi: G(S) \rightarrow G/K$ as follows;

$$\begin{array}{ccccc} \psi: G(S) = \exp(N^\delta(G\{x\})) & \xrightarrow[\cong]{\exp^{-1}} & N^\delta(G\{x\}) & \xrightarrow{\text{proj.}} & G\{x\} \cong G/G_x \\ & & \exp_y(v) \longmapsto (y, v) & \longmapsto & y. \end{array}$$

The inverse image ψ^{-1} is actually equal to $\exp(N_x^\delta(G\{x\})) = S$. □

2.2 Universal Example for Proper Actions

We refer to [BCH, Section 1] in this section.

Definition 2.9 (Universal example for proper actions) Let G be a second countable locally compact Hausdorff group. A universal example for proper actions of G , denoted by $\mathcal{E}G$, is a proper G -space satisfying the following property; If X is any proper G -space, then there exists a G -equivariant map $f: X \rightarrow \mathcal{E}G$, and any such maps are G -homotopic, that is, homotopic through G -equivariant maps.

Lemma 2.10

- (1) *There exists a universal example for proper actions of G , and it is unique up to G -homotopy.*
- (2) *$\mathcal{E}G$ is contractive as a topological space.*

Outline of Proof. The construction of $\mathcal{E}G$ is as follows; let

$$W := \bigsqcup_{\substack{H \subset G \\ \text{compact subgroup}}} G/H$$

be the disjoint union of the homogeneous spaces G/H , over all compact subgroups of G . Then take the join with itself infinitely many times;

$$\mathcal{E}G = W * W * \dots$$

From this construction it follows that $\mathcal{E}G$ is contractive.

Moreover if we have two, $\mathcal{E}G$ and $(\mathcal{E}G)'$, then we have G -maps $f: (\mathcal{E}G)' \rightarrow \mathcal{E}G$ and $f': \mathcal{E}G \rightarrow (\mathcal{E}G)'$, each of which is unique up to G -homotopy. Moreover $f \circ f'$ is G -homotopic to $\text{id}_{\mathcal{E}G}$ and $f' \circ f$ is G -homotopic to $\text{id}_{(\mathcal{E}G)'}$ respectively.

3 K -theoretical Preparation to Describe the Index Map

3.1 Tensor products, Group Algebras and Crossed Products

The index of G -equivariant K -homology class takes value in the K -homology of the group C^* -algebra. We will introduce some kinds of C^* -algebras used to describe the index map.

Definition 3.1 Let A and B be C^* -algebras then let $A \odot B$ denote the algebraic tensor product regarding A and B as vector spaces over $\mathbb{C}$. $A \odot B$ is naturally equipped with a structure of $*$ -algebra;

$$\begin{aligned} (a_1 \otimes b_1) \cdot (a_2 \otimes b_2) &:= a_1 a_2 \otimes b_1 b_2 \\ (a_1 \otimes b_1)^* &:= a_1^* \otimes b_1^*. \end{aligned}$$

In order to define the tensor products of C^* -algebras $A \otimes B$, we need to introduce a C^* -norm on $A \odot B$ and take the completion.

Definition 3.2 (Maximal tensor products)

- Define a C^* -norm $\|\cdot\|_{\text{Max}}$ on $A \odot B$ by

$$\|x\|_{\text{Max}} := \sup \{ \|x\|_\alpha \mid \|\cdot\|_\alpha \text{ is a } C^*\text{-seminorm on } A \odot B \}.$$

The norm $\|x\|_{\text{Max}}$ is finite because every C^* -seminorm $\|\cdot\|_\alpha$ on $A \odot B$ satisfies $\|a \otimes b\|_\alpha \leq \|a\|_A \|b\|_B$. [We, Corollary T.6.2]

- Then define $A \otimes_{\text{Max}} B$ called maximal or universal tensor product as the completion of $A \odot B$ under the norm $\|x\|_{\text{Max}}$.

Definition 3.3 (Minimal tensor products) Let $\rho_A: A \rightarrow \mathcal{L}(\mathcal{H}_A)$ and $\rho_B: B \rightarrow \mathcal{L}(\mathcal{H}_B)$ be representations of A and B on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Assume that ρ_A and ρ_B are faithful. There exist such representations by the Gelfand - Naimark theorem.

- Define a representation $\rho_A \odot \rho_B: A \odot B \rightarrow \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ by

$$(\rho_A \odot \rho_B)(a \otimes b) := \rho_A(a) \otimes \rho_B(b) \in \mathcal{L}(\mathcal{H}_A) \odot \mathcal{L}(\mathcal{H}_B) \subset \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B).$$

$\rho_A \odot \rho_B$ is again a faithful representation. [We, Proposition T.5.1.]

- Define a C^* -norm $\|\cdot\|_{\text{min}}$ on $A \odot B$ by

$$\|x\|_{\text{min}} := \|(\rho_A \odot \rho_B)(x)\|_{\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)} \quad \text{for } x \in A \odot B.$$

This norm is independent of the choice of faithful representations ρ_A and ρ_B . [We, Theorem T.5.15]

- Define $A \otimes_{\text{min}} B$ called the spatial or minimal tensor product as a completion of the algebraic tensor product $A \odot B$ under the norm $\|\cdot\|_{\text{min}}$.

Remark 3.4 In the case when both $A = A^0 \hat{\oplus} A^1$ and $B = B^0 \hat{\oplus} B^1$ are $\mathbb{Z}/2\mathbb{Z}$ -graded, for instance $C_0(\mathbb{V})$, where $\mathbb{V} = \mathbb{V}^0 \hat{\oplus} \mathbb{V}^1$, is a graded vector bundle, it is natural to consider the graded tensor product

$$A \hat{\otimes} B = ((A^0 \otimes B^0) \oplus (A^1 \otimes B^1)) \hat{\oplus} ((A^0 \otimes B^1) \oplus (A^1 \otimes B^0)).$$

The product of $A \hat{\otimes} B$ is given by

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{\deg(b_1) \deg(a_2)} a_1 a_2 \hat{\otimes} b_1 b_2$$

for homogeneous elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Definition 3.5 A C^* -algebra A is called nuclear if there is only one C^* -norm on $A \odot B$ for any each C^* -algebra B . In such case, the tensor product of A and B will be denoted by simply $A \otimes B$.

Example 3.6 Every commutative C^* -algebra is nuclear.[We, Theorem T.6.20]

In particular, for a locally compact Hausdorff space X and a C^* -algebra B , $C_0(X) \otimes B \cong C_0(X) \otimes_{\min} B$ is naturally isomorphic to the C^* -algebra $C_0(X; B)$ consisting of all B -valued continuous functions on X vanishing at infinity.[We, Proposition T.5.21]

Next, we define the two types of crossed product for covariant system, which are generalizations of group C^* -algebra, namely, $C_{\text{Max}}^*(G; B)$ called the maximal crossed product and $C_{\text{red}}^*(G; B)$ called the reduced crossed product.

Definition 3.7 Let G be a locally compact Hausdorff topological group. There exists an unique left invariant Haar measure $\mathbf{m}$ on G up to constant. From now on, $\mathbf{d}\mathbf{m}(\gamma)$ is denoted by $\mathbf{d}\gamma$. G is called unimodular if $\mathbf{m}$ is also right invariant. For instance, any compact groups and any discrete groups are unimodular. Then it is satisfied that

$$\mathbf{d}(\gamma_1\gamma) = \mathbf{d}(\gamma\gamma_1) = \mathbf{d}\gamma \quad \text{and} \quad \mathbf{d}(\gamma^{-1}) = \gamma^{-1} \cdot \mathbf{d}\gamma \cdot \gamma^{-1} = \mathbf{d}\gamma$$

for any $\gamma, \gamma_1 \in G$.

Definition 3.8 (Group algebras $C_c(G; B)$) Let B be an C^* -algebra and $\beta: G \rightarrow \text{Aut}(B)$ be a continuous homomorphism. $\text{Aut}(B)$ is equipped with the point-wise norm-convergence topology. Such triple (B, G, β) is called covariant system. We omit β when it is obvious and in such case, B is called a G - C^* -algebra or simply a G -algebra.

- Define $C_c(G; B)$ as an algebra consisting of all compactly supported continuous B -valued functions on G equipped with the following convolution product;

$$(f * g)(\gamma) := \int_G f(\gamma_1) \cdot \beta[\gamma_1](g(\gamma_1^{-1}\gamma)) \mathbf{d}\gamma_1 = \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1) \cdot \beta[\gamma_1](g(\gamma_1^{-1}\gamma)) \mathbf{d}\gamma_1$$

and with the involution $f^*(\gamma) := \beta[\gamma](f(\gamma^{-1})^*)$.

- A pair (π, ρ) is called a covariant representation for the covariant system (B, G, β) if π and ρ are representations of B and G on a same Hilbert space $\mathcal{H}$ and satisfy that

$$\rho(\gamma)\pi(b)\rho(\gamma)^* = \pi(\beta[\gamma](b)) \quad \text{for } b \in B \text{ and } \gamma \in G.$$

Lemma 3.9 If a covariant representation (π, ρ) for a covariant system (B, G, β) is given, we obtain a representation $\sigma_{(\pi, \rho)}$ of the group algebra $C_c(G; B)$ on $\mathcal{H}$ as follows;

$$\sigma_{(\pi, \rho)}[f] := \int_G \pi(f(\gamma))\rho(\gamma) \mathbf{d}\gamma.$$

Proof. The functoriality is verified by the direct calculation as below;

$$\begin{aligned} \sigma_{(\pi, \rho)}[f * g] &= \int_G \pi \left\{ \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1) \cdot \beta[\gamma_1]g(\gamma_2) \mathbf{d}\gamma_1 \right\} \rho(\gamma) \mathbf{d}\gamma \\ &= \int_G \int_{\gamma_1\gamma_2=\gamma} \pi \{f(\gamma_1)\} \cdot \rho(\gamma_1)\pi \{g(\gamma_2)\} \rho(\gamma_1)^{-1} \cdot \rho(\gamma) \mathbf{d}\gamma_1 \mathbf{d}\gamma \\ &= \int_G \left\{ \int_G \pi(f(\gamma_1))\rho(\gamma_1) \mathbf{d}\gamma_1 \right\} \pi(g(\gamma_2))\rho(\gamma_2) \mathbf{d}\gamma \\ &= \sigma_{(\pi, \rho)}[f]\sigma_{(\pi, \rho)}[g]. \end{aligned}$$

□

Definition 3.10 (Crossed Product $C^*(G; B)$)

- Define $C_{\text{Max}}^*(G; B)$ called the maximal crossed product as the completion of $C_c(G; B)$ under the the maximal norm $\|\cdot\|_{\text{Max}}$:

$$\|f\|_{\text{Max}} := \sup_{\pi, \rho} \|\sigma_{\pi, \rho}[f]\|_{op},$$

where $\sup_{\pi, \rho}$ is taken over all covariant representation for (B, G, β) and $\|\cdot\|_{op}$ denotes the operator norm on Hilbert spaces. $C_{\text{Max}}^*(G; B)$ is sometimes denoted by $C_{\text{Max}}^*(G; B, \beta)$ or $B \rtimes_{\beta, \text{Max}} G$ and Max is sometimes omitted.

- To define $C_{\text{red}}^*(G; B)$ as the Consider the regular representation σ of $C_c(G; B)$ on $L^2(G; B)$ given by the formula;

$$(\sigma[f]g)(\gamma) := \int_{\gamma_1 \gamma_2 = \gamma} \beta[\gamma^{-1}](f(\gamma_1)) \cdot g(\gamma_2) d\gamma_1$$

for $f \in C_c(G; B)$ and $g \in L^2(G; B)$. Then define the reduced crossed product $C_{\text{red}}^*(G; B)$ as the completion of $C_c(G; B)$ under the the operator norm of regular representation;

$$\|f\|_{\text{red}} := \|\sigma[f]\|_{op}.$$

$C_{\text{red}}^*(G; B)$ is sometimes denoted by $C_{\text{red}}^*(G; B, \beta)$ or $B \rtimes_{\beta, \text{red}} G$.

Remarks 3.11

- (1) The assumption that G is unimodular is not actually needed for these definitions. The involution of a non-unimodular group G is replaced by $f^*(\gamma) := \beta[\gamma](f(\gamma^{-1})^*) \Delta(\gamma^{-1})$, where $\Delta: G \rightarrow U(1)$ denotes the modular function on G , which is equal to 1 everywhere if G is unimodular.
- (2) In general $C_{\text{red}}^*(G; B)$ is realized as a quotient of $C_{\text{Max}}^*(G; B)$ and in particular there exists a natural surjective homomorphism $C_{\text{Max}}^*(G; B) \rightarrow C_{\text{red}}^*(G; B)$.
- (3) In the particular case of $(B, G, \beta) = (\mathbb{C}, G, 1)$, $C_{\text{Max}}^*(G; \mathbb{C})$ and $C_{\text{red}}^*(G; \mathbb{C})$ are denoted by $C_{\text{Max}}^*(G)$ and $C_{\text{red}}^*(G)$ respectively and called maximal C^* -algebra of G and reduced C^* -algebra of G .
- (4) In the case when G acts trivially on B , namely when $\beta = 1$, there are natural isomorphisms $C_{\text{Max}}^*(G; B) \cong C_{\text{Max}}^*(G) \otimes_{\text{Max}} B$ and $C_{\text{red}}^*(G; B) \cong C_{\text{red}}^*(G) \otimes_{\text{min}} B$.
- (5) In this paper basically $C^*(-)$ denotes either of $C_{\text{Max}}^*(-)$ or $C_{\text{red}}^*(-)$.

Definition 3.12 (Hilbert $C^*(G; B)$ -module) Let (G, B, β) be a covariant system and $\mathcal{E}$ be a Hilbert B -module. We do not consider the action of G on $\mathcal{E}$. We will define a Hilbert $C^*(G; B)$ -module denoted by $C^*(G; \mathcal{E})$. First, define on $C_c(G; \mathcal{E})$ the structure of a pre-Hilbert module over $C_c(G; B)$ as follows.

- The action of $C_c(G; B)$ from the right is given by the convolution product;

$$[s \cdot f](\gamma) := \int_{\gamma_1 \gamma_2 = \gamma} s(\gamma_1) \cdot \beta[\gamma_1](f(\gamma_2)) d\gamma_1 \quad \text{for } s \in C_c(G; \mathcal{E}), f \in C_c(G; B).$$

- The $C_c(G; B)$ -valued scalar product is given by

$$\langle s_1, s_2 \rangle_{C_c(G; \mathcal{E})}(\gamma) := \int_G \beta[\gamma_1^{-1}] \langle s_1(\gamma_1), s_2(\gamma_1 \gamma) \rangle_{\mathcal{E}} d\gamma_1 \quad \text{for } s_1, s_2 \in C_c(G; \mathcal{E}).$$

- The completion of $C_c(G; \mathcal{E})$ under the norm $\sqrt{\|\langle s, s \rangle_{C_c(G; \mathcal{E})}\|_{C^*(G; B)}}$ provides a Hilbert $C^*(G; B)$ -module, which is denoted by $C^*(G; \mathcal{E})$.

Definition 3.13 (Canonical Trace) There exists the canonical trace on the group C^* -algebra $C^*(G)$, which is defined on its dense sub-algebra $C_c(G)$ as follows;

$$\mathrm{tr}: C_c(G) \rightarrow \mathbb{C} \quad f \mapsto f(e), \quad \text{where } e \text{ denotes the identity element in } G.$$

It induces a linear map $\mathrm{tr}_*: K_0(C^*(G)) \rightarrow \mathbb{R}$ called the trace map in K -theory, by defining

$$\mathrm{tr}([f_1 \otimes p_1] - [f_2 \otimes p_2]) := \mathrm{tr}(f_1)\mathrm{Tr}(p_1) - \mathrm{tr}(f_2)\mathrm{Tr}(p_2)$$

for formal differences of matrices $f_i \otimes p_i \in M_n(C^*(G))$ with entries in $C^*(G)$. Note that $\mathrm{Tr}(p_i)$ denote the usual traces for matrices $p_i \in M_n(\mathbb{C})$ with entries in $\mathbb{C}$. We sometimes let tr_* be denoted simply by tr .

3.2 Equivariant KK -Theory

Let G be a locally compact Hausdorff group, and let A and B be G -algebras. Let us introduce Kasparov equivariant KK -group denoted by $KK^G(A, B)$.

Roughly, this notion is a generalization both of K -homology and K -cohomology. For instance consider first in the simplest and important case when A and B are of the form $C_0(Z)$. Let X and Y be a complete G -Riemannian manifold. $KK^G(\mathbb{C}, C_0(X))$ is isomorphic to $K_0^G(C_0(X)) \simeq K_G^0(X)$ called G -equivariant K -cohomology of X , which is, roughly, an Abelian group consisting of all isomorphic classes of Hermitian G -vector bundles over X .

On the other hand, $KK^G(C_0(Y), \mathbb{C})$ is isomorphic to $K_G^0(C_0(Y)) \simeq K_0^G(Y)$ called G -equivariant K -homology of Y , which is, roughly, an Abelian group consisting of all isomorphic classes of G -invariant Dirac operator acting on L^2 sections of a vector bundle over X .

$KK^G(C_0(Y), C_0(X))$ is, roughly, consisting of G -equivariant families of G -invariant Dirac operators on Y parametrized by X .

More over it is important that there exists the following Kasparov intersection product

$$KK^G(A, D) \otimes KK^G(D, B) \rightarrow KK^G(A, B).$$

3.2.1 Kasparov Modules

Definition 3.14 (Hilbert modules) Let B be a C^* -algebra.

- (1) A Hilbert B -module is a $\mathbb{C}$ -vector space equipped with an associative left $\mathbb{C}$ -action, right B -action and a complete B -valued scalar product $\langle \cdot, \cdot \rangle$ satisfying the following conditions;

- $\langle \cdot, \cdot \rangle$ is linear in the second variable; $\langle x, c_1 y_1 + c_2 y_2 \rangle = c_1 \langle x, y_1 \rangle + c_2 \langle x, y_2 \rangle$.
- $\langle x, y \cdot b \rangle = \langle x, y \rangle b$.
- $\langle y, x \rangle = \langle x, y \rangle^*$.
- $\langle x, x \rangle \geq 0$. Moreover, $\langle x, x \rangle = 0$ only if $x = 0$.

for $x, y, y_i \in \mathcal{E}$, $c_i \in \mathbb{C}$ and $b \in B$. The completeness of the scalar product $\langle \cdot, \cdot \rangle$ means that the norm $\|x\| := \sqrt{\|\langle x, x \rangle\|_B}$ is complete.

- (2) Let G be a locally compact Hausdorff group, and in the case when B is a graded G - C^* -algebra, the grading of $\mathcal{E}$ is given by the grading map $\varepsilon: \mathcal{E} \rightarrow \mathcal{E}$ with $\varepsilon^2 = 1$ compatible with the grading on B , that is,

$$\varepsilon(x \cdot b) = \varepsilon(x) \cdot \varepsilon(b) \quad \text{and} \quad \varepsilon(\langle x, y \rangle) = \langle \varepsilon(x), \varepsilon(y) \rangle$$

hold for any $b \in B$ and $x, y \in \mathcal{E}$ and here ε also denotes the grading of B .

If we consider an action of G on $\mathcal{E}$, we always assume the unitarity, continuity of $G \times \mathcal{E} \rightarrow \mathcal{E}$ and compatibility with other structures of graded Hilbert B -module, that is,

$$\gamma(x \cdot b) = \gamma(x) \cdot \gamma(b), \quad \gamma(\langle x, y \rangle) = \langle \gamma(x), \gamma(y) \rangle \quad \text{and} \quad \varepsilon(\gamma(x)) = \gamma(\varepsilon(x))$$

hold for any $\gamma \in G$, $b \in B$ and $x, y \in \mathcal{E}$.

Definition 3.15 (Notations $\mathbb{L}$ and $\mathbb{K}$) Let B be a C^* -algebra and $\mathcal{E}$ be a countably generated Hilbert B -module. We define the following notations;

- $\mathbb{L}_B(\mathcal{E})$ is an algebra consisting of bounded B -linear operators T on $\mathcal{E}$ which have adjoint T^* satisfying $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for every $x, y \in \mathcal{E}$. $\mathbb{L}_B(\mathcal{E})$ itself has a structure of C^* -algebra with respect to the operator norm and the product given by the composition of maps.

Similarly, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are countably generated Hilbert B -modules, $\mathbb{L}_B(\mathcal{E}_1, \mathcal{E}_2)$ is an algebra of bounded B -linear operators $T: \mathcal{E}_1 \rightarrow \mathcal{E}_2$ which have adjoint $T^*: \mathcal{E}_2 \rightarrow \mathcal{E}_1$.

- $\mathbb{K}_B(\mathcal{E})$ is a closed ideal in $\mathbb{L}_B(\mathcal{E})$ generated by operators whose B -rank are 1 namely, operators of the form

$$\theta_{\xi, \eta}(x) := \xi \langle \eta, x \rangle_{\mathcal{E}}$$

for fixed $\xi, \eta \in \mathcal{E}$. Operators belonging to $\mathbb{K}_B(\mathcal{E})$ is called a B -compact operators.

Similarly, $\mathbb{K}_B(\mathcal{E}_1, \mathcal{E}_2)$ is a closed ideal in $\mathbb{L}_B(\mathcal{E}_1, \mathcal{E}_2)$ generated by operators whose B -rank are 1.

We may omit subscript B to avoid complication.

Definition 3.16 (Tensor products of Hilbert modules) Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert A -module and B -module equipped with A and B -valued scalar product $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ respectively.

- (1) **(Exterior tensor products)** Let $A \otimes B$ denote the minimal tensor product. Then let $\mathcal{E} \odot \mathcal{F}$ be the algebraic tensor product and regard it as a right $A \odot B$ -module by the formula;

$$(x \otimes y)(a \otimes b) = xa \otimes yb \quad \text{for } x \in \mathcal{E}, y \in \mathcal{F}, a \in A, b \in B.$$

The $A \otimes B$ -valued scalar product on $\mathcal{E} \odot \mathcal{F}$ is given by

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{\mathcal{E} \otimes \mathcal{F}} := \langle x_1, x_2 \rangle_{\mathcal{E}} \otimes \langle y_1, y_2 \rangle_{\mathcal{F}}.$$

Then define $\mathcal{E} \otimes \mathcal{F}$ as the completion of $\mathcal{E} \odot \mathcal{F}$ under the norm $\|z\|_{\mathcal{E} \otimes \mathcal{F}} := \sqrt{\|\langle z, z \rangle_{\mathcal{E} \otimes \mathcal{F}}\|}$.

- (2) **(Interior tensor product)** If a homomorphism $\varphi: A \rightarrow \mathbb{L}_B(\mathcal{F})$ is given, then let $\mathcal{E} \odot \mathcal{F}$ be the algebraic tensor product and consider a subspace $\mathcal{N} \subset \mathcal{E} \odot \mathcal{F}$ generated by elements of the form

$$xa \otimes y - x \otimes \varphi(a)y \quad \text{for } x \in \mathcal{E}, y \in \mathcal{F}, a \in A.$$

Define a B -valued scalar product on $\mathcal{E} \odot \mathcal{F}$ by the formula;

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_{\mathcal{E} \otimes \mathcal{F}} := \left\langle y_1, \varphi \left(\langle x_1, x_2 \rangle_{\mathcal{E}} \right) y_2 \right\rangle_{\mathcal{F}} \quad \text{for } x_1, x_2 \in \mathcal{E}, y_1, y_2 \in \mathcal{F}.$$

Note that $\langle z, z \rangle_{\mathcal{E} \otimes \mathcal{F}} = 0$ for any $z \in \mathcal{N}$. Then define $\mathcal{E} \otimes_{\varphi} \mathcal{F}$ as the completion of $(\mathcal{E} \odot \mathcal{F}) / \mathcal{N}$ under the norm $\|z\|_{\mathcal{E} \otimes \mathcal{F}} := \sqrt{\|\langle z, z \rangle_{\mathcal{E} \otimes \mathcal{F}}\|}$.

Examples 3.17

- (1) A C^* -algebra B itself can be regarded as a Hilbert B -module with the B -valued scalar product $\langle b_1, b_2 \rangle := b_1^* b_2$.
- (2) [?, Definition 2] Let $\{V_i\}$ be a countable family of finite dimensional unitary representation spaces of G . We suppose that every isomorphism classes of unitary representation of G occurs an infinitely many times in $\{V_i\}$. Let us construct the scalar product on $V_i \otimes B$ by

$$\langle x_1 \otimes b_1, x_2 \otimes b_2 \rangle_{V_i \otimes B} := \langle x_1, x_2 \rangle_{V_i} b_1^* b_2 \in B.$$

The Hilbert space over B is defined by the direct sum $\mathcal{H}_B := \bigoplus_{i \in \mathbb{N}} (V_i \otimes B)$ completed under

the norm $\|\{z_i\}\| := \sqrt{\sum_{i \in \mathbb{N}} \|\langle z_i, z_i \rangle_{V_i \otimes B}\|}$.

- (3) Let G be a second countable locally compact Hausdorff group. Let X be a G -compact proper complete G -Riemannian manifold. Then $B = C_0(X)$ is a G -algebra with the action

$$[\gamma \cdot b](x) = b(\gamma^{-1}x) \quad \text{for } b \in C_0(X).$$

Let E be a $\mathbb{Z}/2\mathbb{Z}$ -graded Hermitian G -vector bundle over X . Then $\mathcal{E} = C_0(E)$ is a Hilbert B -module with the action of $B = C_0(X)$ given by the point-wise multiplication and the B -valued scalar product given by the point-wise Hermitian scalar product of E , namely,

$$\langle s_1, s_2 \rangle(x) := \langle s_1(x), s_2(x) \rangle_E$$

for $s_1, s_2 \in \mathcal{E} = C_0(E)$, $x \in X$. The grading on $\mathcal{E}$ comes from the grading on E , and the G -action is given by

$$[\gamma \cdot s](x) = \gamma(s(\gamma^{-1}x)).$$

Note that $\mathcal{E}$ is not a Hilbert module over $\mathbb{C}$ since it does not have complete $\mathbb{C}$ -valued scalar product.

Following theorem by Kasparov is useful.

Theorem 3.18 (Stabilization Theorem) [?, Theorem 2] *Let $\mathcal{E}$ be a countably generated Hilbert B -module. Then, $\mathcal{E} \oplus \mathcal{H}_B \simeq \mathcal{H}_B$.*

Definition 3.19 (G -continuous operators) An action of G on $\mathcal{E}$ induces an action on $\mathbb{L}(\mathcal{E})$ given by

$$[\gamma(T)](x) = [\gamma \circ T \circ \gamma^{-1}](x) = \gamma(T(\gamma^{-1}x))$$

for $T \in \mathbb{L}_B(\mathcal{E})$ and $x \in \mathcal{E}$. An operator $T \in \mathbb{L}_B(\mathcal{E})$ is said to be G -continuous if the map $G \rightarrow \mathbb{L}_B(\mathcal{E})$; $\gamma \mapsto \gamma(T)$ is continuous.

Lemma 3.20 *Suppose that $\mathcal{E}$ is a Hilbert B -module equipped with a continuous unitary action of G .*

- (1) *The sub-space consisting of G -continuous operators is a closed sub-algebra in $\mathbb{L}(\mathcal{E})$.*
- (2) *The action of G restricted on $\mathbb{K}(\mathcal{E})$ is always continuous.*

Proof. Let $\{T_i\} \subset \mathbb{L}(\mathcal{E})$ be a sequence of G -continuous operators and $T = \lim_{i \rightarrow \infty} T_i$ be the limit. Choose any convergent sequence $\{\gamma_k\}$ with limit $\gamma = \lim_{k \rightarrow \infty} \gamma_k$. For any $\varepsilon > 0$, there exists $i \in \mathbb{N}$ such that $\|T - T_i\| < \varepsilon/3$. Since T_i is G -continuous, $\|\gamma_k(T_i) - \gamma(T_i)\| < \varepsilon/3$ for sufficiently large $k \in \mathbb{N}$. Then, we have

$$\begin{aligned} \|\gamma_k(T) - \gamma(T)\| &\leq \|\gamma_k(T - T_i)\| + \|\gamma_k(T_i) - \gamma(T_i)\| + \|\gamma(T_i - T)\| \\ &= \|T - T_i\| + \|\gamma_k(T_i) - \gamma(T_i)\| + \|T_i - T\| < \varepsilon. \end{aligned}$$

Therefore, $T = \lim_{i \rightarrow \infty} T_i$ is G -continuous.

To verify the second statement, since we have (1), it is sufficient to consider only in the case of rank one operators, which generate $\mathbb{K}(\mathcal{E})$. Take $\xi, \eta \in \mathcal{E}$ and consider an operator $\theta_{\xi, \eta} = \xi \langle \eta, \cdot \rangle_{\mathcal{E}} \in \mathbb{K}(\mathcal{E})$. Observe that

$$[\gamma(\theta_{\xi, \eta})](x) = \gamma(\xi \langle \eta, \gamma^{-1}x \rangle_{\mathcal{E}}) = (\gamma\xi)(\gamma \langle \eta, \gamma^{-1}x \rangle_{\mathcal{E}}) = (\gamma\xi) \langle \gamma\eta, x \rangle_{\mathcal{E}} = \theta_{\gamma\xi, \gamma\eta}(x).$$

Since $\theta_{\xi, \eta}$ is continuous in $\xi, \eta \in \mathcal{E}$, it follows that $\theta_{\xi, \eta}$ is G -continuous. $\square$

Example 3.21 Note that the action of G on $\mathbb{L}(\mathcal{E})$ is not necessarily continuous. Consider the case when $G = \mathbb{R}$, $B = \mathbb{C}$ and $\mathcal{E} = L^2(\mathbb{R})$ equipped with the usual L^2 -inner product. The action of $\mathbb{R}$ on $L^2(\mathbb{R})$ is given by $\gamma[f](x) = f(x - \gamma)$ for $f \in L^2(\mathbb{R})$ and $\gamma \in \mathbb{R}$. Let $T \in \mathbb{L}(L^2(\mathbb{R}))$ be such an operator that

$$T[f](x) := \chi(x)f(x), \quad \text{where} \quad \chi(x) := \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

Then this T is not G -continuous. In fact, since we have

$$(\gamma(T))[f](x) = \chi(x - \gamma)f(x) \quad \text{and} \quad (\gamma(T) - T)[f](x) = \{\chi(x - \gamma) - \chi(x)\}f(x)$$

and since for each $n \in \mathbb{N}$, there exist $f_n \in L^2(\mathbb{R})$ such that $\|f_n\|_{L^2(\mathbb{R})} = 1$ and $\text{supp}(f_n) \subset [0, 1/n]$, the operator norm $\|\gamma(T) - T\|_{op}$ is always equal to 1. So $\{\gamma(T) - T\}_{\gamma=1/n}$ does not converge in the operator norm as $\gamma \rightarrow 0 \in \mathbb{R}$.

Definition 3.22 If A is another graded C^* -algebra and supposed that a homomorphism $\varphi: A \rightarrow \mathbb{L}_B(\mathcal{E})$ is given, $\mathcal{E}$ is also acted on by A through φ and is called a bi-module over (A, B) . In addition we define the following notations;

- $Q_A(\mathcal{E})$ is a sub-algebra in $\mathbb{L}_B(\mathcal{E})$ generated by operators T such that

$$[T, \varphi(a)] \in \mathbb{K}_B(\mathcal{E})$$

and that $\varphi(a)T$ is G -continuous for any $a \in A$.

- $I_A(\mathcal{E})$ is the ideal in $Q_A(\mathcal{E})$ generated by operators $T \in \mathbb{L}_B(\mathcal{E})$ such that for any $a \in A$,

$$\varphi(a)T, T\varphi(a) \in \mathbb{K}_B(\mathcal{E}).$$

Note that two conditions $\varphi(a)T \in \mathbb{K}_B(\mathcal{E})$ and $T\varphi(a) \in \mathbb{K}_B(\mathcal{E})$ are equivalent since $T \in Q_A(\mathcal{E})$.

Note that the commutator $[\cdot, \cdot]$ is the graded commutator that is, $[x, y] = xy - (-1)^{\deg(x)\deg(y)}yx$ for homogeneous elements and note that the following equations hold.

- $[x, y] + (-1)^{pq}[y, x] = 0,$

- $[x, yz] = [x, y]z + (-1)^{pq}y[x, z],$
- $(-1)^{rp}[[x, y], z] + (-1)^{pq}[[y, z], x] + (-1)^{qr}[[z, x], y] = 0,$

where $p = \deg(x)$, $q = \deg(y)$ and $r = \deg(z)$.

Definition 3.23 (Kasparov modules) [K88, 2.2. Definition.] Let G be a locally compact Hausdorff group, and let A and B be G -algebras. A Kasparov G -module over (A, B) is a triple $(\mathcal{E}, \varphi, T)$ such that

- $\mathcal{E}$ is a countably generated graded Hilbert B -module.
- $\varphi: A \rightarrow \mathbb{L}_B(\mathcal{E})$ is a graded $*$ -homomorphism.
- $T \in Q_A(\mathcal{E})$ is an operator of degree 1 such that for any $\gamma \in G$,

$$T^2 - 1, \quad T - T^* \quad \text{and} \quad \gamma(T) - T \in I_A(\mathcal{E}).$$

We sometimes omit φ when the action of A on $\mathcal{E}$ is canonical or clear by the context.

Definition 3.24 Let $(\mathcal{E}_1, \varphi_1, T_1)$ and $(\mathcal{E}_2, \varphi_2, T_2)$ be Kasparov G -modules over (A, B) . $(\mathcal{E}_1, \varphi_1, T_1)$ and $(\mathcal{E}_2, \varphi_2, T_2)$ are said to be isomorphic if there exists a grading preserving unitary isomorphism of Hilbert B -modules $u: \mathcal{E}_1 \rightarrow \mathcal{E}_2$ compatible with the actions φ_i of A and the operators T_i that is, u satisfies

$$\varphi_2(a) = u\varphi_1(a)u^{-1} \quad \text{and} \quad T_2 = uT_1u^{-1}.$$

$\mathbb{E}^G(A, B)$ is defined to be the set of all isomorphism classes of Kasparov G -modules over (A, B) .

$\mathbb{D}^G(A, B) \subset \mathbb{E}^G(A, B)$ is defined to be the subset consisting of Kasparov G -modules $(\mathcal{E}, \varphi, T)$ over (A, B) satisfying

$$[T, \varphi(a)] = \varphi(a)(T^2 - 1) = \varphi(a)(T - T^*) = \varphi(a)(\gamma(T) - T) = 0 \in \mathbb{L}(\mathcal{E}).$$

Such elements in $\mathbb{D}^G(A, B)$ are called degenerate Kasparov G -modules.

Definition 3.25 Two Kasparov G -modules $(\mathcal{E}_0, \varphi_0, T_0)$ and $(\mathcal{E}_1, \varphi_1, T_1)$ are said to be homotopic if there exists a Kasparov G -module $(\mathcal{E}, \varphi, T) \in \mathbb{E}(A, C([0, 1]; B))$ such that

$$\mathcal{E} \widehat{\otimes}_{e_t} B \simeq \mathcal{E}_t, \quad \varphi(a) \widehat{\otimes}_{e_t} \text{id}_B = \varphi_t(a) \quad \text{and} \quad T \widehat{\otimes}_{e_t} \text{id}_B = T_t \quad \text{for } t = 0, 1,$$

where $e_t: C([0, 1]; B) \rightarrow B$ is the evaluation map at $t \in [0, 1]$. $(\mathcal{E}, \varphi, T)$ itself is called a homotopy between $(\mathcal{E}_0, \varphi_0, T_0)$ and $(\mathcal{E}_1, \varphi_1, T_1)$. Note that $C([0, 1]; B) \simeq C([0, 1]) \widehat{\otimes} B$ and its grading is given by the grading of B and the trivial grading of $C([0, 1])$.

Let IB denote $C([0, 1]; B) \simeq C([0, 1]) \widehat{\otimes} B$, and if $\mathcal{E}$ is a Hilbert B -module, let $I\mathcal{E}$ denote the Hilbert IB -module $C([0, 1]; \mathcal{E}) \simeq C([0, 1]) \widehat{\otimes} \mathcal{E}$.

Remark 3.26 If we have an operator homotopy T_t with $(\mathcal{E}, \varphi, T_t) \in \mathbb{E}^G(A, B)$ for any $t \in [0, 1]$, $(\mathcal{E}, \varphi, T_0)$ and $(\mathcal{E}, \varphi, T_1)$ are homotopic by the homotopy $(I\mathcal{E}, 1 \widehat{\otimes} \varphi, \{T_t\}_{t \in [0, 1]})$.

Lemma 3.27 Any degenerate Kasparov G -module $(\mathcal{E}, \varphi, T) \in \mathbb{D}(A, B)$ is homotopic to the zero module $(0, 0, 0)$.

Proof. Take a Hilbert IB -module $C_0([0, 1]; \mathcal{E}) \simeq C_0([0, 1]) \hat{\otimes} \mathcal{E}$ equipped with the grading coming from $\mathcal{E}$. $C_0([0, 1])$, a Hilbert $C([0, 1])$ -module, is equipped with the trivial G -action and the trivial grading. Then a homotopy between $(\mathcal{E}, \varphi, T)$ and $(0, 0, 0)$ is given by

$$(C_0([0, 1]; \mathcal{E}), 1 \hat{\otimes} \varphi, 1 \hat{\otimes} T) \in \mathbb{D}(A, IB) \subset \mathbb{E}(A, IB).$$

In fact

$$C_0([0, 1]; \mathcal{E}) \hat{\otimes}_{e_0} B \simeq \mathcal{E}, \quad (1 \hat{\otimes} \varphi(a)) \otimes_{e_0} \text{id}_B = \varphi(a) \quad \text{and} \quad (1 \hat{\otimes} T) \otimes_{e_0} \text{id}_B = T$$

obviously hold. $\square$

Lemma 3.28 [?, Theorem 12] *Let $(\mathcal{E}, \varphi, T_0), (\mathcal{E}, \varphi, T_1) \in \mathbb{E}^G(A, B)$ be Kasparov G -modules over (A, B) . If $\varphi(a)[T_0, T_1]\varphi(a^*) \leq 0$ modulo $\mathbb{K}(\mathcal{E})$, then $(\mathcal{E}, \varphi, T_0)$ and $(\mathcal{E}, \varphi, T_1)$ are homotopic.*

Proof. $[T_0, T_1]$ belong to $Q_A(\mathcal{E})$ and $[T_0, T_1] \geq 0$ modulo $I_A(\mathcal{E})$ by the assumption. Write $[T_0, T_1] = P + K$, where $P \geq 0 \in Q_A(\mathcal{E})$ and $K \in I_A(\mathcal{E})$. Put for $t \in [0, 1]$,

$$T_t := (1 + \tfrac{1}{2}(\sin \pi t)P)^{-\frac{1}{2}} ((\cos \tfrac{\pi}{2}t)T_0 + (\sin \tfrac{\pi}{2}t)T_1) \in Q_A(\mathcal{E}).$$

Since $T_0^2 - 1, T_1^2 - 1 \in I_A(\mathcal{E})$ and T_0 and T_1 are of degree 1, it follows that modulo $I_A(\mathcal{E})$,

$$[T_0, P] = [T_0, [T_0, T_1]] = T_0(T_0T_1 + T_1T_0) - (T_0T_1 + T_1T_0)T_0 = (T_0^2 - 1)T_1 - T_1(T_0^2 - 1) = 0.$$

Similarly, it follows that $[T_1, P] = 0$ modulo $I_A(\mathcal{E})$. Therefore,

$$\begin{aligned} T_t^2 - 1 &= (1 + \tfrac{1}{2}(\sin \pi t)P)^{-1} \{ (\cos^2 \tfrac{\pi}{2}t)T_0^2 + (\sin^2 \tfrac{\pi}{2}t)T_1^2 + \tfrac{1}{2}(\sin \pi t)[T_0, T_1] \} - 1 \\ &= (1 + \tfrac{1}{2}(\sin \pi t)P)^{-1} \{ (\cos^2 \tfrac{\pi}{2}t)(T_0^2 - 1) + (\sin^2 \tfrac{\pi}{2}t)(T_1^2 - 1) + \tfrac{1}{2}(\sin \pi t)([T_0, T_1] - P) \} \\ &= 0 \quad \text{modulo } I_A(\mathcal{E}), \end{aligned}$$

which means that $T_t^2 - 1 \in I_A(\mathcal{E})$. Similarly, we obtain $T_t - T_t^* \in I_A(\mathcal{E})$ and $\gamma(T_t) - T_t \in I_A(\mathcal{E})$ for any $\gamma \in G$. Then, T_t provides an operator homotopy between T_0 and T_1 . $\square$

Definition 3.29 (Equivariant KK -group) Define $KK^G(A, B)$ as the quotient of $\mathbb{E}^G(A, B)$ by the equivalence relation of homotopy.

Lemma 3.30 $KK^G(A, B)$ has the structure of Abelian group with the addition defined by the direct sum, that is,

$$(\mathcal{E}_1, \varphi_1, T_1) \oplus (\mathcal{E}_2, \varphi_2, T_2) := (\mathcal{E}_1 \oplus \mathcal{E}_2, \varphi_1 \oplus \varphi_2, T_1 \oplus T_2).$$

Proof. The identity element is the 0-module $(0, 0, 0)$. Only the existence of the inverse element is non-trivial. Let $(\mathcal{E}, \varphi, T) \in \mathbb{E}(A, B)$. Recalling that $\mathcal{E}$ is equipped with the $\mathbb{Z}/2\mathbb{Z}$ -grading, define $(\mathcal{E}', \varphi', T') = (\mathcal{E}', \varphi', T')$ as follows

- $\mathcal{E}' := -\mathcal{E}$ denotes the same B -module equipped with opposite grading.
- $\varphi'(a) := u \circ \varphi(\varepsilon(a))u^{-1}$, where $u: \mathcal{E} \rightarrow \mathcal{E}$ is the identity map and $\varepsilon: A \rightarrow A$ denotes the grading operator on A . Note that u reverses the grading.
- $T' := -uTu^{-1}$.

Then $(\mathcal{E} \oplus \mathcal{E}', \varphi \oplus \varphi', \begin{pmatrix} T & 0 \\ 0 & -uTu^{-1} \end{pmatrix})$ is homotopic to a degenerate element $(\mathcal{E} \oplus \mathcal{E}', \varphi \oplus \varphi', \begin{pmatrix} 0 & u^{-1} \\ u & 0 \end{pmatrix})$ by the operator homotopy $\begin{pmatrix} (\cos \frac{\pi}{2}t)T & (\sin \frac{\pi}{2}t)u^{-1} \\ (\sin \frac{\pi}{2}t)u & -(\cos \frac{\pi}{2}t)uTu^{-1} \end{pmatrix}$. Hence, $(\mathcal{E}, \varphi, T) \oplus (\mathcal{E}', \varphi', T') = 0 \in KK^G(A, B)$. $\square$

Definition 3.31 (Functorial properties)

- (1) Let $f: A_1 \rightarrow A_2$ be a homomorphism of graded G -algebras. Then it induces a homomorphism $f^*: KK^G(A_2, B) \rightarrow KK^G(A_1, B)$ by the formula;

$$f^*: (\mathcal{E}, \varphi, T) \mapsto (\mathcal{E}, \varphi \circ f, T).$$

- (2) Let $h: B_1 \rightarrow B_2$ be a homomorphism of graded G -algebras. Then it induces a homomorphism $h_*: KK^G(A, B_1) \rightarrow KK^G(A, B_2)$ by the formula;

$$h_*: (\mathcal{E}, \varphi, T) \mapsto (\mathcal{E} \widehat{\otimes}_h B_2, \varphi \widehat{\otimes} 1, T \widehat{\otimes} 1).$$

- (3) Let D be a σ -unital G -algebra. Then define the homomorphism $\tau_D: KK^G(A, B) \rightarrow KK^G(A \widehat{\otimes} D, B \widehat{\otimes} D)$ by the formula;

$$\tau_D: (\mathcal{E}, \varphi, T) \mapsto (\mathcal{E} \widehat{\otimes} D, \varphi \widehat{\otimes} 1, T \widehat{\otimes} 1).$$

Lemma 3.32 *Let A, A_1, A_2, B, B_1 and B_2 be graded G -algebras. If f_0 and $f_1: A_1 \rightarrow A_2$ are homotopic, that is, there exists a family of homomorphisms $f_t: A_1 \rightarrow A_2$ for $t \in [0, 1]$ between f_0 and f_1 . Then $(f_0)^* = (f_1)^*: KK^G(A_2, B) \rightarrow KK^G(A_1, B)$. Similarly, if h_0 and $h_1: B_1 \rightarrow B_2$ are homotopic, then $(h_0)_* = (h_1)_*: KK^G(A, B_1) \rightarrow KK^G(A, B_2)$.*

In particular, $KK^G(A, B)$ is homotopy invariant in A and B .

Proof. Let $(\mathcal{E}, \varphi, T) \in \mathbb{E}^G(A_2, B)$. Then there exists a homotopy $(I\mathcal{E}, \{\varphi \circ f_t\}, 1 \widehat{\otimes} T) \in \mathbb{E}^G(A_1, IB)$ between $(\mathcal{E}, \varphi \circ f_0, T)$ and $(\mathcal{E}, \varphi \circ f_1, T) \in \mathbb{E}^G(A_1, B)$.

Let $(\mathcal{E}, \varphi, T) \in \mathbb{E}^G(A, B_1)$. Then there exists a homotopy $(\mathcal{E} \widehat{\otimes}_{\{h_t\}} IB_2, \{\varphi \widehat{\otimes}_{h_t} 1\}, \{T \widehat{\otimes}_{h_t} 1\}) \in \mathbb{E}^G(A, IB_2)$ between $(\mathcal{E} \widehat{\otimes}_{h_0} B_2, \varphi \widehat{\otimes}_{h_0} 1, T \widehat{\otimes}_{h_0} 1)$ and $(\mathcal{E} \widehat{\otimes}_{h_1} B_2, \varphi \widehat{\otimes}_{h_1} 1, T \widehat{\otimes}_{h_1} 1) \in \mathbb{E}^G(A_1, B)$. $\square$

3.2.2 Kasparov Product

We will construct the Kasparov product

$$\widehat{\otimes}_D: KK^G(A, D) \otimes KK^G(D, B) \rightarrow KK^G(A, B).$$

Let $(\mathcal{E}_1, \varphi_1, T_1) \in \mathbb{E}^G(A, D)$ and $(\mathcal{E}_2, \varphi_2, T_2) \in \mathbb{E}^G(D, B)$, then we want to define $(\mathcal{E}, \varphi, T) \in \mathbb{E}^G(A, B)$ so that $\mathcal{E}_{12} = \mathcal{E}_1 \widehat{\otimes}_{\varphi_2} \mathcal{E}_2$, $\varphi_{12} = \varphi_1 \widehat{\otimes}_{\varphi_2} 1$ and T will be a suitable operator having the information of T_1 and T_2 . It is natural to consider $T_1 \widehat{\otimes}_{\varphi_2} 1 \in \mathbb{L}(\mathcal{E}_{12})$, on the other hand, $1 \widehat{\otimes}_{\varphi_2} T_2$ does not make sense, that is, “ $(1 \widehat{\otimes} T_2)(x \widehat{\otimes} y)$ ” and “ $(1 \widehat{\otimes} T_2)(x \widehat{\otimes}_{\varphi_2}(d)y)$ ” do not necessarily coincide unless $[T_2, \varphi(d)] = 0$ for any $d \in D$. We will need a suitable operator $T_{12} \in \mathbb{L}(\mathcal{E}_{12})$ corresponding to “ $1 \widehat{\otimes} T_2$ ”, called T_2 -connection.

Definition 3.33 (T_2 -connection) [?, A.1 Definition.][K88, 2.6. Definition.] Let $\mathcal{E}_2$ be a Hilbert B -module, $\varphi_2: D \rightarrow \mathbb{L}_B(\mathcal{E}_2)$ be a left action of D , $T_2 \in \mathbb{L}_B(\mathcal{E}_2)$, like in the definition of $\mathbb{E}^G(D, B)$, and let $\mathcal{E}_1$ be a Hilbert D -module. Set $\mathcal{E}_{12} := \mathcal{E}_1 \widehat{\otimes}_{\varphi_2} \mathcal{E}_2$. $T_{12} \in \mathbb{L}_B(\mathcal{E}_{12})$ is said to be a T_2 -connection for $\mathcal{E}_1$ on $\mathcal{E}_{12}$ if for any $\xi \in \mathcal{E}_1$,

$$\begin{aligned} \theta_\xi T_2 - (-1)^{\deg \xi \deg T_2} T_{12} \theta_\xi &\in \mathbb{K}_B(\mathcal{E}_2, \mathcal{E}_{12}) \\ \text{and} \quad \theta_\xi T_2^* - (-1)^{\deg \xi \deg T_2} T_{12}^* \theta_\xi &\in \mathbb{K}_B(\mathcal{E}_2, \mathcal{E}_{12}), \end{aligned} \quad (3.1)$$

where $\theta_\xi \in \mathbb{L}_B(\mathcal{E}_2, \mathcal{E}_{12})$ is defined by $\theta_\xi(\eta) := \xi \widehat{\otimes} \eta$ and its adjoint is given by $\theta_\xi^*(x \widehat{\otimes} y) := \varphi_2(\langle \xi, x \rangle_{\mathcal{E}_1}) y$.

Example 3.34 In the case of $[T_2, \varphi(d)] = 0$ for any $d \in D$, as the simplest example, $1 \widehat{\otimes} T_2$ makes sense and it provides a T_2 -connection on $\mathcal{E}_{12}$.

Lemma 3.35 (Existence of T_2 -connection) [K88, 2.7. Lemma.] Assume $(\mathcal{E}_2, \varphi_2, T_2) \in \mathbb{E}^G(D, B)$ and let $\mathcal{E}_1$ be countably generated Hilbert D -module. Then there exists a T_2 -connection T_{12} on $\mathcal{E}_{12} = \mathcal{E}_1 \widehat{\otimes}_{\varphi_2} \mathcal{E}_2$.

Proof. Due to the stabilization theorem 3.18 by Kasparov, $\mathcal{E}_1$ can be regarded as a sub-module in $\mathcal{H}_D$, in other words, there exists an operator $V \in \mathbb{L}_D(\mathcal{E}_1, \mathcal{H}_D)$ of degree 1 such that $V^*V = \text{id}_{\mathcal{E}_1}$. Define $T_0 \in \mathbb{L}(\mathcal{H}_D \widehat{\otimes}_{\varphi} \mathcal{E}_2) \simeq \mathbb{L}(\mathcal{H} \widehat{\otimes} \mathcal{E}_2)$ by $T_0 := \text{id}_{\mathcal{H}} \widehat{\otimes}_{\mathbb{C}} T_2$. Since $\mathcal{H} \widehat{\otimes} \mathcal{E}_2$ is the exterior tensor product unlike $\mathcal{E}_1 \widehat{\otimes}_{\varphi_2} \mathcal{E}_2$, T_0 makes sense. Then $T_{12} := (V^* \widehat{\otimes}_{\varphi} 1) T_0 (V \widehat{\otimes}_{\varphi} 1)$ provides a T_2 -connection. $\square$

Lemma 3.36 Let $\mathcal{E}_2$ be a Hilbert B -module, $\varphi: D \rightarrow \mathbb{L}_B(\mathcal{E}_2)$ be a left action of D , $T_2 \in \mathbb{L}_B(\mathcal{E}_2)$, and let $\mathcal{E}_1$ be a Hilbert D -module. and T_{12} be a T_2 -connection on $\mathcal{E}_{12} = \mathcal{E}_1 \widehat{\otimes}_{\varphi_2} \mathcal{E}_2$.

- (1) T_{12}^* is a T_2^* -connection.
- (2) Let $T'_2 \in \mathbb{L}_B(\mathcal{E}_2)$. If T'_{12} is a T'_2 -connection, then $T_{12} + T'_{12}$ is a $(T_2 + T'_2)$ -connection and
- (3) $T_{12}T'_{12}$ is a $(T_2T'_2)$ -connection.
- (4) The space consisting of 0-connections on $\mathcal{E}_{12}$ is equal to

$$\Omega := \{T \in \mathbb{L}_B(\mathcal{E}_{12}) \mid T(K \widehat{\otimes} 1) \text{ and } (K \widehat{\otimes} 1)T \in \mathbb{K}_B(\mathcal{E}_{12}) \text{ for any } K \in \mathbb{K}_D(\mathcal{E}_1)\}.$$

- (5) The space consisting of T_2 -connections is affine space with the associated vector space Ω .
- (6) Assume that $T_2 \in \mathbb{L}_B(\mathcal{E}_2)$ and $T_{12} \in \mathbb{L}_B(\mathcal{E}_{12})$ are normal elements. Let f be a continuous function on the union of their spectra. Then, $f(T_{12})$ is a $f(T_2)$ -connection
- (7) For any $K \in \mathbb{K}_D(\mathcal{E}_1)$, $[T_{12}, K \widehat{\otimes} 1] \in \mathbb{K}_B(\mathcal{E}_{12})$.
- (8) Assume that $\varphi(d)T_2$ and $T_2\varphi(d)$ are G -continuous for any $d \in D$. Then $T_{12} \in Q_{\mathbb{K}_D(\mathcal{E}_1)}(\mathcal{E}_{12})$. If $T_2 \in I_D(\mathcal{E}_2)$, then $T_{12} \in I_{\mathbb{K}_D(\mathcal{E}_1)}(\mathcal{E}_{12})$.

Note that the algebra $\mathbb{K}_D(\mathcal{E}_1)$ acts on $\mathcal{E}_{12}$ through $\mathbb{K}_D(\mathcal{E}_1) \rightarrow \mathbb{L}_B(\mathcal{E}_{12}); K \mapsto K \widehat{\otimes} 1$.

Proof. (1)-(3) The conditions corresponding to (3.1) for T_2^* are the same equations, so (1) is obvious. Since (3.1) are linear, also (2) follows directly from the definition. (3) follows from

$$\begin{aligned} & \theta_{\xi} T_2 T'_2 - (-1)^{\deg \xi \deg(T_2 T'_2)} T_{12} T'_{12} \theta_{\xi} \\ = & \left(\theta_{\xi} T_2 - (-1)^{\deg \xi \deg T_2} T_{12} \theta_{\xi} \right) T'_2 \\ & + (-1)^{\deg \xi \deg T_2} T_{12} \left(\theta_{\xi} T'_2 - (-1)^{\deg \xi \deg T'_2} T'_{12} \theta_{\xi} \right) \in \mathbb{K}_B(\mathcal{E}_2, \mathcal{E}_{12}) \end{aligned}$$

and the similar equation corresponding to the second row in (3.1).

(4) Let T_0 be a 0-connection on $\mathcal{E}_{12}$ and $\xi, \eta \in \mathcal{E}_1$ be arbitrary element. By (3.1), we have $T_0 \theta_{\xi} \theta_{\eta}^* \in \mathbb{K}_B(\mathcal{E}_2, \mathcal{E}_{12})$ and $\theta_{\xi} \theta_{\eta}^*(x \widehat{\otimes} y) = \xi \widehat{\otimes}_{\varphi_2} \langle \eta, x \rangle_{\mathcal{E}_1} y = \xi \langle \eta, x \rangle_{\mathcal{E}_1} \widehat{\otimes} y$, which means operators of the form $\theta_{\xi} \theta_{\eta}^* = \theta_{\xi, \eta} \widehat{\otimes} 1$ generate $\mathbb{K}_D(\mathcal{E}_1) \otimes 1$. ($\theta_{\xi, \eta}$ is defined in Definition 3.15.) Hence

$$T_0(K \widehat{\otimes} 1) \in \mathbb{K}_B(\mathcal{E}_{12}) \text{ for any } K \in \mathbb{K}_D(\mathcal{E}_1).$$

Since the second row of (3.1) is equivalent to

$$T_2 \theta_{\xi}^* - (-1)^{\deg \xi \deg T_2} \theta_{\xi}^* T_{12} \in \mathbb{K}_B(\mathcal{E}_{12}, \mathcal{E}_2), \quad (3.2)$$

it similarly follows that

$$(K \widehat{\otimes} 1)T_0 \in \mathbb{K}_B(\mathcal{E}_{12}) \text{ for any } K \in \mathbb{K}_D(\mathcal{E}_1).$$

To conclude, we obtain that $T_0 \in \Omega$. Conversely, if $T \in \Omega$, then $(T\theta_\xi)(T\theta_\xi)^* \in \mathbb{K}_B(\mathcal{E}_{12})$ and hence $T\theta_\xi \in \mathbb{K}_B(\mathcal{E}_2, \mathcal{E}_{12})$ for any $\xi \in \mathcal{E}_1$. Similarly, we have $T^*\theta_\xi$ and to conclude, T is a 0-connection.

(5)-(6) (5) follows from (2) and (4). (6) follows from (2) and (3).

(7) We may consider only the case of $\deg T_{12} = \deg T_2$. This is because the homogeneous part of T_{12} of different degree from the degree of T_2 is a 0-connection and hence belongs to the vector space Ω so this part satisfies the claim. Since operators of the form $\theta_\xi \theta_\eta^*$ generate $\mathbb{K}_D(\mathcal{E}_1) \otimes 1$, it is sufficient to verify in the case of $K \widehat{\otimes} 1 = \theta_\xi \theta_\eta^*$ for $\xi, \eta \in \mathcal{E}_1$. By (3.1),

$$\begin{aligned} [T_{12}, \theta_\xi \theta_\eta^*] &= T_{12} \theta_\xi \theta_\eta^* - (-1)^{(\deg \xi + \deg \eta) \deg T_{12}} \theta_\xi \theta_\eta^* T_{12} \\ &= \left(T_{12} \theta_\xi - (-1)^{\deg \xi \deg T_2} \theta_\xi T_2 \right) \theta_\eta^* \\ &\quad + (-1)^{\deg \xi \deg T_2} \theta_\xi \left(T_2 \theta_\eta^* - (-1)^{\deg \eta \deg T_2} \theta_\eta^* T_2 \right) \\ &\in \mathbb{K}_B(\mathcal{E}_{12}). \end{aligned}$$

(8) By the definition of $Q_{\mathbb{K}_D(\mathcal{E}_1)}(\mathcal{E}_{12})$, what we have to verify is that $[T_{12}, K \widehat{\otimes} 1] \in \mathbb{K}_B(\mathcal{E}_{12})$ and that $(K \widehat{\otimes} 1)T_{12}$ is G -continuous for any $K \in \mathbb{K}_D(\mathcal{E}_1)$. The first statement is the consequence of (7). To verify the second one, we may consider only the case of $\deg T_{12} = \deg T_2$ and it is sufficient to verify them for $K \widehat{\otimes} 1 = \theta_\xi \theta_\eta^*$ by the same reason as the case of (7).

Let $\xi \in \mathcal{E}_1$ and $\{e_m\}_{m \in \mathcal{M}} \subset D$ be an approximate unit of D . Then we have

$$\begin{aligned} \theta_\xi T_2 &= \varinjlim_m \theta_{\xi e_m} T_2 = \varinjlim_m \theta_\xi \circ \varphi(e_m) T_2, \\ T_2 \theta_\xi^* &= \varinjlim_m T_2 \theta_{\xi e_m}^* = \varinjlim_m T_2 \varphi(e_m^*) \circ \theta_\xi. \end{aligned}$$

Hence $\theta_\xi T_2$ and $T_2 \theta_\xi^*$ are G -continuous for any $\xi \in \mathcal{E}_1$ by the assumption. Since any compact operators are G -continuous by lemma 3.20, it follows from (3.2) that $\theta_\xi^* T_{12}$ is G -continuous. Therefore $\theta_\xi \theta_\eta^* T_{12}$ is G -continuous.

If $T_2 \in I_D(\mathcal{E}_2)$, that is, if $\varphi(d)T_2$ and $T_2\varphi(d)$ are compact operators for any $d \in D$, it follows that $\theta_\xi T_2$ and $T_2 \theta_\xi^*$ are compact operators in the same way. Then we obtain $T_{12} \theta_\xi \in \mathbb{K}(\mathcal{E}_2, \mathcal{E}_{12})$ and $\theta_\eta^* T_{12} \in \mathbb{K}(\mathcal{E}_{12}, \mathcal{E}_2)$ therefore, $\theta_\xi \theta_\eta^* T_{12} \in \mathbb{K}_B(\mathcal{E}_{12}, \mathcal{E}_{12})$ and $T_{12} \theta_\xi \theta_\eta^* \in \mathbb{K}_B(\mathcal{E}_{12}, \mathcal{E}_{12})$, which imply $T_{12} \in I_{\mathbb{K}_D(\mathcal{E}_1)}(\mathcal{E}_{12})$. $\square$

Lemma 3.37 [K88, 2.8. Lemma] *Let D be σ -unital G -algebra. For any $(\mathcal{E}, \varphi, T) \in \mathbb{E}^G(D, B)$, there exists $(\mathcal{E}', \varphi', T') \in \mathbb{E}^G(D, B)$ homotopy equivalent to the initial one, where $\mathcal{E}' = D \widehat{\otimes}_\varphi \mathcal{E}$ and $\varphi' = \text{id}_D \widehat{\otimes}_\varphi \text{id}_\mathcal{E}$. In particular, φ' satisfies $D \widehat{\otimes}_{\varphi'} \mathcal{E}' \cong (D \widehat{\otimes}_{\text{id}_D} D) \widehat{\otimes}_\varphi \mathcal{E} \cong D \widehat{\otimes} \mathcal{E} = \mathcal{E}'$.*

Proof. Let $\tilde{D}$ denote the unitalization of D . Set $Z(D, \tilde{D}) := \left\{ f: [0, 1] \rightarrow \tilde{D} \mid f(1) \in D \right\}$. It is an ideal in $I\tilde{D}$ and can be regarded as a Hilbert $I\tilde{D}$ -module. And consider the following Kasparov module $(\tilde{\mathcal{E}}, m \widehat{\otimes}_{\tilde{\varphi}} 1, \tilde{T}) \in \mathbb{E}^G(D, IB)$ constructed as follows;

- $\tilde{\mathcal{E}} := Z(D, \tilde{D}) \widehat{\otimes}_{1 \widehat{\otimes} \tilde{\varphi}} I\mathcal{E}$, where $\tilde{\varphi}: \tilde{D} \rightarrow \mathbb{L}_B(\mathcal{E})$ denotes the unital extension of φ .
- $m: D \rightarrow \mathbb{L}_{I\tilde{D}}(Z(D, \tilde{D}))$ be the multiplication embedding D into $I\tilde{D}$ as a constant functions.
- Consider $1 \widehat{\otimes} T \in \mathbb{L}_{IB}(I\mathcal{E})$ and construct a $(1 \widehat{\otimes} T)$ -connection $\tilde{T}$ on $\tilde{\mathcal{E}}$.

The restriction to $1 \in [0, 1]$ is $(D \widehat{\otimes}_{\varphi} \mathcal{E}, \text{id}_D \widehat{\otimes}_{\varphi} \text{id}_{\mathcal{E}}, \widetilde{T}_1)$. This gives the required form of Kasparov G -module over (D, B) homotopic to the restriction to $0 \in [0, 1]$.

Next, the restriction to $0 \in [0, 1]$ is $(\widetilde{D} \widehat{\otimes}_{\widetilde{\varphi}} \mathcal{E}, i \widehat{\otimes}_{\widetilde{\varphi}} 1, \widetilde{T}_0)$, where $i: D \rightarrow \widetilde{D}$ denotes the inclusion and $\widetilde{T}_0$ denotes the restriction of $\widetilde{T}$. We claim that $(\widetilde{D} \widehat{\otimes}_{\widetilde{\varphi}} \mathcal{E}, i \widehat{\otimes}_{\widetilde{\varphi}} \text{id}_{\mathcal{E}}, \widetilde{T}_0) = (\mathcal{E}, \varphi, T)$ as elements in $KK^G(D, B)$.

Firstly, we have a canonical isomorphism $\widetilde{D} \widehat{\otimes}_{\widetilde{\varphi}} \mathcal{E} \cong \mathcal{E}$ given by $\widetilde{D} \widehat{\otimes}_{\widetilde{\varphi}} \mathcal{E} \rightarrow \mathcal{E}; d \widehat{\otimes} x \mapsto \widetilde{\varphi}(d)x$ and $\mathcal{E} \rightarrow \widetilde{D} \widehat{\otimes}_{\widetilde{\varphi}} \mathcal{E}; x \mapsto 1 \widehat{\otimes} x$. Under this identification, $i \widehat{\otimes}_{\widetilde{\varphi}} \text{id}_{\mathcal{E}} = \varphi$ in fact, $(i \widehat{\otimes}_{\widetilde{\varphi}} \text{id}_{\mathcal{E}})(d \otimes x) = d \otimes x = 1 \widehat{\otimes} \varphi(d)x \mapsto \varphi(d)x$.

Moreover $\widetilde{T}_0$ can be regarded as a T -connection on $\widetilde{D} \widehat{\otimes}_{\widetilde{\varphi}} \mathcal{E}$ by its construction, and it implies $T - \widetilde{T}_0 \in \mathbb{K}_B(\mathcal{E})$. This follows from the definition 3.33 substituting $\mathcal{E}_1 = \widetilde{D}$ and $\xi = 1 \in \widetilde{D}$. So we have an operator homotopy $T + t(\widetilde{T}_0 - T)$ between T and $\widetilde{T}_0$ and hence, $(\mathcal{E}, \varphi, T)$ and $(\mathcal{E}, \varphi, \widetilde{T}_0)$ are homotopic in $\mathbb{E}^G(D, B)$. $\square$

Definition 3.38 (Intersection products) [?, A.3. Theorem.][?, Definition. 10] (non-equivariant case), [K88, 2.10. Definition.] (equivariant case) Let A, D, B be G -algebras, and $(\mathcal{E}_1, \varphi_1, T_1) \in \mathbb{E}^G(A, D)$, $(\mathcal{E}_2, \varphi_2, T_2) \in \mathbb{E}^G(D, B)$. Then, $(\mathcal{E}_{12}, \varphi_{12}, T_{12})$ is called a Kasparov product of $(\mathcal{E}_1, \varphi_1, T_1)$ and $(\mathcal{E}_2, \varphi_2, T_2)$ if the followings are satisfied;

- (a) $\mathcal{E}_{12} := \mathcal{E}_1 \widehat{\otimes}_{\varphi_2} \mathcal{E}_2$ and $\varphi_{12} := \varphi_1 \widehat{\otimes}_{\varphi_2} 1$.
- (b) $(\mathcal{E}_{12}, \varphi_{12}, T_{12}) \in \mathbb{E}^G(A, B)$
- (c) $T_{12} \in \mathbb{L}_B(\mathcal{E}_{12})$ is a T_2 -connection on $\mathcal{E}_{12}$
- (d) $\varphi_{12}(a) [T_1 \widehat{\otimes}_{\varphi_1} 1, T_{12}] \varphi_{12}(a^*) \geq 0$ modulo $\mathbb{K}_B(\mathcal{E}_{12})$ for any $a \in A$.

Theorem 3.39 [?, A.3 Theorem.][?, Theorem 12] (non-equivariant case), [K88, 2.11 Theorem] (equivariant case) *If A is separable, the Kasparov product of $(\mathcal{E}_1, \varphi_1, T_1) \in \mathbb{E}^G(A, D)$ and $(\mathcal{E}_2, \varphi_2, T_2) \in \mathbb{E}^G(D, B)$ always exists and it is unique up to homotopy. In particular, it defines a bilinear pairing $\widehat{\otimes}_D: KK^G(A, D) \otimes KK^G(D, B) \rightarrow KK^G(A, B)$.*

In order to verify the existence, we use the following technical lemma.

Lemma 3.40 [K88, 1.4 Theorem] *Let J be a σ -unital G -algebra, $\mathcal{M}(J)$ be the multiplier algebra equipped with the norm topology. Let...*

- $Z_1 \subset \mathcal{M}(J)$ be a σ -unital sub-algebra which is a G -algebra,
- Z_2 be an arbitrary σ -unital sub-algebra in $\mathcal{M}(J)$ without the group action,
- $\Delta \subset \mathcal{M}(J)$ be a separable subset such that $[z_1, \delta] \in Z_1$ for any $z_1 \in Z_1$ and $\delta \in \Delta$, and
- $\psi: G \rightarrow \mathcal{M}(J)$ be a bounded function.

Assume that

$$z_1 z_2, z_1 \psi(\gamma), \psi(\gamma) z_1 \in J$$

for any $z_1 \in Z_1, z_2 \in Z_2$ and $\gamma \in G$.

Then, there exist G -continuous positive elements $M_1, M_2 \in \mathcal{M}(J)$ of degree 0 such that

- $M_1 + M_2 = 1$,
- $M_i z_i, [M_i, \delta], M_2 \psi(\gamma), \psi(\gamma) M_2, \gamma(M_i) - M_i \in J$ for any $z_i \in Z_i, \delta \in \Delta, \gamma \in G$ ($i = 1, 2$),
- two maps $G \rightarrow J; \gamma \mapsto M_2 \psi(\gamma)$ and $\gamma \mapsto \psi(\gamma) M_2$ are continuous.

□

Proof of Theorem 3.39. Construction. Put $\mathcal{E}_{12} := \mathcal{E}_1 \widehat{\otimes}_{\varphi_2} \mathcal{E}_2$ and $\varphi_{12} := \varphi_1 \widehat{\otimes}_{\varphi_2} 1$. Let $\tilde{T}_{12} \in \mathbb{L}(\mathcal{E}_{12})$ be a T_2 -connection of degree 1. By lemma 3.36, $\tilde{T}_{12} \in Q_{\mathbb{K}_D(\mathcal{E}_1)}(\mathcal{E}_{12})$. Let...

- $J := \mathbb{K}(\mathcal{E}_{12})$, then $\mathcal{M}(J) = \mathbb{L}(\mathcal{E}_{12})$,
- $Z_1 := \mathbb{K}(\mathcal{E}_{12}) + \mathbb{K}(\mathcal{E}_1) \widehat{\otimes} 1$,
- Z_2 be the minimal C^* -sub-algebra generated by $\left\{ \tilde{T}_{12} - \tilde{T}_{12}^*, \tilde{T}_{12}^2 - 1, [\tilde{T}_{12}, T_1 \widehat{\otimes} 1], [\tilde{T}_{12}, \varphi_{12}(a)] \right\}_{a \in A}$,
- $\Delta := \left\{ T_1 \widehat{\otimes} 1, \tilde{T}_{12}, \varphi_{12}(a) \right\}_{a \in A} \subset \mathbb{L}(\mathcal{E}_{12})$, and
- $\psi: G \rightarrow \mathbb{L}(\mathcal{E}_{12}); \gamma \mapsto \gamma(\tilde{T}_{12}) - \tilde{T}_{12}$.

By Lemma 3.36, they satisfy the condition of Lemma 3.40. Apply it to them to obtain positive and 0-degree operators $M_1, M_2 \in \mathbb{L}(\mathcal{E}_{12})$. Put

$$T_{12} := \sqrt{M_1}(T_1 \widehat{\otimes} 1) + \sqrt{M_2}\tilde{T}_{12}.$$

We claim that T_{12} satisfies the conditions (b)-(d) of Definition 3.38.

(b). Notice that the positive elements $\sqrt{M_1}$ and $\sqrt{M_2}$ also satisfy the conclusion of Lemma 3.40 except for $M_1 + M_2 = 1$. This is because we can approximate the continuous function $\lambda \mapsto \sqrt{\lambda}$ on the interval $[0, 1]$ by polynomials $p_\varepsilon(\lambda)$ due to Weierstrass approximation theorem. Considering $p_\varepsilon(\lambda) - p_\varepsilon(0)$ instead, we may assume $p_\varepsilon(\lambda)$ are of the form $\lambda q_\varepsilon(\lambda)$. Then, for example, $\sqrt{M_1}z_1$ is approximated by $p_\varepsilon(M_1)z_1 = q_\varepsilon(M_1)M_1z_1 \in \mathbb{K}(\mathcal{E}_{12})$ and hence, we obtain $\sqrt{M_1}z_1 \in \mathbb{K}(\mathcal{E}_{12})$. In addition, remark that M_1 and M_2 commute since $M_2 = 1 - M_1$. Using these facts, it is easy to verify that $(\mathcal{E}_{12}, \varphi_{12}, T_{12})$ satisfies the condition (b). For example one can check that, modulo $I_A(\mathcal{E}_{12})$,

$$\begin{aligned} T_{12}^2 - 1 &= \left\{ M_1(T_1 \widehat{\otimes} 1)^2 + M_2\tilde{T}_{12}^2 + \sqrt{M_1}\sqrt{M_2}[T_1 \widehat{\otimes} 1, \tilde{T}_{12}] \right\} - (M_1 + M_2) \\ &= M_1((T_1 \widehat{\otimes} 1)^2 - 1) + M_2(\tilde{T}_{12}^2 - 1) \\ &= 0. \end{aligned}$$

(c). By (4) and (6) of Lemma 3.36, M_1 is, and hence $\sqrt{M_1}$ is a 0-connection. Since $[T_1 \widehat{\otimes} 1, M_1] \in \mathbb{K}(\mathcal{E}_{12})$, $\sqrt{M_1}(T_1 \widehat{\otimes} 1)$ is also a 0-connection. By (2) and (3) of Lemma 3.36, T_{12} is a T_2 -connection.

(d). Since $[\sqrt{M_1}, T_1 \widehat{\otimes} 1]$, $[\sqrt{M_2}, T_1 \widehat{\otimes} 1]$ and $\sqrt{M_2}[T_1 \widehat{\otimes} 1, \tilde{T}_{12}] \in \mathbb{K}(\mathcal{E}_{12})$, we have

$$\begin{aligned} [T_1 \widehat{\otimes} 1, T_{12}] &= \left[T_1 \widehat{\otimes} 1, \sqrt{M_1} \right] (T_1 \widehat{\otimes} 1) + (-1)^{\deg(\sqrt{M_1}) \deg(T_1)} \sqrt{M_1} [T_1 \widehat{\otimes} 1, T_1 \widehat{\otimes} 1] \\ &\quad + \left[T_1 \widehat{\otimes} 1, \sqrt{M_2} \right] \tilde{T}_{12} + (-1)^{\deg(\sqrt{M_2}) \deg(T_1)} \sqrt{M_2} [T_1 \widehat{\otimes} 1, \tilde{T}_{12}] \\ &= \sqrt{M_1} [T_1 \widehat{\otimes} 1, T_1 \widehat{\otimes} 1] \\ &= 2\sqrt{M_1}(T_1^2 \widehat{\otimes} 1) \text{ modulo } \mathbb{K}(\mathcal{E}_{12}) \end{aligned}$$

and hence,

$$\begin{aligned} \varphi_{12}(a) [T_1 \widehat{\otimes} 1, T_{12}] \varphi_{12}(a^*) &= 2\varphi_{12}(a)\sqrt{M_1}(T_1^2 \varphi_1(a^*) \widehat{\otimes} 1) \\ &= 2\varphi_{12}(a)\sqrt{M_1}\varphi_{12}(a^*) \geq 0 \text{ modulo } \mathbb{K}(\mathcal{E}_{12}). \end{aligned}$$

Uniqueness. We first check that the Kasparov product is uniquely determined as a map from $\mathbb{E}^G(A, D) \times \mathbb{E}^G(D, B)$ to $KK^G(A, B)$. Let T_{12} and $T'_{12} \in \mathbb{L}(\mathcal{E}_{12})$ be operators satisfying the definition of the intersection product of $(\mathcal{E}_1, \varphi_1, T_1)$ and $(\mathcal{E}_2, \varphi_2, T_2)$. Let...

- $J := \mathbb{K}(\mathcal{E}_{12})$ and $Z_1 := \mathbb{K}(\mathcal{E}_{12}) + \mathbb{K}(\mathcal{E}_1) \widehat{\otimes} 1$ be the same as above,
- Z_2 be the minimal C^* -sub-algebra generated by $\{T_{12} - T'_{12}, [T_{12}, T_1 \widehat{\otimes} 1], [T'_{12}, T_1 \widehat{\otimes} 1], \}$
- $\Delta := \{T_1 \widehat{\otimes} 1, T_{12}, T'_{12}, \varphi_{12}(a)\}_{a \in A} \subset \mathbb{L}(\mathcal{E}_{12})$, and
- $\psi: G \rightarrow \mathbb{L}(\mathcal{E}_{12}); \gamma \mapsto \gamma(T_{12}) - T_{12}$.

We obtain M'_1 and M'_2 . Similarly to the construction, put

$$T''_{12} := \sqrt{M'_1}(T_1 \widehat{\otimes} 1) + \sqrt{M'_2}T_{12}.$$

Then we have $(\mathcal{E}, \varphi_{12}, T''_{12}) \in \mathbb{E}^G(A, B)$ and $\varphi_{12}(a)[T_{12}, T''_{12}] \varphi_{12}(a^*) = \varphi_{12}(a)[T'_{12}, T''_{12}] \varphi_{12}(a^*) \geq 0$ modulo $\mathbb{K}(\mathcal{E}_{12})$ for any $a \in A$. Due to Lemma 3.28, $(\mathcal{E}, \varphi_{12}, T_{12})$, $(\mathcal{E}, \varphi_{12}, T''_{12})$ and $(\mathcal{E}, \varphi_{12}, T'_{12})$ are homotopic.

Homotopy Invariance. Let $(\bar{\mathcal{E}}_1, \bar{T}_1) \in \mathbb{E}^G(A, ID)$ be a homotopy between $(\mathcal{E}_1^0, T_1^0)$ and $(\mathcal{E}_1^1, T_1^1) \in \mathbb{E}^G(A, D)$ and let $(\mathcal{E}_2, T_2) \in \mathbb{E}^G(D, B)$. Let $(\bar{\mathcal{E}}_{12}, \bar{T}_{12}) \in \mathbb{E}^G(A, IB)$ be the Kasparov product of $(\bar{\mathcal{E}}_1, \bar{T}_1)$ and $\tau_{C([0,1])}(\mathcal{E}_2, T_2) \in \mathbb{E}^G(ID, IB)$. Then, $(\bar{\mathcal{E}}_{12}, \bar{T}_{12}) \in \mathbb{E}^G(A, IB)$ provides a homotopy from the Kasparov product of $(\mathcal{E}_1^0, T_1^0)$ and $(\mathcal{E}_2, T_2)$ to the Kasparov product of $(\mathcal{E}_1^1, T_1^1)$ and $(\mathcal{E}_2, T_2)$. Therefore, the Kasparov product is invariant under the homotopy of the first factor. In the similar way, homotopy invariance in the second factor can be verified. $\square$

Definition 3.41 [K88, 2.12. Definition.] Assume A_2 and B_1 are σ -unital. Then,

$$\widehat{\otimes}_D: KK^G(A_1, B_1 \widehat{\otimes} D) \otimes KK^G(D \widehat{\otimes} A_2, B_2) \rightarrow KK^G(A_1 \widehat{\otimes} A_2, B_1 \widehat{\otimes} B_2)$$

$$\text{is defined by} \quad x \otimes y \mapsto \left(\tau_{A_2}(x) \right) \widehat{\otimes}_{(B_1 \widehat{\otimes} D \widehat{\otimes} A_2)} \left(\tau_{B_1}(y) \right).$$

Examples 3.42 If a homomorphism $f: D_1 \rightarrow D_2$ is given, we can consider $\mathbf{f} := (D_2, f, 0) \in \mathbb{E}^G(D_1, D_2)$.

- (1) For $x \in KK^G(D_2, B)$, by lemma 3.37, we may choose $(\mathcal{E}_2, \varphi_2, T_2) \in \mathbb{E}^G(D_2, B)$ representing x such that $D_2 \widehat{\otimes}_{\varphi_2} \mathcal{E}_2 \cong \mathcal{E}_2$ holds. Then T_2 is a T_2 -connection on $D_2 \widehat{\otimes}_{\varphi_2} \mathcal{E}_2 \cong \mathcal{E}_2$. So $(\mathcal{E}_2, \varphi \circ f, T_2)$ is a Kasparov product of $(D_2, f, 0)$ and $(\mathcal{E}_2, \varphi_2, T_2)$. Recall that $(\mathcal{E}_2, \varphi \circ f, T_2)$ represents $f^*(x) \in KK^G(D_1, B)$. To conclude, we have $\mathbf{f} \widehat{\otimes}_{D_2} x = f^*(x)$.
- (2) For $y \in KK^G(A, D_1)$, choose $(\mathcal{E}_1, \varphi_1, T_1) \in \mathbb{E}^G(A, D_1)$ representing y . Then $T_1 \widehat{\otimes} 1$ is a 0-connection on $\mathcal{E}_1 \widehat{\otimes}_f D_2$. So $(\mathcal{E}_1 \widehat{\otimes}_f D_2, \varphi \widehat{\otimes}_f 1, T_1 \widehat{\otimes} 1)$ is a Kasparov product of $(\mathcal{E}_1, \varphi_1, T_1)$ and $(D_2, f, 0)$. Recall that $(\mathcal{E}_1 \widehat{\otimes}_f D_2, \varphi \widehat{\otimes}_f 1, T_1 \widehat{\otimes} 1)$ represents $g_*(y) \in KK^G(A, D_2)$. To conclude, we have $y \widehat{\otimes}_{D_1} \mathbf{f} = f_*(y)$.

Proposition 3.43 (Functoriality of the Kasparov product) [?, Proposition 13], [K88, 2.14. Theorem.] Assume that A, A_1 and A_2 are separable.

- (1) Let $f: A_1 \rightarrow A_2$ be a homomorphism. Then for any $x \in KK^G(A_2, D)$ and $y \in KK^G(D, B)$,

$$f^*(x) \widehat{\otimes}_D y = f^*(x \widehat{\otimes}_D y).$$

- (2) Let $g: D_1 \rightarrow D_2$ be a homomorphism. Then for any $x \in KK^G(A, D_1)$ and $y \in KK^G(D_2, B)$,

$$g_*(x) \widehat{\otimes}_{D_2} y = x \widehat{\otimes}_{D_1} g^*(y).$$

- (3) Let $h: B_1 \rightarrow B_2$ be a homomorphism. Then for any $x \in KK^G(A, D)$ and $y \in KK^G(D, B_1)$,

$$x \widehat{\otimes}_D h_*(y) = h_*(x \widehat{\otimes}_D y).$$

Proof. (1) Let x and y be represented by $(\mathcal{E}_1, \varphi_1, T_1) \in \mathbb{E}^G(A_2, D)$ and $(\mathcal{E}_2, \varphi_2, T_2) \in \mathbb{E}^G(D, B)$, and let their Kasparov product $x \hat{\otimes}_D y$ be represented by $(\mathcal{E}_{12}, \varphi_{12}, T_{12})$. Then, $(\mathcal{E}_{12}, \varphi_{12} \circ f, T_{12})$, which represents $f^*(x \hat{\otimes}_D y)$, is a Kasparov product of $(\mathcal{E}_1, \varphi_1 \circ f, T_1)$ and $(\mathcal{E}_2, \varphi_2, T_2)$, which represent $f^*(x)$ and y .

(2) Let x and y be represented by $(\mathcal{E}_1, \varphi_1, T_1) \in \mathbb{E}^G(A, D_1)$ and $(\mathcal{E}_2, \varphi_2, T_2) \in \mathbb{E}^G(D_2, B)$. Then, $x \hat{\otimes}_{D_1} g^*(y)$ is represented by $(\mathcal{E}_1 \hat{\otimes}_{\varphi_2 \circ g} \mathcal{E}_2, \varphi_1 \hat{\otimes} 1, T)$ for some T_2 -connection T for $\mathcal{E}_1$. Note that there is a canonical isomorphism

$$\mathcal{E}_1 \hat{\otimes}_{\varphi_2 \circ g} \mathcal{E}_2 \cong (\mathcal{E}_1 \hat{\otimes}_g D_2) \hat{\otimes}_{\varphi_2} \mathcal{E}_2.$$

Under this isomorphism, let T' be denote the T_2 -connection for $(\mathcal{E}_1 \hat{\otimes}_g D_2)$ corresponding to T . Then,

$$((\mathcal{E}_1 \hat{\otimes}_g D_2) \hat{\otimes}_{\varphi_2} \mathcal{E}_2, (\varphi_1 \hat{\otimes} 1) \hat{\otimes} 1, T')$$

is a Kasparov product of $(\mathcal{E} \hat{\otimes}_g D_2, \varphi_1 \hat{\otimes} 1, T_1)$ and $(\mathcal{E}_2, \varphi_2, T_2)$, which represent $g_*(x)$ and y .

(3) Let x and y be represented by $(\mathcal{E}_1, \varphi_1, T_1) \in \mathbb{E}^G(A, D)$ and $(\mathcal{E}_2, \varphi_2, T_2) \in \mathbb{E}^G(D, B_1)$, and let their Kasparov product $x \hat{\otimes}_D y$ be represented by $(\mathcal{E}_{12}, \varphi_{12}, T_{12})$. Then, $(\mathcal{E}_{12} \hat{\otimes}_h B_2, \varphi_{12} \hat{\otimes} 1, T_{12} \hat{\otimes} 1)$, which represents $h_*(x \hat{\otimes}_D y)$, is a Kasparov product of $(\mathcal{E}_1, \varphi_1, T_1)$ and $(\mathcal{E}_2 \hat{\otimes}_h B_2, \varphi_2 \hat{\otimes} 1, T_2 \hat{\otimes} 1)$, which represent x and $h_*(y)$. $\square$

Proposition 3.44 (Associativity of the Kasparov product) [?, Theorem 21], [K88, 2.14. Theorem.] *Assume that A and D_1 are separable. For any $x \in KK^G(A, D_1)$, $y \in KK^G(D_1, D_2)$ and $z \in KK^G(D_2, B)$,*

$$(x \hat{\otimes}_{D_1} y) \hat{\otimes}_{D_2} z = x \hat{\otimes}_{D_1} (y \hat{\otimes}_{D_2} z).$$

We need the following lemma for the proof.

Lemma 3.45 *Let $(\mathcal{E}_1, \varphi_1, T_1) \in \mathbb{E}(A, D_1)$, $(\mathcal{E}_2, \varphi_2, T_2) \in \mathbb{E}(D_1, D_2)$, $(\mathcal{E}_3, \varphi_3, T_3) \in \mathbb{E}(D_2, B)$. Set $\mathcal{E}_{12} := \mathcal{E}_1 \hat{\otimes}_{\varphi_2} \mathcal{E}_2$, $\mathcal{E}_{23} := \mathcal{E}_2 \hat{\otimes}_{\varphi_3} \mathcal{E}_3$ and $\mathcal{E}_{123} := \mathcal{E}_1 \hat{\otimes}_{\varphi_2} \mathcal{E}_{23} \cong \mathcal{E}_{12} \hat{\otimes}_{\varphi_3} \mathcal{E}_3$. Let T_{12} be a T_2 -connection on $\mathcal{E}_{12}$ and T_{23} be a T_3 -connection on $\mathcal{E}_{23}$.*

If T_{123} be a T_{23} -connection on $\mathcal{E}_{123}$, then $[T_{12} \hat{\otimes}_{\varphi_3} 1, T_{123}]$ is a $[T_2 \hat{\otimes}_{\varphi_3} 1, T_{23}]$ -connection on $\mathcal{E}_{123}$.

Proof. We may assume that every operators are homogeneous due to Lemma 3.36. Let

$$\begin{aligned} \widetilde{T_{123}} &:= T_{23} \oplus T_{123} \in \mathbb{L}(\mathcal{E}_{23} \oplus \mathcal{E}_{123}) \\ \widetilde{T_{12}} &:= T_2 \oplus T_{12} \in \mathbb{L}(\mathcal{E}_2 \oplus \mathcal{E}_{12}) \\ \widetilde{F_{123}} &:= [T_2 \hat{\otimes} 1, T_{23}] \oplus [T_{12} \hat{\otimes} 1, T_{123}] = [\widetilde{T_{12}} \hat{\otimes} 1, \widetilde{T_{123}}] \in \mathbb{L}(\mathcal{E}_{23} \oplus \mathcal{E}_{123}). \end{aligned}$$

For $\xi \in \mathcal{E}_1$, let $\tilde{\theta}_\xi := \begin{pmatrix} 0 & \theta_\xi^* \\ \theta_\xi & 0 \end{pmatrix} \in \mathbb{L}(\mathcal{E}_{23} \oplus \mathcal{E}_{123})$. It is sufficient to verify that $[\widetilde{F_{123}}, \tilde{\theta}_\xi] \in \mathbb{K}(\mathcal{E}_{23} \oplus \mathcal{E}_{123})$.

$$(-1)^{rp} \left[[\widetilde{T_{12}} \hat{\otimes} 1, \widetilde{T_{123}}], \tilde{\theta}_\xi \right] + (-1)^{pq} \left[[\widetilde{T_{123}}, \tilde{\theta}_\xi], \widetilde{T_{12}} \hat{\otimes} 1 \right] + (-1)^{qr} \left[[\tilde{\theta}_\xi, \widetilde{T_{12}} \hat{\otimes} 1], \widetilde{T_{123}} \right] = 0$$

where $p = \deg(\widetilde{T_{12}})$, $q = \deg(\widetilde{T_{123}})$ and $r = \deg(\xi)$. Since T_{123} and T_{12} are connections respectively, both $[\widetilde{T_{123}}, \tilde{\theta}_\xi]$ and $[\tilde{\theta}_\xi, \widetilde{T_{12}} \hat{\otimes} 1]$ belong to $\mathbb{K}(\mathcal{E}_{23} \oplus \mathcal{E}_{123})$, and consequently, the first term $[[\widetilde{T_{12}} \hat{\otimes} 1, \widetilde{T_{123}}], \tilde{\theta}_\xi]$ belongs to $\mathbb{K}(\mathcal{E}_{23} \oplus \mathcal{E}_{123})$. $\square$

Proof of Proposition 3.44. Let x, y and z be represented by $(\mathcal{E}_1, \varphi_1, T_1) \in \mathbb{E}^G(A, D_1)$, $(\mathcal{E}_2, \varphi_2, T_2) \in \mathbb{E}^G(D_1, D_2)$ and $(\mathcal{E}_3, \varphi_3, T_3) \in \mathbb{E}^G(D_2, B)$. We may assume that T_1, T_2 and T_3 are self adjoint replacing T_j by $\frac{1}{2}(T_j + T_j^*)$ with their difference $\frac{1}{2}(T_j - T_j^*)$ belonging to $I_A(\mathcal{E}_1)$, $I_{D_1}(\mathcal{E}_2)$ and $I_{D_2}(\mathcal{E}_3)$. As in the previous lemma, let $\mathcal{E}_{12} := \mathcal{E}_1 \hat{\otimes}_{\varphi_2} \mathcal{E}_2$, $\mathcal{E}_{23} := \mathcal{E}_2 \hat{\otimes}_{\varphi_3} \mathcal{E}_3$ and $\mathcal{E}_{123} := \mathcal{E}_1 \hat{\otimes}_{\varphi_2} \mathcal{E}_{23} (\cong \mathcal{E}_{12} \hat{\otimes}_{\varphi_3} \mathcal{E}_3)$.

Let the Kasparov products $x \hat{\otimes}_{D_1} y$, $y \hat{\otimes}_{D_2} z$ and $x \hat{\otimes}_{D_1} (y \hat{\otimes}_{D_2} z)$ be represented by

$$(\mathcal{E}_{12}, \varphi_1 \hat{\otimes} 1, T_{12}), (\mathcal{E}_{23}, \varphi_2 \hat{\otimes} 1, T_{23}) \quad \text{and} \quad (\mathcal{E}_{123}, \varphi_1 \hat{\otimes} 1 \hat{\otimes} 1, T_{123}).$$

with every operators being self-adjoint. By the previous lemma, $[T_{12} \hat{\otimes} 1, T_{123}]$ is a $[T_2 \hat{\otimes} 1, T_{23}]$ -connection on $\mathcal{E}_{123}$. By (1) and (6) of Lemma 3.36, $[T_{12} \hat{\otimes} 1, T_{123}]_{\pm}$ is a $[T_2 \hat{\otimes} 1, T_{23}]_{\pm}$ -connection, where $F_+ := \frac{1}{2} (F + \sqrt{F^* F})$ denotes the positive part of the self adjoint operator F , and F_- denotes its negative part. Since $\varphi_2(d_1)[T_2 \hat{\otimes} 1, T_{23}]\varphi_2(d_1^*) \geq 0$ modulo $\mathbb{K}(\mathcal{E}_{23})$ for any $d_1 \in D_1$, it follows that $[T_{12} \hat{\otimes} 1, T_{123}]_-$ is a 0-connection for the action of $\mathcal{E}_1$. Let...

- $J := \mathbb{K}(\mathcal{E}_{123})$,
- $Z_1 := \mathbb{K}(\mathcal{E}_{123}) + \mathbb{K}(\mathcal{E}_{12}) \hat{\otimes} 1 + \mathbb{K}(\mathcal{E}_1) \hat{\otimes} 1 \hat{\otimes} 1$,
- Z_2 be the minimal C^* -sub-algebra generated by $\{[T_{123}, T_{12} \hat{\otimes} 1]_-, [T_{123}, T_1 \hat{\otimes} 1 \hat{\otimes} 1]\}$
- $\Delta := \{T_{123}, T_{12} \hat{\otimes} 1, T_1 \hat{\otimes} 1 \hat{\otimes} 1, \varphi_1(a) \hat{\otimes} 1 \hat{\otimes} 1\}_{a \in A} \subset \mathbb{L}(\mathcal{E}_{123})$, and
- $\psi: G \rightarrow \mathbb{L}(\mathcal{E}_{123}); \gamma \mapsto \gamma(T_{123}) - T_{123}$.

Apply Lemma 3.40 to obtain positive and 0-degree operators $M_1, M_2 \in \mathbb{L}(\mathcal{E}_{123})$. Put

$$T'_{123} := \sqrt{M_1}(T_{12} \hat{\otimes} 1) + \sqrt{M_2}T_{123}.$$

We can verify that $(\mathcal{E}_{123}, \varphi_1 \hat{\otimes} 1 \hat{\otimes} 1, T'_{123})$ represents $(x \hat{\otimes}_{D_1} y) \hat{\otimes}_{D_2} z$ in the similar argument in the proof of Theorem 3.39 (Construction). For example, since $\sqrt{M_1}(K \hat{\otimes} 1) \in \mathbb{K}(\mathcal{E}_{123})$ for any $K \in \mathbb{K}(\mathcal{E}_{12})$ and $\sqrt{M_i}$ commutes with T_{123} and $T_{12} \hat{\otimes} 1$, we have that modulo $\mathbb{K}(\mathcal{E}_{123})$,

$$(T_{123} - T'_{123})(K \hat{\otimes} 1) = -(T_{12} \hat{\otimes} 1)\sqrt{M_1}(K \hat{\otimes} 1) + T_{123} (1 + \sqrt{M_2})^{-1} (1 - M_2)(K \hat{\otimes} 1) = 0$$

for any $K \in \mathbb{K}(\mathcal{E}_{12})$. Note that $1 + \sqrt{M_2}$ is invertible in $\mathbb{L}(\mathcal{E}_{123})$ since $1 \leq 1 + \sqrt{M_2} \leq 2$. Similarly, we have that $(K \hat{\otimes} 1)(T_{123} - T'_{123}) \in \mathbb{K}(\mathcal{E}_{123})$. These imply that $(T_{123} - T'_{123})$ is a 0-connection on $\mathcal{E}_{123}$ for $\mathcal{E}_{12}$. Since T_{123} is a T_3 -connection on $\mathcal{E}_{123}$ for $\mathcal{E}_{12}$, so is T'_{123} .

Besides, we can verify that $(\mathcal{E}_{123}, \varphi_1 \hat{\otimes} 1 \hat{\otimes} 1, T'_{123})$ is homotopic to $(\mathcal{E}_{123}, \varphi_1 \hat{\otimes} 1 \hat{\otimes} 1, T_{123})$. This is because we have, modulo $\mathbb{K}(\mathcal{E}_{123})$,

$$\begin{aligned} (\varphi_1(a) \hat{\otimes} 1 \hat{\otimes} 1) [T_{123}, T'_{123}] (\varphi_1(a^*) \hat{\otimes} 1 \hat{\otimes} 1) &= \sqrt{M_1} (\varphi_1(a) \hat{\otimes} 1 \hat{\otimes} 1) [T_{123}, T_{12} \hat{\otimes} 1] (\varphi_1(a^*) \hat{\otimes} 1 \hat{\otimes} 1) \\ &\quad + 2\sqrt{M_2} (\varphi_1(a) \hat{\otimes} 1 \hat{\otimes} 1) T_{123}^2 (\varphi_1(a^*) \hat{\otimes} 1 \hat{\otimes} 1) \geq 0 \end{aligned}$$

for any $a \in A$, noting that we have already had that T_{123} satisfies the condition (d) of Definition 3.38. Then, $(\mathcal{E}_{123}, \varphi_1 \hat{\otimes} 1 \hat{\otimes} 1, T'_{123})$ also represents $x \hat{\otimes}_{D_1} (y \hat{\otimes}_{D_2} z)$. To conclude, we obtain that $(x \hat{\otimes}_{D_1} y) \hat{\otimes}_{D_2} z = x \hat{\otimes}_{D_1} (y \hat{\otimes}_{D_2} z)$. $\square$

4 Geometrical Preparation for Description of the Index Map

4.1 Hermitian G -Vector Bundles, Connections, and Characteristic Classes

Definition 4.1 (Hermitian G -vector bundles, G -invariant connections)

Let X be a proper G -Riemannian manifold.

- (1) A smooth G -vector bundle E is a smooth vector bundle over X on which G smoothly acts and the action is compatibility with the the action on the base space X , and the action induced on fibers are linear isomorphisms, that is,

$$\pi(\gamma\sigma) = \gamma\pi(\sigma)$$

holds for any $\sigma \in E$ and $\gamma \in G$, where $\pi: E \rightarrow X$ denotes the projection, and $\gamma: E_x \rightarrow E_{\gamma x}$ is a linear isomorphism. If $E = E^0 \hat{\otimes} E^1$ is graded, the action of G is assumed to preserve the gradings.

- (2) Let $C^\infty(E)$ denote the space consisting of all smooth sections $s: X \rightarrow E$. G acts on $C^\infty(E)$ by the formula;

$$\gamma[s](x) = \gamma(s(\gamma^{-1}x)).$$

- (3) A G -vector bundle is called Hermitian G -vector bundle if E is equipped with a G -invariant smooth Hermitian metric $h(\cdot, \cdot) = \langle \cdot, \cdot \rangle_{E_x}$, that is, $\langle \gamma\sigma_1, \gamma\sigma_2 \rangle_{E_{\gamma x}} = \langle \sigma_1, \sigma_2 \rangle_{E_x}$ holds for any $\sigma_1, \sigma_2 \in E_x$ and $\gamma \in G$.

- (4) A connection ∇ on a Hermitian G -vector bundle is called a G -invariant connection if ∇ is compatible with the Hermitian metric and commutes with the action of G , that is,

$$d \langle s(x), t(x) \rangle = \langle \nabla s(x), t(x) \rangle + \langle s(x), \nabla t(x) \rangle \quad \text{and} \quad \gamma[\nabla s] = \nabla(\gamma[s])$$

hold for any $s, t \in C^\infty(E)$ and $\gamma \in G$.

Definition 4.2 A Hermitian G -vector bundle E over X defines the following element $[E]$ in KK -theory;

$$[E] = (C_0(E), 0) \in KK^G(C_0(X), C_0(X))$$

with the action of $C_0(X)$ on $C_0(E)$ being the point-wise multiplication.

Here, $C_0(X)$ denotes the C^* -algebra consisting of all $\mathbb{C}$ -valued continuous functions on X vanishing at infinity equipped with the sup-norm, and $C_0(E)$ denotes the Hilbert $C_0(X)$ module consisting of all continuous sections of E vanishing at infinity equipped with the scalar product given by the point wise Hermitian metric.

Lemma 4.3 *Let G be a second countable locally compact Hausdorff group. Let X be a G -compact proper complete G -Riemannian manifold.*

(1) *Any smooth G -vector bundle E over X admits a smooth G -invariant Hermitian metric.*

(2) *Any Hermitian G -vector bundle E over X admits a G -invariant connection.*

Proof. Choose any cut-off function $c \in C_c(X)$. The only we have to do is to take the average using c . For the first part, fix any smooth Hermitian metric h on E which is not necessarily G -invariant. Then set

$$\tilde{h}_{E_x}(\sigma_1, \sigma_2) := \int_G c(g^{-1}x) h_{E_{g^{-1}x}}(g^{-1}\sigma_1, g^{-1}\sigma_2) dg \quad \text{for } \sigma_1, \sigma_2 \in E_x.$$

Note that for any fixed $x \in X$, $\text{supp}\{c(\gamma^{-1}x)\} \subset G$ is compact due to the properness of the action so that the integral above makes sense. $\tilde{h}$ is a desired metric. In fact for any fixed $\gamma \in G$,

$$\begin{aligned} \tilde{h}_{E_{\gamma x}}(\gamma\sigma_1, \gamma\sigma_2) &= \int_G c(g^{-1}\gamma x) h_{E_{g^{-1}\gamma x}}(g^{-1}\gamma\sigma_1, g^{-1}\gamma\sigma_2) dg \\ &= \int_G c(g^{-1}\gamma x) h_{E_{g^{-1}\gamma x}}(g^{-1}\gamma\sigma_1, g^{-1}\gamma\sigma_2) d(\gamma^{-1}g) \\ &= \tilde{h}_{E_x}(\sigma_1, \sigma_2). \end{aligned}$$

It follows from $c \geq 0$ that $\tilde{h}$ is positive definite.

As for the second part, let E be a Hermitian G -vector bundle over X and fix a connection compatible with the Hermitian metric $\nabla: C^\infty(E) \rightarrow C^\infty(T^*X \otimes E)$ which is not necessarily commutes with the action of G . Then set

$$\tilde{\nabla}s(x) := \int_G c(g^{-1}x)g\nabla g^{-1}s(x) \, dg \quad \text{for } s \in C^\infty(E).$$

□

Note that the action of G on E and X induces the action on $E \otimes T^*X$. $\tilde{\nabla}$ is a desired connection. In fact for any fixed $\gamma \in G$,

$$\begin{aligned} \gamma[\tilde{\nabla}s](x) &= \gamma(\tilde{\nabla}s(\gamma^{-1}x)) \\ &= \int_G c(g^{-1}\gamma^{-1}x)\gamma g\nabla g^{-1}s(\gamma^{-1}x) \, dg \\ &= \int_G c(\{\gamma g\}^{-1}x)\{\gamma g\}\nabla\{\gamma g\}^{-1}\gamma s(\gamma^{-1}x) \, d\{\gamma g\} \\ \tilde{\nabla}(\gamma[s])(x) &= \int_G c(g^{-1}x)g\nabla g^{-1}\gamma s(\gamma^{-1}x) \, dg, \\ \text{therefore } \gamma[\tilde{\nabla}s](x) &= \tilde{\nabla}(\gamma[s])(x). \end{aligned}$$

The compatibility with the Hermitian metric follows from;

$$\begin{aligned} \langle \tilde{\nabla}s_1(x), s_2(x) \rangle + \langle s_1(x), \tilde{\nabla}s_2(x) \rangle &= \int_G c(g^{-1}) \langle g\nabla g^{-1}s_1(x), s_2(x) \rangle + \langle s_1(x), g\nabla g^{-1}s_2(x) \rangle \, dg \\ &= \int_G c(g^{-1}) \langle \nabla g^{-1}(s_1(x), g^{-1}s_2(x)) \rangle + \langle s_1(x), g^{-1}\nabla g^{-1}(s_2(x)) \rangle \, dg \\ &= \int_G c(g^{-1})d \langle g^{-1}s_1(x), g^{-1}s_2(x) \rangle \, dg \\ &= \int_G c(g^{-1})d \langle s_1(x), s_2(x) \rangle \, dg \\ &= d \langle s_1(x), s_2(x) \rangle. \end{aligned}$$

The Lipnitz rule follows from;

$$\begin{aligned} \tilde{\nabla}(fs)(x) &= \int_G c(g^{-1}x)g\nabla fg^{-1}s(x) \, dg \\ &= df \otimes \int_G c(g^{-1}x)gg^{-1}s(x) \, dg + f(x) \int_G c(g^{-1}x)g\nabla g^{-1}s(x) \, dg \\ &= df \otimes s(x) + f(x)\tilde{\nabla}s(x). \end{aligned}$$

Remark 4.4 Let E be a smooth Hermitian G -vector bundle, then G acts on $\text{End}(E)$ by

$$\gamma[T] = \gamma T \gamma^{-1} \quad \text{for } T \in \text{End}(E), \gamma \in G.$$

Then, $\gamma(T(\sigma)) = \gamma[T](\gamma\sigma)$ and $\gamma[S \circ T] = \gamma[S] \circ \gamma[T]$ hold for any $S, T \in \text{End}(E)_x$ and $\sigma \in E_x$.

Definition 4.5 (Chern classes and Chern character) Let E be a smooth Hermitian G -vector bundle over X . Choose a G -invariant connection ∇ on E and let $\Theta^\nabla \in C^\infty(\wedge^2(T^*X) \otimes \mathfrak{u}(E))$

denote the curvature 2-form. Then define the Chern classes $c_k(E, \nabla) \in \Omega^{2k}(X)$ and the Chern character $\text{ch}(E, \nabla) \in \Omega^*(X)$ by the formula;

$$\sum_k c_k(E, \nabla) \lambda^k = \det \left(\frac{i\lambda \Theta^\nabla}{2\pi} - \text{id}_E \right) \quad \text{and} \quad \text{ch}(E, \nabla) = \text{tr} \left(\exp \left(\frac{i\Theta^\nabla}{2\pi} \right) \right).$$

We may omit ∇ when it is clear from the context. As usual, $c_k(E, \nabla)$ and $\text{ch}(E, \nabla)$ are closed form.

Remark 4.6 The following two lemmas are valid also for any other characteristic classes constructed using the Chern Weil theory.

Lemma 4.7 *Let ∇ be a G -invariant connection on E . Then Θ^∇ , $c_k(E, \nabla)$ and $\text{ch}(E, \nabla)$ are G -invariant differential form.*

Proof. Note that $\Theta^\nabla \wedge s = d^\nabla d^\nabla s$, where $d^\nabla: C^\infty(\bigwedge^k T^*X \otimes E) \rightarrow C^\infty(\bigwedge^{k+1} T^*X \otimes E)$ denotes the exterior covariant derivative defined by

$$d^\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s \quad \text{for } \omega \in \Omega^k(X), s \in C^\infty(E),$$

which commutes with the action of G . Hence, Θ^∇ is G -invariant and we can conclude that $c_k(E, \nabla)$ and $\text{ch}(E, \nabla)$ are G -invariant since they are expressed by polynomials in $\text{tr}(\Theta^\nabla)$, $\text{tr}((\Theta^\nabla)^2)$, $\text{tr}((\Theta^\nabla)^3), \dots$. $\square$

Lemma 4.8 *Let E be a smooth Hermitian G -vector bundle over X . Let ∇^0 and ∇^1 be G -invariant connections on E . Then there exist G -invariant differential forms of odd degree $\theta_k \in \Omega^{2k-1}(X)$ for each k and $\theta \in \Omega^{\text{odd}}(X)$ such that*

$$d\theta_k = c_k(E, \nabla^1) - c_k(E, \nabla^0) \quad \text{and} \quad d\theta = \text{ch}(E, \nabla^1) - \text{ch}(E, \nabla^0).$$

In particular $\int_X c(x) \text{ch}(E, \nabla)$ and $\int_X c(x) P(c_1(E, \nabla), c_2(E, \nabla), \dots)$, where P is an arbitrary multi-variable polynomial, are independent of the choice of G -invariant connection ∇ (and a cut-off function c) due to the lemma 2.5.

Proof. Since $(\nabla^1 - \nabla^0)(fs) = f(\nabla^1 - \nabla^0)s$ for $f \in C^\infty(X)$ and $s \in C^\infty(E)$, there exists $\eta \in C^\infty(\bigwedge^1 T^*X \otimes \text{End}(E))$ such that

$$(\nabla^1 - \nabla^0)s = \eta \wedge s.$$

Note that η is G -invariant since ∇^0 and ∇^1 are G -invariant connections. Consider a smooth family of connections for $t \in [0, 1]$

$$\nabla^t := \nabla^0 + t\eta.$$

We need a local expression of the curvature 2-forms. Fix a local trivialization $E|_U \rightarrow U \times \mathbb{C}^N$ and express $\nabla^0 = d + \omega$. Then we have

$$\nabla^t = d + \omega + t\eta \quad \text{and} \quad \Theta(t) := \Theta^{\nabla^t} = d\omega(t) + \omega(t) \wedge \omega(t),$$

where $\omega(t) = \omega + t\eta$. Note that also $\Theta(t)$ are G -invariant since ∇^t are commute with the action of G .

We will omit the symbol $\wedge$ for the rest of the proof if there is no chance of leading to misunderstandings. Due to the Bianchi identity for ∇^t , that is, $d\Theta(t) = \Theta(t)\omega(t) - \omega(t)\Theta(t)$, we have

$$\begin{aligned}
d\Theta(t)^m &= \sum_{j=1}^m \Theta(t)^{m-j} (d\Theta(t)) \Theta(t)^{j-1} \\
&= \sum_{j=1}^m \Theta(t)^{m-j} (\Theta(t)\omega(t) - \omega(t)\Theta(t)) \Theta(t)^{j-1} \\
&= \sum_{j=1}^m \Theta(t)^{m-j+1} \omega(t) \Theta(t)^{j-1} - \sum_{j=1}^m \Theta(t)^{m-j} \omega(t) \Theta(t)^j \\
&= \Theta(t)^m \omega(t) - \omega(t) \Theta(t)^m.
\end{aligned} \tag{4.1}$$

In particular $\text{tr}(\Theta(t)^m)$ are closed forms due to the symmetry of the trace;

$$d\text{tr}(\Theta(t)^m) = \text{tr}(d\Theta(t)^m) = \text{tr}\{\Theta(t)^m \omega(t) - \omega(t) \Theta(t)^m\} = 0. \tag{4.2}$$

Due to (4.1), the symmetry of the trace and the fact that ω and $\eta\Theta(t)^m$ are of odd degree,

$$\begin{aligned}
&\text{tr}\{(\eta\omega(t) + \omega(t)\eta)\Theta(t)^m\} \\
&= \text{tr}\{\eta\omega(t)\Theta(t)^m - \eta\Theta(t)^m\omega(t)\} \\
&= \text{tr}\{\eta \wedge (\omega(t)\Theta(t)^m - \Theta(t)^m\omega(t))\} \\
&= -\text{tr}\{\eta \wedge d\Theta(t)^m\}.
\end{aligned} \tag{4.3}$$

Using (4.3) and again the symmetry of the trace,

$$\begin{aligned}
\frac{d}{dt} \left(\text{tr}(\Theta(t)^k) \right) &= \text{tr} \left(\frac{d}{dt} \Theta(t)^k \right) = \text{tr} \left\{ \sum_{j=1}^k \Theta(t)^{k-j} \left(\frac{d}{dt} \Theta(t) \right) \Theta(t)^{j-1} \right\} \\
&= \sum_{j=1}^k \text{tr} \left\{ \left(\frac{d}{dt} \Theta(t) \right) \Theta(t)^{k-1} \right\} \\
&= k \text{tr} \left\{ (d\eta + \eta\omega(t) + \omega(t)\eta) \Theta(t)^{k-1} \right\} \\
&= k \left(\text{tr} \left\{ d\eta \wedge \Theta(t)^{k-1} \right\} - \text{tr} \left\{ \eta \wedge d\Theta(t)^{k-1} \right\} \right) \\
&= k \text{tr} \left\{ d \left(\eta \wedge \Theta(t)^{k-1} \right) \right\} \\
&= d \left(k \text{tr} \left(\eta \wedge \Theta(t)^{k-1} \right) \right).
\end{aligned}$$

$$\begin{aligned}
\text{Then we obtain} \quad \text{tr}(\Theta(1)^k) - \text{tr}(\Theta(0)^k) &= d \left(k \text{tr} \left(\eta \int_0^1 \Theta(t)^{k-1} dt \right) \right) \\
&= d\zeta_k,
\end{aligned} \tag{4.4}$$

where $\zeta_k = k \text{tr} \left(\eta \int_0^1 \Theta(t)^{k-1} dt \right)$, which are G invariant since η and $\Theta(t)$ are globally defined G -invariant differential forms.

Next, for $\lfloor \frac{\dim(X)}{2} \rfloor$ -variable polynomials P , let us introduce G -invariant differential forms

$$P\langle t \rangle := P(\text{tr}(\Theta(t)), \text{tr}(\Theta(t)^2), \text{tr}(\Theta(t)^3), \dots).$$

And we claim that every P satisfies the following two conditions;

(A) $P \langle t \rangle$ is a closed form; $dP \langle t \rangle = 0$.

(B) There exists a G -invariant differential form $\zeta_P \in \Omega^*(X)$ such that $d\zeta_P = P \langle 1 \rangle - P \langle 0 \rangle$.

If this is true, we have finished the proof of the lemma 4.8 since $c_k(E, \nabla)$ and $\text{ch}(E, \nabla)$ are expressed by polynomials in $\text{tr}(\Theta^\nabla)$, $\text{tr}((\Theta^\nabla)^2)$, $\text{tr}((\Theta^\nabla)^3), \dots$.

Let us verify the claim. If two polynomials P and Q satisfy these conditions, obviously so does $aP + bQ$. We will verify that so does the product PQ . In fact, we have

$$d(P \langle t \rangle Q \langle t \rangle) = dP \langle t \rangle Q \langle t \rangle + P \langle t \rangle dQ \langle t \rangle = 0,$$

so PQ satisfies the condition (A). Besides, we have

$$\begin{aligned} P \langle 1 \rangle Q \langle 1 \rangle - P \langle 0 \rangle Q \langle 0 \rangle &= (P \langle 1 \rangle - P \langle 0 \rangle)Q \langle 1 \rangle + P \langle 0 \rangle (Q \langle 1 \rangle - Q \langle 0 \rangle) \\ &= d\zeta_P Q \langle 1 \rangle + P \langle 0 \rangle d\zeta_Q \\ &= d(\zeta_P Q \langle 1 \rangle + P \langle 0 \rangle \zeta_Q) \\ &= d\zeta_{PQ}. \end{aligned}$$

Since all of the $\zeta_P, Q \langle 1 \rangle, P \langle 0 \rangle$ and ζ_Q are G -invariant, so does $\zeta_{PQ} := \zeta_P Q \langle 1 \rangle + P \langle 0 \rangle \zeta_Q$, which implies that PQ satisfies the condition (B).

Moreover, it follows from (4.2) and (4.4) that all of the monomial P_k of the form

$$P_k(\text{tr}(\Theta(t)), \text{tr}(\Theta(t)^2), \text{tr}(\Theta(t)^3), \dots) = \text{tr}(\Theta(t)^k),$$

which generate all the polynomials, satisfy the two conditions above. Therefore we conclude that every polynomial P satisfies (A) and (B). $\square$

4.2 Properly Supported G -Invariant Elliptic Operators

Let X be a proper G -Riemannian manifold, and $\mathbb{V}^0, \mathbb{V}^1$ be Hermitian G -vector bundle. (Later on, we will consider a $\mathbb{Z}/2\mathbb{Z}$ -graded bundle $\mathbb{V} = \mathbb{V}^0 \hat{\oplus} \mathbb{V}^1$.) The goal of this subsection is to define a K -homology classes determined by properly supported G -invariant elliptic operators. We refer to [K, Section 3] and [Hö71].

Let $\pi: T^*X \rightarrow X$ be the projection. For a short while, we forget the action of G .

Definition 4.9

Pseudo Differential Operators. A pseudo-differential operator $A_0: C_c^\infty(\mathbb{V}^0) \rightarrow C^\infty(\mathbb{V}^1)$ is given by a sum of an integral operator with a smooth kernel

$$Fs(x) = \int_X k(x, y) s(y) dy. \quad \text{for } s \in C_c^\infty(\mathbb{V}_0),$$

where k is a smooth section $X \times X \rightarrow \mathbb{V}^1 \boxtimes (\mathbb{V}^0)^*$, and an operator of the following form;

$$A_0 s(x) = \frac{1}{(2\pi)^n} \int_{X \times T_x^* X} e^{i\Phi(y, (x, \xi))} \mathbf{a}(y, (x, \xi)) s(y) dy d\xi \quad \text{for } s \in C_c^\infty(\mathbb{V}_0), \quad (4.5)$$

where $\mathbf{a}: X \times T^*X \rightarrow \text{Hom}(\pi^*\mathbb{V}^0, \pi^*\mathbb{V}^1)$. Let $m \in \mathbb{R}$. A pseudo differential operator is of order m if $\mathbf{a}$ satisfies the condition that for any compact subset $K \subset X \times X$ and multi indices $a = (a_1, \dots, a_n), a' = (a'_1, \dots, a'_n), b = (b_1, \dots, b_n)$, where $n = \dim X$, there exists a constant $C_{a, a', b, K}$ is a constant depending on K, a, a' and b such that for any $(x, y) \in K, \xi \in T_x X$,

$$\left\| \frac{\partial^{a'}}{\partial y^{a'}} \frac{\partial^{a|}}{\partial x^a} \frac{\partial^{b|}}{\partial \xi^b} \mathbf{a}(y, (x, \xi)) \right\| \leq C_{a, a', b, K} \sqrt{1 + \|\xi\|^2}^{m - |b|}. \quad (4.6)$$

$\mathbf{a}$ is assumed to vanish outside a neighborhood U of the diagonal of $X \times X$ so that a small neighborhood of the image of the zero-section of TX is diffeomorphic to U by $(x, v) \mapsto (x, \exp_x(v))$ and that the phase function Φ is defined on this neighborhood U .

Here Φ , called the phase function, is a $\mathbb{C}$ -valued function given by $\Phi(y, (x, \xi)) = \langle \xi, \exp_x^{-1}(y) \rangle_{T_x^* X}$. For instance if $X = \mathbb{R}$, $\Phi(y, (x, \xi)) = \langle \xi, y - x \rangle$.

The space consisting of all pseudo-differential operators $A_0: C_c^\infty(\mathbb{V}^0) \rightarrow C^\infty(\mathbb{V}^1)$ will be denoted by $\Psi\text{DO}^m(\mathbb{V}^0, \mathbb{V}^1)$.

We used a coordinate-free presentation above, however, if in the usual way of definition of pseudo-differential operator, firstly we define the operator on $\mathbb{R}^n$ by

$$A_0 s(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\langle x-y, \xi \rangle} \mathbf{a}(y, (x, \xi)) s(y) dy d\xi$$

with $\mathbf{a}(y, (x, \xi))$ satisfying the same condition as (4.6). Then define $\Psi\text{DO}^m(\mathbb{R}; \mathbb{V}^0, \mathbb{V}^1)$ as the space consisting of sum pf integral operator with smooth kernel and operators given as above. For Riemannian manifold X , an operator A_0 belongs to $\Psi\text{DO}^m(X; \mathbb{V}^0, \mathbb{V}^1)$ if its restriction to any open subset diffeomorphic to $\mathbb{R}^n$ belongs to $\Psi\text{DO}^m(\mathbb{R}^n; \mathbb{V}^0, \mathbb{V}^1)$.

Distributional Kernels. The distributional kernel for $A_0 \in \Psi\text{DO}^m(\mathbb{V}^0, \mathbb{V}^1)$ is a distributional section $K_{A_0}: X \times X \rightarrow \mathbb{V}^1 \boxtimes (\mathbb{V}^0)^*$ such that

$$A_0 s(x) = \int_X K_{A_0}(x, y) s(y) dy \quad \text{for } s \in C_c^\infty(\mathbb{V}_0).$$

Formally, the kernel K_{A_0} for an operator of the form (4.5) is given by the formula;

$$K_{A_0}(x, y) = \frac{1}{(2\pi)^n} \int_{T_x^* X} e^{i\Phi(y, (x, \xi))} \mathbf{a}(y, (x, \xi)) d\xi.$$

in the distributional sense and precisely, K_{A_0} is determined by

$$K_{A_0}(w) = \frac{1}{(2\pi)^n} \int_{X \times T^* X} e^{i\Phi(y, (x, \xi))} \mathbf{a}(y, (x, \xi)) w(x, y) dy dx d\xi \quad (4.7)$$

holds for any $w \in C_c^\infty((\mathbb{V}^1)^* \boxtimes \mathbb{V}^0)$. The simplest example for distributional kernel is the Dirac-delta distributional function for the identity operator $K_{\text{id}}(x, y) = \delta_{x, y}$.

Properly supported Pseudo-differential Operators. $A_0 \in \Psi\text{DO}^m(\mathbb{V}^0, \mathbb{V}^1)$ is said to be properly supported or has proper support if both of the maps p_1 and $p_2: \text{supp}(K_{A_0}) \rightarrow X$ are proper, where p_i denotes the projection onto the i -th factor, in other words, both of the subsets

$$(\text{supp}(K_{A_0})) \cap (K \times X) \quad \text{and} \quad (\text{supp}(K_{A_0})) \cap (X \times K) \subset X \times X$$

are compact for any compact subset $K \subset X$. The sub-space of $\Psi\text{DO}^m(\mathbb{V}^0, \mathbb{V}^1)$ consisting of all properly supported pseudo-differential operators will be denoted by $\text{P-}\Psi\text{DO}^m(\mathbb{V}^0, \mathbb{V}^1)$.

Symbol Functions. For $m \in \mathbb{R}$, we define $\text{Symb}^m(\mathbb{V}^0, \mathbb{V}^1)$ as the space consisting of all sections $\sigma: T^*X \rightarrow \text{Hom}(\pi^*\mathbb{V}^0, \pi^*\mathbb{V}^1)$ satisfying that for any compact subset $K \subset X$ and multi indices $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n)$, where $n = \dim X$, there exists a constant $C_{a, b, K}$ is a constant depending on K, a and b such that

$$\left\| \frac{\partial^{|a|}}{\partial x^a} \frac{\partial^{|b|}}{\partial \xi^b} \sigma(x, \xi) \right\| \leq C_{a, b, K} \sqrt{1 + \|\xi\|^2}^{m-|b|} \quad \text{holds for any } x \in K, \xi \in T_x X. \quad (4.8)$$

An element $\sigma \in \text{Sym}^m(\mathbb{V}^0, \mathbb{V}^1)$ is called a symbol function of order m and $\sigma \in [\text{Symb}^m(\mathbb{V}^0, \mathbb{V}^1)] := \text{Symb}^m(\mathbb{V}^0, \mathbb{V}^1)/\text{Symb}^{m-1}(\mathbb{V}^0, \mathbb{V}^1)$ is called a principal symbol. For example, every polynomial function in ξ of order m is an element of $\text{Symb}^m(\mathbb{V}^0, \mathbb{V}^1)$.

Principal Symbol of Pseudo-differential Operators. Let $A_0 \in \text{P-}\Psi\text{DO}^m(\mathbb{V}^0, \mathbb{V}^1)$ be a properly supported pseudo-differential operator of order m . Take a sufficiently small Euclidean neighborhood W and local co-ordinate $\psi: W \rightarrow \mathbb{R}^n$. Then the symbol σ_{A_0} of $A_0 \in \text{P-}\Psi\text{DO}(\mathbb{V}^0, \mathbb{V}^1)$ is defined by the formula

$$\sigma_{A_0}(x, \xi) = e^{-i\langle \psi(x), \xi \rangle} A_0 e^{i\langle \psi(x), \xi \rangle} \quad \text{for } x \in W, \xi \in \mathbb{R}^n. \quad (4.9)$$

σ_{A_0} belongs to $\text{Symb}^m(\mathbb{V}^0, \mathbb{V}^1)$, and σ_{A_0} regarded as an element of $[\text{Symb}^m(\mathbb{V}^0, \mathbb{V}^1)]$ is called the principal symbol of A_0 . We may abbreviate principal symbol as just symbol.

Remark 4.10

- (1) Let us consider an operator on $\mathbb{R}^n$. If $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a diffeomorphism, then the principal symbol of the operator $(\varphi^*)^{-1}A_0\varphi^*$ is

$$\sigma_{(\varphi^*)^{-1}A_0\varphi^*}(x, \xi) = \sigma_{A_0}(\varphi(x), \varphi^*(\xi)) \mod \text{Symb}^{m-1}(\mathbb{V}^0, \mathbb{V}^1).$$

This implies that the principal symbol $\sigma_{A_0} \in [\text{Symb}^m(\mathbb{V}^0, \mathbb{V}^1)]$ defined by (4.9) does not depend on the choice of local co-ordinates.

- (2) If A_0 is given by (4.5), its principal symbol is calculated from $\mathfrak{a}$ by

$$\sigma_{A_0}(x, \xi) = \mathfrak{a}(x, (x, \xi)) \in [\text{Symb}^m(\mathbb{V}^0, \mathbb{V}^1)].$$

Conversely a symbol function $\sigma \in \text{Symb}^m(\mathbb{V}^0, \mathbb{V}^1)$ has an amplitude $\mathfrak{a}: X \times T^*X \rightarrow \text{Hom}(\pi^*\mathbb{V}^0, \pi^*\mathbb{V}^1)$ defined by

$$\mathfrak{a}(y, (x, \xi)) := \chi(x, y)\sigma(x, \xi), \quad (4.10)$$

where $\chi: X \times X \rightarrow [0, 1]$ is a smooth function satisfying that $\chi(x, x) = 1$ and $\chi(x, y) = 0$ if $\text{dist}(x, y) > r$, where r is the injective radius of X , which is bounded below. We can construct a pseudo differential operator by (4.5) whose principal symbol is equal to the given $\sigma \in [\text{Symb}^m(\mathbb{V}^0, \mathbb{V}^1)]$.

- (3) If $A_0 \in \Psi\text{DO}^{m_0}(\mathbb{V}^0, \mathbb{V}^1)$ and $A_1 \in \Psi\text{DO}^{m_1}(\mathbb{V}^1, \mathbb{V}^2)$, then we have

$$\begin{aligned} \sigma_{A_1 A_0} &= \sigma_{A_1} \sigma_{A_0} \mod \text{Symb}^{m_0+m_1-1}(\mathbb{V}^0, \mathbb{V}^2) \\ \text{and } \sigma_{A_0^*} &= (\sigma_{A_0})^* \mod \text{Symb}^{m_0-1}(\mathbb{V}^0, \mathbb{V}^1). \end{aligned}$$

- (4) Let us start with a symbol $\sigma \in [\text{Symb}^m(\mathbb{V}^0, \mathbb{V}^1)]$. When we construct an amplitude (4.10), if we replace r by the minimum among the injective radius and 1 so that $\text{supp}(\chi)$ is in bounded distance from the diagonal in $X \times X$, any pseudo-differential operator constructed by the formula (4.10) and (4.5) from a given symbol is always properly supported. This is because if w in (4.7) has a support outside the r -neighborhood of the diagonal in $X \times X$, $\mathfrak{a}(y, (x, \xi))w(x, y) = 0$ for any $(x, y) \in X \times X$ therefore K_{A_0} has a support contained in the r -neighborhood of the diagonal in $X \times X$. So both $(\text{supp}(K_{A_0})) \cap (K \times X)$ and $(\text{supp}(K_{A_0})) \cap (X \times K)$ are contained in a $2r$ -neighborhood of $K \times K$ which is bounded.
- (5) For any properly supported pseudo-differential operator $A_0 \in \Psi\text{DO}^m(\mathbb{V}^0, \mathbb{V}^1)$, the image of $C_c(\mathbb{V}^0)$ is contained in $C_c(\mathbb{V}^1)$. For these reasons we basically assume that A_0 is properly supported.

Definition 4.11 (Ellipticity) An operator $A_0 \in \text{P-}\Psi\text{DO}^m(\mathbb{V}^0, \mathbb{V}^1)$ is called Elliptic if the principal symbol $\sigma_A(x, \xi)$ is invertible at infinity in ξ , more precisely, there exists a symbol $\tau \in [\text{Symb}^{-m}(\mathbb{V}^1, \mathbb{V}^0)]$ such that

$$\lim_{\|\xi\| \rightarrow \infty} \|\tau(x, \xi)\sigma_{A_0}(x, \xi) - \text{id}_{\mathbb{V}^0}\| = \lim_{\|\xi\| \rightarrow \infty} \|\sigma_{A_0}(x, \xi)\tau(x, \xi) - \text{id}_{\mathbb{V}^1}\| = 0 \quad (4.11)$$

uniformly in $x \in K$ for any compact subsets $K \subset X$.

Remark 4.12

- (1) Without loss of generality we may consider the case of elliptic operators of order 0 and furthermore the condition (4.11) may be replaced by the condition that the principal symbol is unitary at infinity, that is,

$$\lim_{\|\xi\| \rightarrow \infty} \|\sigma_{A_0}^*(x, \xi)\sigma_{A_0}(x, \xi) - \text{id}_{\mathbb{V}^0}\| = \lim_{\|\xi\| \rightarrow \infty} \|\sigma_{A_0}(x, \xi)\sigma_{A_0}^*(x, \xi) - \text{id}_{\mathbb{V}^1}\| = 0 \quad (4.12)$$

uniformly in $x \in K$ for any compact subsets $K \subset X$. This is because $\sigma_{A_0} \in [\text{Symb}^0(\mathbb{V}^0, \mathbb{V}^1)]$ can be modified to

$$\sigma' := (\sigma_{A_0}\sigma_{A_0}^*)^{-1/2} \sigma_{A_0}$$

at infinity so that σ' satisfies the condition (4.12) through a homotopy by a family of symbols $\{(\sigma_{A_0}\sigma_{A_0}^*)^{-t/2} \sigma_{A_0}\}_{t \in [0,1]}$ which are invertible, at infinity in ξ .

- (2) In the case of elliptic operators A on $\mathbb{V} = \mathbb{V}^0 \hat{\oplus} \mathbb{V}^1$ of the form $A = \begin{pmatrix} 0 & A_0^* \\ A_0 & 0 \end{pmatrix}$, the requirement (4.12) will be replaced by $\lim_{\|\xi\| \rightarrow \infty} \|\sigma_A(x, \xi)^2 - \text{id}_{\mathbb{V}}\| = 0$.

Let us recall the action of G .

Definition 4.13 (G -invariant operators) G acts on $\text{P-}\Psi\text{DO}^m(\mathbb{V}^0, \mathbb{V}^1)$ by the formula;

$$\gamma[A_0] = \gamma A_0 \gamma^{-1}.$$

Let $\text{P-}\Psi\text{DO}_G^m(\mathbb{V}^0, \mathbb{V}^1)$ denote the subspace of $\text{P-}\Psi\text{DO}^m(\mathbb{V}^0, \mathbb{V}^1)$ consisting of all G -invariant operators.

Remark 4.14 It directly follows that the action of G on the operators are compatible with the action on the sections, that is, $\gamma[A_0 s] = \gamma[A_0] \gamma[s]$ for $s \in C_c^\infty(\mathbb{V}^0)$.

Note that the action of G on $\mathbb{V}^0 \hat{\oplus} \mathbb{V}^1$ induces the action on the bundle $\text{Hom}(\pi^* \mathbb{V}^0, \pi^* \mathbb{V}^1)$ over T^*X . Any G -invariant operator $A_0 \in \text{P-}\Psi\text{DO}_G^m(\mathbb{V}^0, \mathbb{V}^1)$ has the G -invariant principal symbol. Conversely we have the following;

Lemma 4.15 *If $\sigma: T^*X \rightarrow \text{Hom}(\pi^* \mathbb{V}^0, \pi^* \mathbb{V}^1)$ is a G -invariant principal symbol of order m , then there exists a properly supported G -invariant operator $A \in \text{P-}\Psi\text{DO}_G^m(\mathbb{V}^0, \mathbb{V}^1)$ whose principal symbol is σ .*

Proof. When we take an amplitude $\mathfrak{a}$ of σ by (4.10), choose $\chi: X \times X \rightarrow [0, 1]$ so that $\chi(x, y)$ depends only on $\text{dist}(x, y)$. More precisely, let $\bar{\chi}: \mathbb{R}_{\geq 0} \rightarrow [0, 1]$ be a smooth function such that $\bar{\chi}(t) = 1$ for $t \leq r/4$ and $\bar{\chi}(t) = 0$ for $t \geq r/2$, where r is a constant less than the injective radius. and set $\chi(x, y) := \bar{\chi}(\text{dist}(x, y))$. Since the action of G is isometric, the amplitude $\mathfrak{a}(y, (x, \xi))$ constructed by (4.10) using this χ is G -invariant. Then the pseudo-differential operator $A \in \text{P-}\Psi\text{DO}^m(\mathbb{V}^0, \mathbb{V}^1)$ constructed by the formula (4.5) is G -invariant. By the construction, it is properly supported. $\square$

We will construct a K -homology classes $[A] := (L^2(\mathbb{V}), A) \in KK^G(C_0(X), \mathbb{C})$ induced by properly supported G -invariant elliptic operators. in the last of this subsection. In order to argue the L^2 boundedness of elliptic operators, let us introduce the following theorem by Hörmander;

Theorem 4.16 [Hö71, Theorem 2.2.1] *Let $P \in \text{P-}\Psi\text{DO}^0(\mathbb{V}^0, \mathbb{V}^1)$ be a properly supported pseudo-differential operator of order 0. Suppose that for any compact subset $K \subset X$*

$$\limsup_{\|\xi\| \rightarrow \infty} \|\sigma_P(x, \xi)\| < C \quad \text{for } x \in K.$$

Then there exists a properly supported pseudo differential operator $B \in \text{P-}\Psi\text{DO}^0(\mathbb{V}^0, \mathbb{V}^1)$ of order 0 and a self adjoint integral operator R with continuous and properly supported kernel such that

$$PP^* + BB^* - C^2 = R.$$

Moreover, if the support of the distributional kernel of P is compact in $X \times X$, then both B and R will also have compactly supported distributional kernels. $\square$

Lemma 4.17 *Let $P \in \mathbb{L}(L^2(\mathbb{V}^0), L^2(\mathbb{V}^1))$ be a bounded operator with distributional kernel supported in a compact subset $K \times K \subset X \times X$. Then $\tilde{P} := \int_G \gamma[P] d\gamma$ is well defined as a bounded operator in $\mathbb{L}(L^2(\mathbb{V}^0), L^2(\mathbb{V}^1))$ and the inequation $\|\tilde{P}\|_{\text{op}} \leq C_K \|P\|_{\text{op}}$ holds, where C_K is a constant depending on K .*

Proof. We will follow the proof of [CM, Lemma 1.4 - 1.5]. Fix an arbitrary smooth section with compact support $s \in C_c^\infty(\mathbb{V}^0)$ and let us consider $F = F_s \in L^2(G; L^2(\mathbb{V}^1))$ given by

$$F_s(\gamma) := \gamma[P]s.$$

Note that for any $\gamma \in G$ the distributional kernel of $\gamma[P]$ is contained in $\gamma(K) \times \gamma(K)$. This is because for any $s \in C_c^\infty(\mathbb{V}^0)$, $\text{supp}(\gamma[P]s) \subset \gamma(K)$ and $\gamma[P]s = 0$ whenever $\text{supp}(s) \cap \gamma(K) = \emptyset$. In particular, since the action of G on X is proper, F has compact support in G and hence $F \in L^2(G; L^2(\mathbb{V}^1))$. In addition, since the action of G on X is proper, $\gamma(K) \cap \eta(K) = \gamma(K \cap \gamma^{-1}\eta(K)) = \emptyset$ if $\gamma^{-1}\eta \in G$ is outside some compact neighborhood $Z \subset G$ in particular, $\langle F(\gamma), F(\eta) \rangle_{L^2(\mathbb{V}^1)} = 0$ for such γ and $\eta \in G$. Remind that Z is determined only by K so independent of s . Then,

$$\begin{aligned} \left\| \int_G F(\gamma) d\gamma \right\|_{L^2(\mathbb{V}^1)}^2 &= \left\| \int_G F(\gamma) d\gamma \right\|_{L^2(\mathbb{V}^1)} \left\| \int_G F(\eta) d\eta \right\|_{L^2(\mathbb{V}^1)} \\ &\leq \int_G \int_G \left| \langle F(\gamma), F(\eta) \rangle_{L^2(\mathbb{V}^1)} \right| d\gamma d\eta \\ &\leq \int_G \|F(\gamma)\|_{L^2(\mathbb{V}^1)} \left(\int_G \chi_Z(\gamma^{-1}\eta) \|F(\eta)\|_{L^2(\mathbb{V}^1)} d\eta \right) d\gamma \\ &\leq \|F\|_{L^2(G)} \|\chi_Z\|_{L^2(G)} \|F\|_{L^2(G)} \\ &\leq |Z| \|F\|_{L^2(G)}^2, \end{aligned}$$

where $\chi_Z: G \rightarrow [0, 1]$ is the characteristic function of C , that is $\chi_Z(\gamma) = 1$ for $x \in Z$ and $\chi_Z(\gamma) = 0$ for $x \notin Z$.

On the other hand, take a compactly supported smooth function $f \in C_c^\infty(X; [0, 1])$ such that $f(x) = 1$ for $x \in K$ and $f(x) = 0$ if $\text{dist}(x, K) > r$. Since the support of the distributional kernel

of P is contained in $K \times K$ and hence $P = Pf$, we have

$$\begin{aligned}
\|F\|_{L^2(G)}^2 &= \int_G \|F(\gamma)\|_{L^2(\mathbb{V}^1)}^2 d\gamma = \int_G \|\gamma Pf\gamma^{-1}s\|_{L^2(\mathbb{V}^1)}^2 d\gamma \\
&\leq \int_G \|P\|_{\text{op}}^2 \|f\gamma^{-1}s\|_{L^2(\mathbb{V}^0)}^2 d\gamma \\
&\leq \|P\|_{\text{op}}^2 \int_G \int_X |f(x)|^2 \|\gamma^{-1}s(x)\|_{\mathbb{V}^0}^2 dx d\gamma \\
&\leq \|P\|_{\text{op}}^2 \int_G \int_X |f(\gamma^{-1}x)|^2 \|s(x)\|_{\mathbb{V}^0}^2 dx d\gamma \\
&\leq \|P\|_{\text{op}}^2 \sup_{x \in X} \left(\int_G |f(\gamma^{-1}x)|^2 d\gamma \right) \|s\|_{L^2(\mathbb{V}^0)}^2.
\end{aligned}$$

Since the action of G is proper, $\{\gamma \in G \mid \gamma^{-1}x \in \text{supp}(f)\} \subset G$ is compact so the value $\int_G |f(\gamma^{-1}x)|^2 d\gamma$ is always finite for any fixed $x \in X$. Besides, since X/G is compact, we have

$$\sup_{x \in X} \left(\int_G |f(\gamma^{-1}x)|^2 d\gamma \right) = \sup_{[x] \in X/G} \left(\int_G |f(\gamma^{-1}x)|^2 d\gamma \right) < \infty$$

Remind that this value depends only on K and independent of s . We conclude that

$$\left\| \int_G \gamma[P]s d\gamma \right\|_{L^2(\mathbb{V}^1)}^2 = \left\| \int_G F(\gamma) d\gamma \right\|_{L^2(\mathbb{V}^1)}^2 \leq |Z| \|F\|_{L^2(G)}^2 \leq |Z|C \cdot \|P\|_{\text{op}}^2 \|s\|_{L^2(\mathbb{V}^0)}^2,$$

where $C = \sup_{x \in X} \left(\int_G |f(\gamma^{-1}x)|^2 d\gamma \right) < \infty$. □

Lemma 4.18 *If $P \in \Psi\text{DO}^m(\mathbb{V}^0, \mathbb{V}^1)$ with compact support, then $\tilde{P} := \int_G \gamma[P]d\gamma \in \text{P-}\Psi\text{DO}_G^m(\mathbb{V}^0, \mathbb{V}^1)$.*

Proof. It is clear that $\tilde{P} \in \Psi\text{DO}_G^n(\mathbb{V}^0, \mathbb{V}^1)$ for some n . Let the support of distributional kernel of P is contained in $K \times K$ for some compact $K \subset X$. Let $\mathbf{a}$ be its amplitude. Then the amplitude of $\gamma[P]$ is given by $\gamma[\mathbf{a}](y, (x, \xi)) = \gamma\mathbf{a}(\gamma^{-1}y, \gamma^{-1}(x, \xi))$. Now we have $\int_G \gamma[\mathbf{a}](y, (x, \xi))d\gamma$, which is the amplitude of $\tilde{P}$, satisfies the condition (4.6) since the integral is over a subset $\{\gamma \in G \mid (\gamma^{-1}x, \gamma^{-1}y) \neq 0\} \subset G$ which is compact by the properness of the action. The distributional kernel of $\gamma[P]$ is contained in $\gamma(K \times K)$ and hence, the distributional kernel of $\tilde{P}$ is contained in $G(K \times K)$ so $\tilde{P}$ is properly supported. □

Lemma 4.19 *Let $A_0 \in \text{P-}\Psi\text{DO}_G^0(\mathbb{V}^0, \mathbb{V}^1)$ be an elliptic operator of order 0 which is properly supported and G -invariant. If X/G is compact, then A_0 is L^2 -bounded.*

Proof. A_0 satisfies the assumptions of theorem 4.16. Let $c \in C_c^\infty(X)$ be an arbitrary cut-off function. And let us consider $P := cA_0$. it has the distributional kernel supported in a compact subset $(\text{supp}(c) \times X) \cap \text{supp}(K_{A_0}) \subset X \times X$ and still satisfies the assumptions of the proposition. Then we have

$$0 \leq PP^* + BB^* = C^2 + R.$$

And it follows that P is bounded since BB^* is positive and $C^2 + R$ is bounded.

Next, note that $A_0 = \int_G \gamma[cA_0] d\gamma = \int_G \gamma[P] d\gamma$ since A_0 itself is G -invariant. Then by the previous lemma 4.17, A_0 is L^2 bounded. □

Lemma 4.20 [K, Lemma 3.7] Let $A = \begin{pmatrix} 0 & A_0^* \\ A_0 & 0 \end{pmatrix} \in \text{P-}\Psi\text{DO}_G^0(\mathbb{V})$ be a properly supported G -invariant elliptic operator on $\mathbb{V} = \mathbb{V}^0 \hat{\oplus} \mathbb{V}^1$. Then for any $f \in C_0(X)$, the multiplication operators by f commutes with A modulo compact operators $\mathbb{K}(L^2(\mathbb{V}))$, and $f(1 - A^2)$ is compact.

Proof. It is enough to prove when f has a compact support. The commutator $Af - fA$ is a pseudo-differential operator of negative order because the principal symbol of order 0 is

$$\sigma_{Af-fA} = \sigma_A f - f \sigma_A = 0 \in [\text{Symb}^0(\mathbb{V})].$$

Also, the symbol of $f(1 - A^2)$ satisfies

$$\sigma_{f(1-A^2)}(x) = f(x)(1 - \sigma_{A^2}(x, \xi)) \rightarrow 0 \quad \text{as} \quad \|\xi\| \rightarrow \infty.$$

Let P be $Af - fA$ or $f(1 - A^2)$. Then P has the compactly supported distributional kernel and satisfies the assumption of theorem 4.16 for any small constant $C > 0$. We obtain

$$PP^* + BB^* = R$$

with R having the compactly supported distributional kernel. Hence R is a compact operator, which implies that so is P . $\square$

Remark 4.21 Note that the symbol $\sigma_A = \begin{pmatrix} 0 & \sigma_{A_0}^* \\ \sigma_A & 0 \end{pmatrix}$ of a properly supported G -invariant elliptic operator A of order 0 induces a map $\sigma_A: \pi^*\mathbb{V} \rightarrow \pi^*\mathbb{V}$, where $\pi: T^*X \rightarrow X$ denotes the projection. Additionally, note that the space of $C_0(T^*X)$ -compact operators on $C_0(\pi^*\mathbb{V})$ is $C_0(\text{End}(\pi^*\mathbb{V}))$ and that $C_0(X)$ acts on $C_0(\pi^*\mathbb{V})$ by the point-wise multiplication. Hence, for any $f \in C_0(X)$, both $\sigma_A f - f \sigma_A (= 0)$ and $f(\sigma_A^2 - 1)$ are compact as an operator on a Hilbert $C_0(T^*X)$ -module.

Definition 4.22 (The K-homology classes determined by elliptic operators)

- (1) Let $A = \begin{pmatrix} 0 & A_0^* \\ A_0 & 0 \end{pmatrix} \in \text{P-}\Psi\text{DO}_G^0(\mathbb{V})$ be a properly supported G -invariant elliptic operator of order 0 on a graded Hermitian G -vector bundle $\mathbb{V} = \mathbb{V}^0 \hat{\oplus} \mathbb{V}^1$ over X . Due to the lemma 4.19 and 4.20, we can define a K -cohomology class

$$[A] := (L^2(\mathbb{V}), A) \in KK^G(C_0(X), \mathbb{C})$$

with the action of $C_0(X)$ on $L^2(\mathbb{V})$ being the point-wise multiplication.

- (2) Its symbol $\sigma_A = \begin{pmatrix} 0 & \sigma_{A_0}^* \\ \sigma_A & 0 \end{pmatrix}$ determines the following element in KK -theory;

$$[\sigma_A] := (C_0(\pi^*\mathbb{V}), \sigma_A) \in KK^G(C_0(X), C_0(T^*X)).$$

5 Index Map

Let G be a second countable locally compact Hausdorff group. Let X be a G -compact proper complete G -Riemannian manifold. Let $\mathbb{V}$ be a smooth (finite dimensional) Hermitian G -vector bundle over X . We summarize the notations;

- $C_c^\infty(\mathbb{V})$ denotes the space of compactly supported smooth sections $s: X \rightarrow \mathbb{V}$.
- $C_0(\mathbb{V})$ denotes the C^* -algebra consisting of continuous sections $s: X \rightarrow \mathbb{V}$ vanishing at infinity equipped with the sup-norm.
- $L^2(\mathbb{V})$ denotes the Hilbert $C_0(\mathbb{V})$ -module consisting of L^2 -sections $s: X \rightarrow \mathbb{V}$. $C_0(\mathbb{V})$ acts by point-wise multiplication.

5.1 The Topological Index Map

Let

$$A = \begin{pmatrix} 0 & A_0^* \\ A_0 & 0 \end{pmatrix} : L^2(\mathbb{V}) \rightarrow L^2(\mathbb{V})$$

be a properly supported G -invariant elliptic operator of order 0 on a graded Hermitian G -vector bundle $\mathbb{V} = \mathbb{V}^0 \hat{\oplus} \mathbb{V}^1$. Then as in definition 4.22, A defines the K -cohomology class

$$[A] = (L^2(\mathbb{V}), m, A) \in KK^G(C_0(X), \mathbb{C}),$$

where the action m of $C_0(X)$ on $L^2(\mathbb{V})$ is ordinary point-wise multiplication. We will define the index map

$$\mu_G : KK^G(C_0(X), \mathbb{C}) \rightarrow K_0(C^*(G)).$$

Definition 5.1 Let G be a second countable locally compact Hausdorff group, A and B be G -algebras and $(\mathcal{E}, \varphi, T) \in \mathbb{E}^G(A, B)$ be a Kasparov G -module over (A, B) . We will introduce the following notations;

- (1) $\hat{T}$ is defined to be an operator on $C^*(G; \mathcal{E})$ given by

$$\hat{T}[\hat{e}](\gamma) = T(\hat{e}(\gamma)) \quad \text{for } \hat{e} \in C_c(G; \mathcal{E}).$$

- (2) The action $\hat{\varphi}$ of $C^*(G; A)$ on $C^*(G; \mathcal{E})$ is given by the following convolution product;

$$\hat{\varphi}[\hat{a}](\hat{e}) : \gamma \mapsto \int_{\gamma_1 \gamma_2 = \gamma} \varphi(\hat{a}(\gamma_1)) \gamma_1(\hat{e}(\gamma_2)) d\gamma_1 \quad \text{for } \hat{a} \in C_c(G; A), \quad \hat{e} \in C_c(G; \mathcal{E}).$$

Lemma 5.2 [K88, Theorem 3.11] *Let G be a second countable locally compact Hausdorff group. For any G -algebras A and B there exists a natural homomorphism*

$$j^G : KK^G(A, B) \rightarrow KK(C^*(G; A), C^*(G; B))$$

constructed as follows; If $x = (\mathcal{E}, \varphi, T) \in KK^G(A, B)$,

$$j^G(x) = (C^*(G; \mathcal{E}), \hat{\varphi}, \hat{T}).$$

Furthermore if $x \in KK^G(A, B)$ and $y \in KK^G(B, D)$, then $j^G(x \hat{\otimes}_B y) = j^G(x) \hat{\otimes}_{C^*(G; B)} j^G(y)$. $\square$

Lemma 5.3 *Using a cut-off function $c \in C_c(X)$, one can define an idempotent $p \in C_c(G; C_0(X))$ by the formula;*

$$\check{c}(\gamma)(x) = \sqrt{c(x)c(\gamma^{-1}x)}.$$

In particular it defines an element of K -homology denoted by $[c] \in K_0(C^(G; C_0(X)))$. Moreover the element of K -homology $[c] \in K_0(C^*(G; C_0(X)))$ does not depend on the choice of cut-off functions.*

Proof. It is verified by the direct calculation that $\check{c}$ is an idempotent;

$$\begin{aligned} (\check{c} * \check{c})(\gamma)(x) &= \int_{\gamma_1 \gamma_2 = \gamma} \check{c}(\gamma_1)(x) \cdot \gamma_1[p(\gamma_2)](x) d\gamma_1 \\ &= \int_{\gamma_1 \gamma_2 = \gamma} \sqrt{c(x)c(\gamma_1^{-1}x)} \sqrt{c(\gamma_1^{-1}x)c(\gamma_2^{-1}(\gamma_1^{-1}x))} d\gamma_1 \\ &= \sqrt{c(x)c(\gamma^{-1}x)} \int_{\gamma_1 \gamma_2 = \gamma} c(\gamma_1^{-1}x) d\gamma_1 \\ &= \check{c}(\gamma)(x). \end{aligned}$$

As for the independence of the choice of cut-off function, it is followed from the fact that the subset in $C_c(X)$ of all cut-off functions on X is convex. Precisely, if we have two cut-off functions c_0 and c_1 , $c_t := (1-t)c_0 + tc_1$ provides a family of cut-off functions for $t \in [0, 1]$. $\check{c}_t$ provides a homotopy by idempotents connecting $\check{c}_0$ and $\check{c}_1$. $\square$

Definition 5.4 (Topological Index) Define

$$\mu_G: KK^G(C_0(X), \mathbb{C}) \rightarrow K_0(C^*(G))$$

as the composition of

- $j^G: KK^G(C_0(X), \mathbb{C}) \rightarrow KK(C^*(G; C_0(X)), C^*(G))$ with
- $[c] \hat{\otimes}: KK(C^*(G; C_0(X)), C^*(G)) \rightarrow KK(\mathbb{C}, C^*(G)) \simeq K_0(C^*(G))$.

In other words,

$$\mu_G(-) := [c] \hat{\otimes}_{C^*(G; C_0(X))} j^G(-) \in K_0(C^*(G)).$$

5.2 Reduction to the Dirac Operators

[K, Remark 4.4.] One can reduce the general case of elliptic operators to the case of Dirac operators when calculating the index. In this subsection, we will see that any K -homology class $[A] \in KK^G(C_0(X), \mathbb{C})$, where A is a properly supported G -equivariant elliptic operator of order 0 can be realized as a Dirac operator $\mathcal{D}$ on a larger manifold Y obtained by a fiber bundle over X . To be more specific, we will obtain $[A] = (\pi^\sharp)^*[\mathcal{D}]$ with the map $(\pi^\sharp)^*: KK^G(C_0(Y), \mathbb{C}) \rightarrow KK^G(C_0(X), \mathbb{C})$ induced by $\pi^\sharp: C_0(X) \rightarrow C_0(Y)$. If this has been done, it is sufficient to consider the case when $\mathcal{D}$ is a Dirac operator on a Clifford bundle $\mathbb{V}$ over Y and $[\mathcal{D}] = \left(L^2(\mathbb{V}), \frac{\mathcal{D}}{\sqrt{1+\mathcal{D}^2}}\right) \in KK^G(C_0(Y), \mathbb{C})$.

Lemma 5.5 *Let $\mathcal{D}$ be a G -invariant first order Dirac type operator on a spin manifold Y on a bundle $\mathbb{V}$. Put $\mathcal{T} := \frac{\mathcal{D}}{\sqrt{1+\mathcal{D}^2}}$ and $\mathcal{H} := L^2(\mathbb{V})$. Consider the action of $C_0(Y)$ on $\mathcal{H}$ by point-wise multiplication. Then $f(1 - \mathcal{T}^2)$ and $f\mathcal{T} - \mathcal{T}f$ are compact operators on $\mathcal{H}$ for any $f \in C_0(Y)$.*

In particular we can define the following K -homology class determined by $\mathcal{D}$;

$$[\mathcal{D}] := (\mathcal{H}, \mathcal{T}) \in KK^G(C_0(Y), \mathbb{C}).$$

Proof. It is sufficient to consider the case when $f \in C_c^\infty(Y)$. Noting that $1 - \mathcal{T}^2 = (1 + \mathcal{D}^2)^{-1}$ and using the Rellich lemma, $f(1 - \mathcal{T}^2) \in \mathbb{K}(\mathcal{H})$. As for $f\mathcal{T} - \mathcal{T}f$, we use the formula;

$$\mathcal{T} = \frac{2}{\pi} \int_0^{+\infty} \frac{\mathcal{D}}{1 + \lambda^2 + \mathcal{D}^2} d\lambda.$$

This integral converges in the strong topology, not in the operator norm. Then we have

$$\begin{aligned} f\mathcal{T} - \mathcal{T}f &= \frac{2}{\pi} \int_0^{+\infty} \left(f \frac{\mathcal{D}}{1 + \lambda^2 + \mathcal{D}^2} - \frac{\mathcal{D}}{1 + \lambda^2 + \mathcal{D}^2} f \right) d\lambda \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{1 + \lambda^2 + \mathcal{D}^2} \left[(1 + \lambda^2 + \mathcal{D}^2) f \mathcal{D} - \mathcal{D} f (1 + \lambda^2 + \mathcal{D}^2) \right] \frac{1}{1 + \lambda^2 + \mathcal{D}^2} d\lambda \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{1 + \lambda^2 + \mathcal{D}^2} \left[(1 + \lambda^2) (f \mathcal{D} - \mathcal{D} f) - \mathcal{D} (f \mathcal{D} - \mathcal{D} f) \mathcal{D} \right] \frac{1}{1 + \lambda^2 + \mathcal{D}^2} d\lambda \\ &= \frac{2}{\pi} \int_0^{+\infty} \frac{\sqrt{1 + \lambda^2}}{1 + \lambda^2 + \mathcal{D}^2} (f \mathcal{D} - \mathcal{D} f) \frac{\sqrt{1 + \lambda^2}}{1 + \lambda^2 + \mathcal{D}^2} d\lambda \\ &\quad - \frac{2}{\pi} \int_0^{+\infty} \frac{\mathcal{D}}{1 + \lambda^2 + \mathcal{D}^2} (f \mathcal{D} - \mathcal{D} f) \frac{\mathcal{D}}{1 + \lambda^2 + \mathcal{D}^2} d\lambda. \end{aligned}$$

Note that $(f\mathcal{D} - \mathcal{D}f)$ is a multiplication operator by a compactly supported function df due to the Leibniz rule. Therefore

$$(f\mathcal{D} - \mathcal{D}f) \frac{1}{1 + \lambda^2 + \mathcal{D}^2} \quad \text{and} \quad (f\mathcal{D} - \mathcal{D}f) \frac{\mathcal{D}}{1 + \lambda^2 + \mathcal{D}^2}$$

are compact operators by the Rellich lemma. Moreover, since we have

$$\left\| \frac{\sqrt{1 + \lambda^2}}{1 + \lambda^2 + \mathcal{D}^2} \right\| \leq \frac{1}{\sqrt{1 + \lambda^2}} \quad \text{and} \quad \left\| \frac{\mathcal{D}}{1 + \lambda^2 + \mathcal{D}^2} \right\| \leq \frac{1}{\sqrt{1 + \lambda^2}},$$

the last two integrals converge in the operator norm topology and are compact operators. $\square$

Definition 5.6 Let $BX \subset T^*X$ be the unit closed disk bundle in the cotangent bundle, whose boundary is the sphere bundle $\partial BX = SX$. Then construct

$$\Sigma X = BX \cup_{SX} BX$$

gluing two copies of BX together along SX . The zero in the second BX will be denoted by ∞ .

In order to avoid the confusion, let us distinguish the notations of projections $\pi_T: T^*X \rightarrow X$ and $\pi_\Sigma: \Sigma X \rightarrow X$.

The action of G on X induces the proper co-compact action on ΣX because the action is isometric and each fiber of π_Σ is compact. This action commutes with the projection π_Σ . Let $\pi_\Sigma^\sharp: C_0(X) \rightarrow C_0(\Sigma X)$ be the homomorphism induced by the projection π_Σ .

In addition, note that ΣX is of a structure of smooth almost complex manifold.

Definition 5.7 Let $\mathcal{D}_{\Sigma X}$ be the Dolbeault operator on ΣX . It determines the following K -homology class;

$$[\mathcal{D}_{\Sigma X}] := \left(L^2 \left(\bigwedge^{0,*} T^*(\Sigma X) \right), \frac{\mathcal{D}_{\Sigma X}}{\sqrt{1 + \mathcal{D}_{\Sigma X}^2}} \right) \in KK^G(C_0(\Sigma X), \mathbb{C}).$$

Now let $A = \begin{pmatrix} 0 & A_0^* \\ A_0 & 0 \end{pmatrix} \in \text{P-}\Psi\text{DO}_G^0(\mathbb{V})$ be a properly supported elliptic operator of order 0 on a graded Hermitian G -vector bundle $\mathbb{V} = \mathbb{V}^0 \hat{\oplus} \mathbb{V}^1$. Without loss of generality we may assume its symbol σ_A satisfies

$$\sigma_A^2(x, \xi) = 1 \quad \text{for } \|\xi\| \geq 1 \quad \text{and} \quad \|\sigma_A^2\| \leq 1 \quad \text{everywhere.} \quad (5.1)$$

In particular $\sigma_A|_{SX}: (\pi_T^* \mathbb{V})_{SX} \rightarrow (\pi_T^* \mathbb{V})_{SX}$ is invertible then it provides a gluing map from $(\pi_T^* \mathbb{V})_{BX}$ to $(\pi_T^* \mathbb{V})_{BX}$. From now on, we will abbreviate $\sigma_A = \begin{pmatrix} 0 & \sigma_{A_0}^* \\ \sigma_{A_0} & 0 \end{pmatrix}$ to $\sigma = \begin{pmatrix} 0 & \sigma_0^* \\ \sigma_0 & 0 \end{pmatrix}$ in order to avoid the complication.

Definition 5.8

- Let us construct a new bundle $\mathbb{V}(\sigma)$ over ΣX by gluing $(\pi_T^* \mathbb{V}^0)_{BX}$ and $(\pi_T^* \mathbb{V}^1)_{BX}$ together;

$$\mathbb{V}(\sigma) := (\pi_T^* \mathbb{V})_{BX} \cup_{\sigma|_{SX}} (\pi_T^* \mathbb{V})_{BX}.$$

Remark that the gluing map σ reverses the gradings, $\mathbb{V}(\sigma)$ has the grading

$$\begin{aligned} \mathbb{V}(\sigma) &= \mathbb{V}(\sigma)^0 && \hat{\oplus} && \mathbb{V}(\sigma)^1 \\ &:= (\pi_T^* \mathbb{V}^0)_{BX} \cup_{\sigma_0|_{SX}} (\pi_T^* \mathbb{V}^1)_{BX} && \hat{\oplus} && (\pi_T^* \mathbb{V}^1)_{BX} \cup_{\sigma_0^*|_{SX}} (\pi_T^* \mathbb{V}^0)_{BX}. \end{aligned} \quad (5.2)$$

- In addition consider another bundle just given by the pull-back $\pi_\Sigma^*(\mathbb{V}^1)$ over ΣX . Remark that $\pi_\Sigma^*(\mathbb{V}^1) \cong \mathbb{V}(\sigma)^0$ on the second $BX \subset \Sigma X$.

These G -vector bundles $\mathbb{V}(\sigma)^0$ and $\pi_\Sigma^*(\mathbb{V}^1)$ determine elements in KK -theory;

$$\begin{aligned} [\mathbb{V}(\sigma)^0] &= (C_0(\mathbb{V}(\sigma)^0), 0) \\ \text{and } [\pi_\Sigma^*(\mathbb{V}^1)] &= (C_0(\pi_\Sigma^*(\mathbb{V}^1)), 0) \in KK^G(C_0(\Sigma X), C_0(\Sigma X)). \end{aligned}$$

- Define an element of KK -group denoted by $[[\sigma]] \in KK^G(C_0(X), C_0(\Sigma X))$ as

$$\begin{aligned} [[\sigma]] &:= (\pi_\Sigma^\sharp)^* [\mathbb{V}(\sigma)^0 \hat{\oplus} \pi_\Sigma^*(\mathbb{V}^1)] \\ &= (C_0(\mathbb{V}(\sigma)^0 \hat{\oplus} \pi_\Sigma^*(\mathbb{V}^1)), 0) \in KK^G(C_0(X), C_0(\Sigma X)). \end{aligned}$$

Remark that we can define such class because each fiber of ΣX is compact, in other words, π_Σ is a proper map. We cannot define a KK -theory class in $KK^G(C_0(X), C_0(T^*X))$ of the same form.

Our main claim in this subsection is the following;

Proposition 5.9

$$[[\sigma]] \hat{\otimes}_{C_0(\Sigma X)} [\mathcal{D}_{\Sigma X}] = [A] \in KK^G(C_0(X), \mathbb{C}).$$

In particular, let $\mathcal{D}_\sigma$ be a Dirac operator obtained by the twisting the canonical Dolbeault operator $\mathcal{D}_{\Sigma X}$ by the $\mathbb{Z}/2\mathbb{Z}$ -graded bundle $\mathbb{V}(\sigma)^0 \hat{\oplus} \pi_\Sigma^*(\mathbb{V}^1)$. Then it is a realization of initial A through $\pi_*: KK^G(C_0(\Sigma X), \mathbb{C}) \rightarrow KK^G(C_0(X), \mathbb{C})$, in other words,

$$(\pi_\Sigma^\sharp)^*([\mathcal{D}_\sigma]) = [A] \in KK^G(C_0(X), \mathbb{C}).$$

To verify this, we need theorem 5.11 by G. Kasparov stated below.

Firstly, let us introduce the K -cohomology class of T^*X determined by the Dolbeault operator. Since the manifold T^*X has the canonical structure of an almost complex manifold in particular, Spin^c -structure.

Definition 5.10 Let $\mathcal{D}_{T^*X}$ be the canonical Dirac operator on T^*X called the Dolbeault operator acting on $L^2(\bigwedge^{0,*} T^*(T^*X))$. It determines the following K -homology class;

$$[\mathcal{D}_{T^*X}] := \left(L^2(\bigwedge^{0,*} T^*(T^*X)), \frac{\mathcal{D}_{T^*X}}{\sqrt{1 + \mathcal{D}_{T^*X}^2}} \right) \in KK^G(C_0(T^*X), \mathbb{C}).$$

Theorem 5.11 [K, Theorem 4.2]

$$[A] = [\sigma] \hat{\otimes}_{C_0(T^*X)} [\mathcal{D}_{T^*X}] \in KK^G(C_0(X), \mathbb{C}).$$

Recall that $[\sigma] \in KK^G(C_0(X), C_0(T^*X))$ is the KK -theory class given by $[\sigma] = (C_0(\pi^*\mathbb{V}), \sigma)$ as in definition 4.22. In particular, the K -homology class of a G -invariant elliptic operator and hence its the index depend only on its symbol. $\square$

To obtain proposition 5.9, we should rephrase theorem 5.11 in terms of ΣX instead of T^*X . Let us regard T^*X as a subset of ΣX by

$$\begin{aligned} T^*X &= BX \cup_{SX} (T^*X \setminus \text{int}(BX)) \\ &\cong BX \cup_{SX} (BX \setminus \{0\text{-section}\}) = \Sigma X \setminus \{(x, \infty_x)\} \hookrightarrow \Sigma X. \end{aligned} \tag{5.3}$$

Noting that the identification $T^*X \setminus \text{int}(BX) \rightarrow BX \setminus \{0\text{-section}\}; (x, \xi) \mapsto \left(x, \frac{\xi}{\|\xi\|^2}\right)$ is G -equivariant since the action of G does not change the length of co-tangent vectors, we will consider the following G -equivariant inclusion map;

$$\iota: C_0(T^*X) \rightarrow C_0(\Sigma X).$$

Lemma 5.12 *Consider $\iota_*: KK(C_0(X), C_0(T^*X)) \rightarrow KK(C_0(X), C_0(\Sigma X))$. Then, $\iota_*([\sigma]) = [[\sigma]]$.*

Proof. Remark that

$$\begin{aligned} \mathbb{V}(\sigma)^0 &= (\pi_T^* \mathbb{V}^0)_{BX} \cup_{\sigma_0|_{SX}} (\pi_T^* \mathbb{V}^1)_{BX} \\ \pi_\Sigma^* \mathbb{V}^1 &= (\pi_T^* \mathbb{V}^1)_{BX} \cup_{\text{id}|_{SX}} (\pi_T^* \mathbb{V}^1)_{BX}. \end{aligned}$$

Let $\mathbb{V}_T(\sigma)^0$ denote the restriction of $\mathbb{V}(\sigma)^0$ on $T^*X \subset \Sigma X$ using (5.3). Firstly we have

$$\begin{aligned} [\sigma] &= \left(C_0(\pi_T^* \mathbb{V}), \begin{pmatrix} 0 & \sigma_0^* \\ \sigma_0 & 0 \end{pmatrix} \right) \\ &= \left(C_0(\mathbb{V}_T(\sigma)^0 \hat{\oplus} \pi_T^* \mathbb{V}^1), \begin{pmatrix} 0 & \sigma_0^* \\ \sigma_0 & 0 \end{pmatrix} \cup_{SX} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \in KK^G(C_0(X), C_0(T^*X)). \end{aligned}$$

This is due to the unitary isomorphism $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cup_{SX} \begin{pmatrix} \sigma_0 & 0 \\ 0 & 1 \end{pmatrix} : \pi_T^* \mathbb{V} \rightarrow \mathbb{V}_T(\sigma)^0 \hat{\oplus} \pi_T^* \mathbb{V}^1$.

Then, $\iota_*: KK(C_0(X), C_0(T^*X)) \rightarrow KK(C_0(X), C_0(\Sigma X))$ maps $[\sigma]$ to the class

$$\left(Z, \begin{pmatrix} 0 & \sigma_0^* \\ \sigma_0 & 0 \end{pmatrix} \cup_{SX} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \in KK(C_0(X), C_0(\Sigma X)),$$

where

$$Z := \{ s \in C_0(\mathbb{V}(\sigma)^0 \hat{\oplus} \pi_\Sigma^* \mathbb{V}^1) \mid s(x, \infty) = 0 \text{ for any } x \in X. \}.$$

This K -theory class is equal to the class

$$\left(C_0(\mathbb{V}(\sigma)^0 \hat{\oplus} \pi_\Sigma^* \mathbb{V}^1), \begin{pmatrix} 0 & \sigma_0^* \\ \sigma_0 & 0 \end{pmatrix} \cup_{SX} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \in KK(C_0(X), C_0(\Sigma X))$$

by the homotopy given by the following $IC_0(\Sigma X)$ -module;

$$\{ f \in IC_0(\mathbb{V}(\sigma)^0 \hat{\oplus} \pi_\Sigma^* \mathbb{V}^1) \mid f(1)(x, \infty) = 0 \text{ for any } x \in X. \}.$$

Finally, it is equal to

$$[[\sigma]] = (C_0(\mathbb{V}(\sigma)^0 \hat{\oplus} \pi_\Sigma^* \mathbb{V}^1), 0) \in KK^G(C_0(X), C_0(\Sigma X)).$$

by the operator homotopy $\begin{pmatrix} 0 & t\sigma_0^* \\ t\sigma_0 & 0 \end{pmatrix} \cup_{SX} \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$. □

Lemma 5.13 *Consider $\iota^*: KK^G(C_0(\Sigma X), \mathbb{C}) \rightarrow KK^G(C_0(T^*X), \mathbb{C})$. Then, $\iota^*([\mathcal{D}_{\Sigma X}]) = [\mathcal{D}_{T^*X}]$.*

Proof. $\iota^*([\mathcal{D}_{\Sigma X}])$ is represented by the class

$$\left(L^2 \left(\bigwedge^{0,*} T^*(\Sigma X) \right), \frac{\mathcal{D}_{\Sigma X}}{\sqrt{1 + \mathcal{D}_{\Sigma X}^2}} \right) \in KK^G(C_0(T^*X), \mathbb{C})$$

with the point-wise multiplication action by $C_0(T^*X)$ via ι .

For a while, let ι denote also the inclusion of the spaces $\iota: T^*X \hookrightarrow \Sigma X$. For each $k = 0, 1, \dots$, we can find a smooth function $f_k: T^*X \rightarrow \mathbb{R}_{>0}$ so that the following maps are isometric on each fiber;

$$\begin{aligned} \bigwedge^{0,k} T^*(\Sigma X) &\rightarrow \bigwedge^{0,k} T^*(T^*X) \\ \omega &\mapsto f_k \cdot \iota^* \omega. \end{aligned}$$

Assembling these maps, we obtain the fiber-wise isometry

$$\widehat{\varphi} := \bigoplus_k f_k \cdot \iota^*: \bigwedge^{0,*} T^*(\Sigma X) \rightarrow \bigwedge^{0,*} T^*(T^*X).$$

Next, let h denote the Radon-Nikodym derivative $\frac{\text{vol}_{\Sigma X}}{\text{vol}_{T^*X}}$ so that ι induces the isomorphism between measure spaces;

$$(\Sigma X, \text{vol}_{\Sigma X}) \cong (T^*X, h \cdot \text{vol}_{T^*X}).$$

Using $\widehat{\varphi}$ and h , $L^2\left(\bigwedge^{0,*} T^*(\Sigma X)\right)$ can be identified with $L^2\left(\bigwedge^{0,*} T^*(T^*X)\right)$ by

$$\begin{aligned} L^2\left(\bigwedge^{0,*} T^*(\Sigma X)\right) &\rightarrow L^2\left(\bigwedge^{0,*} T^*(T^*X)\right) \\ s &\mapsto \sqrt{h} \cdot \widehat{\varphi}(s). \end{aligned}$$

This isomorphism will be denoted by $\varphi := \sqrt{h} \cdot \widehat{\varphi}$. Under this unitary equivalence, we have

$$\varphi \circ D_{T^*X} \circ \varphi^{-1} = D_{\Sigma X}.$$

Besides, since the functional calculus commutes with the unitary conjugation, we obtain

$$\varphi \circ \frac{\mathcal{D}_{T^*X}}{\sqrt{1 + \mathcal{D}_{T^*X}^2}} \circ \varphi^{-1} = \frac{\mathcal{D}_{\Sigma X}}{\sqrt{1 + \mathcal{D}_{\Sigma X}^2}}.$$

Note that this unitary equivalent map is compatible with the action of $C_0(T^*X)$. Therefore, as elements in $KK(C_0(T^*X), \mathbb{C})$,

$$\begin{aligned} \iota^*([\mathcal{D}_{\Sigma X}]) &= \left(L^2\left(\bigwedge^{0,*} T^*(\Sigma X)\right), \frac{\mathcal{D}_{\Sigma X}}{\sqrt{1 + \mathcal{D}_{\Sigma X}^2}} \right) \\ &= \left(L^2\left(\bigwedge^{0,*} T^*(\Sigma X)\right), \varphi \circ \frac{\mathcal{D}_{T^*X}}{\sqrt{1 + \mathcal{D}_{T^*X}^2}} \circ \varphi^{-1} \right) \\ &= \left(\varphi L^2\left(\bigwedge^{0,*} T^*(\Sigma X)\right), \frac{\mathcal{D}_{T^*X}}{\sqrt{1 + \mathcal{D}_{T^*X}^2}} \right) \\ &= \left(L^2\left(\bigwedge^{0,*} T^*(T^*X)\right), \frac{\mathcal{D}_{T^*X}}{\sqrt{1 + \mathcal{D}_{T^*X}^2}} \right) \\ &= [\mathcal{D}_{T^*X}] \end{aligned}$$

□

Proof of Proposition 5.9. Due to theorem 5.11, lemma 5.12 and lemma 5.13, we obtain

$$\begin{aligned}
[A] &= [\sigma] \hat{\otimes}_{C_0(TX)} [\mathcal{D}_{T^*X}] \\
&= [\sigma] \hat{\otimes}_{C_0(TX)} \iota^*([\mathcal{D}_{\Sigma X}]) \\
&= \iota_*([\sigma]) \hat{\otimes}_{C_0(\Sigma X)} [\mathcal{D}_{\Sigma X}] \\
&= [[\sigma]] \hat{\otimes}_{C_0(\Sigma X)} [\mathcal{D}_{\Sigma X}].
\end{aligned}$$

It implies the second part as

$$\begin{aligned}
[A] = [[\sigma]] \hat{\otimes}_{C_0(\Sigma X)} [\mathcal{D}_{\Sigma X}] &= (\pi_\Sigma^\#)^*([\mathbb{V}(\sigma)^0 \hat{\oplus} \pi_\Sigma^* \mathbb{V}^1]) \hat{\otimes}_{C_0(\Sigma X)} [\mathcal{D}_{\Sigma X}] \\
&= (\pi_\Sigma^\#)^*([\mathbb{V}(\sigma)^0 \hat{\oplus} \pi_\Sigma^* \mathbb{V}^1] \hat{\otimes}_{C_0(\Sigma X)} [\mathcal{D}_{\Sigma X}]) \\
&= (\pi_\Sigma^\#)^*([\mathcal{D}_\sigma]).
\end{aligned}$$

□

Corollary 5.14 *Let L be a Hermitian G -line bundle over X . Consider the induced bundle $\pi_\Sigma^* L$ over ΣX , which determines the KK -theory element $[\pi_\Sigma^* L] = (C_0(\pi_\Sigma^* L), 0) \in KK^G(C_0(\Sigma X), C_0(\Sigma X))$. Then,*

$$\mu_G([L] \hat{\otimes} [A]) = \mu_G([\pi_\Sigma^* L] \hat{\otimes} [\mathcal{D}_\sigma]).$$

Proof. As we will not consider the projection from T^*X , π_Σ will be abbreviated to π . The map between the algebras of continuous functions induced by π will be denoted by $\pi^\# : C_0(X) \rightarrow C_0(\Sigma X)$ and it will also be regarded as a KK -theory class $\pi^\# \in KK^G(C_0(X), C_0(\Sigma X))$. Note that $j^G(\pi^\#) \in KK(C^*(G; C_0(X)), C^*(G; C_0(\Sigma X)))$ represents a natural map $C^*(G; C_0(X)) \rightarrow C^*(G; C_0(\Sigma X))$ given by $f \mapsto \bar{f}(\gamma)(x, \xi) := f(\gamma)(\pi(x, \xi)) = f(\gamma)(x)$.

Let $c \in C_c(X)$ be an arbitrary cut-off function. Then $\pi^\# c = c \circ \pi \in C_c(\Sigma X)$ is a cut-off function for the action of G on ΣX . They determine K -homology classes $[c] \in KK(\mathbb{C}, C^*(G; C_0(X)))$ and $[c \circ \pi] \in KK(\mathbb{C}, C^*(G; C_0(\Sigma X)))$ as in lemma 5.3, and they are related by the formula $[c] \hat{\otimes} j^G(\pi^\#) = [c \circ \pi]$.

Using the proposition and noting that $[\pi^* L]$ is an element in $KK^G(C_0(\Sigma X), C_0(\Sigma X))$ and $(\pi^\#)^*[\pi^* L] = (\pi^\#)_*[L] \in KK^G(C_0(X), C_0(\Sigma X))$, we obtain

$$\begin{aligned}
\mu_G([L] \hat{\otimes} [A]) &= \mu_G([L] \hat{\otimes} (\pi^\#)^*[\mathcal{D}_\sigma]) \\
&= [c] \hat{\otimes} j^G((\pi^\#)_*[L] \hat{\otimes} [\mathcal{D}_\sigma]) \\
&= [c] \hat{\otimes} j^G((\pi^\#)^*[\pi^* L] \hat{\otimes} [\mathcal{D}_\sigma]) \\
&= [c] \hat{\otimes} j^G(\pi^\#) \hat{\otimes} j^G([\pi^* L] \hat{\otimes} [\mathcal{D}_\sigma]) \\
&= [c \circ \pi] \hat{\otimes} j^G([\pi^* L] \hat{\otimes} [\mathcal{D}_\sigma]) \\
&= \mu_G([\pi^* L] \hat{\otimes} [\mathcal{D}_\sigma]).
\end{aligned}$$

□

5.3 Expression of $\mu_G([A])$

Let A be a G -invariant elliptic operator of order 0 with proper support or a Dirac type operator and let $[A] = (L^2(\mathbb{V}), A) \in KK^G(C_0(X), \mathbb{C})$. To see more specific expression of the image of μ , let us first introduce a Hilbert $C^*(G)$ -module $\mathcal{E}^G$ by taking the completion of $C_c(\mathbb{V})$. which will be used to represent $\mu_G([A]) \in KK(\mathbb{C}, C^*(G))$.

First we define on $C_c(\mathbb{V})$ the structure of a pre-Hilbert module over $C_c(G)$ using the action of G on $C_c(\mathbb{V})$ given by $g[s](x) = g(s(g^{-1}x))$ for $g \in G$.

- The action of $C_c(G)$ on $C_c(\mathbb{V})$ from the right is given by

$$s \cdot b = \int_G g[s] \cdot b(g^{-1}) dg \in C_c(\mathbb{V}) \quad (5.4)$$

for $s \in C_c(\mathbb{V})$ and $b \in C_c(G)$. Let ρ^G denote this right action.

- The scalar product valued in $C_c(G)$ is given by

$$\langle s_1, s_2 \rangle_{\mathcal{E}^G}(g) = \langle s_1, g[s_2] \rangle_{L^2(\mathbb{V})} \quad (5.5)$$

for $s_i \in C_c(\mathbb{V})$.

Definition 5.15 Define $\mathcal{E}^G$ as the completion of $C_c(\mathbb{V})$ under the norm $\sqrt{\|\langle s, s \rangle\|_{C^*(G)}}$.

In order to realize $\mathcal{E}^G$ as a submodule of $C^*(G, C_0(X))$ for convenience, we define the map $i^G: C_c(\mathbb{V}) \rightarrow C_c(G; L^2(\mathbb{V}))$ and $q^G: C_c(G; L^2(\mathbb{V})) \rightarrow C_c(\mathbb{V})$ by the formula;

$$\begin{aligned} i^G[s](\gamma) &= \sqrt{c}\gamma[s] \\ q^G[\hat{s}] &= \int_G \gamma [\sqrt{c}\hat{s}(\gamma^{-1})] d\gamma \end{aligned}$$

for $s \in C_c(\mathbb{V})$ and $\hat{s} \in C_c(G; L^2(\mathbb{V}))$, where $c \in C_c(X)$ denotes a cut-off function.

i^G preserves the right G -action and scalar product, here $C_c(G; L^2(\mathbb{V}))$ is equipped with the following $C^*(G)$ -valued inner product $\langle \hat{s}_1, \hat{s}_2 \rangle_{C^*(G; L^2(\mathbb{V}))}(g) = \int_{g_1 g_2 = g} \langle \hat{s}_1(g_1), \hat{s}_2(g_2) \rangle_{L^2(\mathbb{V})} dg_1$.

q^G is the adjoint of i^G that is, $\langle i^G(s), \hat{s} \rangle_{C^*(G; L^2(\mathbb{V}))} = \langle s, q^G(\hat{s}) \rangle_{\mathcal{E}^G}$. Then, i^G determines the inclusion $\mathcal{E}^G \rightarrow C^*(G; L^2(\mathbb{V}))$ and q^G determines the projection $C^*(G; L^2(\mathbb{V})) \rightarrow \mathcal{E}^G$. Recall that $[c]\hat{\otimes}$ is given by the left action by the idempotent $\check{c}(\gamma)(x) = \sqrt{c(x)c(\gamma^{-1}x)} \in C_c(G; C_0(X))$. For $\hat{s} \in C_c(G; L^2(\mathbb{V}))$,

$$\begin{aligned} \check{c} * \hat{s}(\gamma) &= \int_G \check{c}(\gamma_1)\gamma_1[\hat{s}(\gamma_1^{-1}\gamma)]d\gamma_1 = \sqrt{c} \int_G \sqrt{\gamma_1[c]}\gamma_1[\hat{s}(\gamma_1^{-1}\gamma)]d\gamma_1 \\ &= \sqrt{c} \int_G \gamma\gamma_2 [\sqrt{c}\hat{s}(\gamma_2^{-1})] d\gamma_2 \\ &= i^G q^G \hat{s}(\gamma). \end{aligned}$$

Now $j^G([A])$ is represented by $(C^*(G; L^2(\mathbb{V})), A^G)$ here, the operator $A^G: C^*(G; L^2(\mathbb{V})) \rightarrow C^*(G; L^2(\mathbb{V}))$ is given by

$$A^G[\hat{s}](\gamma) = A[\hat{s}(\gamma)]$$

for $\hat{s} \in C_c(G; L^2(\mathbb{V}))$ as in definition 5.1. Hence, $\mu_G([A]) = [c]\hat{\otimes}j^G([A])$ is given by $(q^G C^*(G; L^2(\mathbb{V})), q^G A^G i^G)$. And we have for $s \in C_c(\mathbb{V})$,

$$\begin{aligned} \overline{A^G} s &:= q^G A^G i^G s = q^G (\gamma \mapsto A\sqrt{c}\gamma[s]) = \int_G \sqrt{\gamma[c]}\gamma [A\sqrt{c}\gamma^{-1}[s]] d\gamma \\ &= \int_G \gamma [\sqrt{c}A\sqrt{c}\gamma^{-1}[s]] d\gamma = \int_G \gamma [\sqrt{c}A\sqrt{c}] s d\gamma \end{aligned}$$

Theorem 5.16 [K, Theorem 5.6] $\mu_G([A]) \in KK(\mathbb{C}, C^*(G))$ is represented by $(\mathcal{E}^G, \overline{A^G})$, where $\overline{A^G} = q^G A^G i^G: \mathcal{E}^G \rightarrow \mathcal{E}^G$ and This operator is equal to

$$\overline{A^G} = \int_G \gamma [\sqrt{c}A\sqrt{c}] d\gamma.$$

$\overline{A^G}$ defines a bounded operator both on $L^2(\mathbb{V})$ and $\mathcal{E}^G$ [K, Proposition 5.1].

5.4 The Analytical Index Map

Let A be a properly supported G -invariant elliptic operator of order 0. We will define the analytical index of A using $\mathcal{E}^G$, introduced in the section 5.3.

Lemma 5.17 [K, Proposition 5.5] *Let A be a properly supported G -invariant elliptic operator of order 0. If the symbol σ_A of A is bounded at infinity in the cotangent direction, that is,*

$$\sup_{x \in X} \limsup_{\|\xi\| \rightarrow \infty} \|\sigma_P(x, \xi)\| < C,$$

then A determines the operator in $\mathbb{L}(\mathcal{E})$ and $\|A\|_{\mathbb{L}(\mathcal{E})/\mathbb{K}(\mathcal{E})} \leq C$. In particular if $C = 0$, which means that the symbol vanishes at infinity, $A \in \mathbb{K}(\mathcal{E})$. $\square$

Definition 5.18 (Analytical Index) Let G be a second countable locally compact Hausdorff group. Let X be a G -compact proper complete G -Riemannian manifold. Let A be a properly supported G -invariant elliptic operator of order 0.

$$\text{ind}_G(A) := (\mathcal{E}^G, A) \in C^*(G).$$

The previous lemma implies that $(\mathcal{E}^G, A)$ is a Kasparov module. Moreover, we have

Theorem 5.19 [K, Theorem 5.6]

$$\text{ind}_G(A) = \mu_G([A]).$$

$\square$

5.5 Kronecker Pairing and its Formula

Now we can define the Kronecker pairing $\langle [E], [A] \rangle_G \in \mathbb{R}$ for $[A] \in KK^G(C_0(X), \mathbb{R})$ and $[E] = (C_0(E), 0) \in KK^G(C_0(X), C_0(X))$ in Theorem A.

Definition 5.20

$$\langle [E], [A] \rangle_G := \text{tr}_G \circ \mu_G([E] \hat{\otimes} [A]) \in \mathbb{R},$$

where tr_G is the canonical trace on $C^*(G)$ in the definition 3.13.

Remark 5.21 We will see that this definition is compatible with the case of [HS], that is, the case when a discrete group Γ acts on X freely and X/Γ is compact. In such case, one has

$$K_\Gamma^0(C_0(X)) \simeq K^0(C^*(\Gamma; C_0(X))) \simeq K^0(C(X/\Gamma))$$

and $C(X/\Gamma)$ is unital, one also has a map

$$K^0(C(X/\Gamma)) \rightarrow K^0(\mathbb{C}) \simeq \mathbb{Z}$$

induced by the inclusion of $\mathbb{C}$ into $C(X/\Gamma)$ as constant functions. Let A be a Γ -invariant Dirac type operator on X , and E be a Γ -Hermitian vector bundle over X . Then they defines a Dirac type operator on X/Γ denoted by A' and a Hermitian vector bundle E' over X/Γ . Let $A'_{E'}$ denote the Dirac type operator twisted by E' . The composition $K_\Gamma^0(C_0(X)) \rightarrow \mathbb{Z}$ maps $[E] \hat{\otimes} [A]$ to the ordinary index of Dirac type operator $A'_{E'}$ on X/Γ , which will be denoted by $\text{ind}(A'_{E'})$. And it is equal to the usual K -theoretical pairing $\langle [E'], [A'] \rangle$ for $[E'] \in K^0(X/\Gamma)$ and $[A'] \in K_0(X/\Gamma)$. Therefore, it is equal to the pairing between ordinary cohomology and homology $\langle \text{ch}(E'), \text{ch}(A') \rangle$, which appears in [HS].

On the other hand our definition of the Kronecker pairing is defined by the composition of μ_G and tr_G

$$\text{tr}_G \circ \mu_G: K_F^0(C_0(X)) \rightarrow K_0(C^*(\Gamma)) \rightarrow \mathbb{R}.$$

Due to Atiyah's L^2 -index theorem [At], they coincide with each other, namely, $\langle [E], [A] \rangle_\Gamma = \text{ind}(A_E)$.

We need a formula of Kronecker pairing in the term of the integration of cohomology classes.

Theorem 5.22 [Wa, Proposition 6.11.] *Let X be a complete Riemannian manifold, where a second countable locally compact Hausdorff then the L^2 -index of A is given by the formula*

$$\text{tr}_G \text{ind}_G(A) = \int_{TX} c(x) \text{Td}(T_{\mathbb{C}}X) \wedge \text{ch}(\sigma_A), \quad (5.6)$$

where tr_G is the canonical trace on $C^*(G)$, σ_A denotes the symbol of A , and unimodular group G acts properly, co-compactly and isometrically. If A is a properly supported G -invariant elliptic operator of order 0, $c \in C_0(X)$ is any cut-off function.

Also we obtain a formula of the Kronecker pairing $\langle [E], [A] \rangle_G$ using the cohomology. Choose an arbitrary G -invariant connection ∇ on E and apply the theorem to the Dirac type operator A_E twisted by (E, ∇) to obtain

$$\begin{aligned} \langle [E], [A] \rangle_G &= \text{tr}_G \mu_G([E] \hat{\otimes} [A]) = \text{tr}_G \text{ind}_G(A_E) \\ &= \int_{TX} c(x) \text{Td}(T_{\mathbb{C}}X) \wedge \text{ch}(\sigma_A) \wedge \text{ch}(E, \nabla). \end{aligned}$$

This value is independent of the choice of the G -invariant connection on E according to the lemma 4.8.

Using this formula, we can define higher G - $\hat{\mathcal{A}}$ -genus in corollary B.

Definition 5.23 Let E be a smooth Hermitian G -vector bundle over X . Then we define

$$\hat{\mathcal{A}}_G(X; E) = \int_X c(x) \hat{\mathcal{A}}(TX) \wedge \text{ch}(E) \in \mathbb{R}.$$

This is the spacial case of (5.6) when A is the Dirac operator on the canonical spinor bundle over X twisted by E .

In the case of a free action of a discrete group Γ on X , by remark 2.6, $\hat{\mathcal{A}}_G(X; E)$ is equal to $\int_{X/\Gamma} \hat{\mathcal{A}}(T(X/\Gamma)) \wedge \text{ch}(E)$, namely, the higher $\hat{\mathcal{A}}$ -genus on X/Γ , so this definition 5.23 is one of generalizations of the higher $\hat{\mathcal{A}}$ -genus to the case of proper actions of non-discrete groups.

6 Construction of Almost Flat Bundles and their Indices

Observing that for E of the form $E = L^1 \oplus \dots \oplus L^N$, where L^k are G -line bundles, $\mu_G([E] \hat{\otimes} [A]) = \mu_G([L^1] \hat{\otimes} [A]) + \dots + \mu_G([L^N] \hat{\otimes} [A])$ and $\text{ch}(E) = \text{ch}(L^1) + \dots + \text{ch}(L^N)$, it is sufficient to consider the case that E is a single smooth G -line bundle L . Our strategy to prove the Theorem A is that firstly, construct a family of almost flat bundles L_t over X for $t \in [0, 1]$ using L , and calculate the index of twisted Dirac operators A_{L_t} . In order to do this, we introduce a $U(1)$ -cocycle $\alpha \in C^2(G; U(1))$ coming from the G -line bundle L following [Ma99] and [Ma03]. And we will consider the central extension using α .

6.1 Central Extension

We would like to construct line bundles L_t over X for $t \in [0, 1]$ whose curvature have the norm decreasing to zero. However L_t are not G -line bundles despite L is G -line bundle. We will introduce $U(1)$ -cocycle $\alpha \in C^2(G; U(1))$ coming from $C^2(G; \mathbb{R}) \xrightarrow{\exp(i\theta)} C^2(G; U(1))$ and central $U(1)$ -extensions G_{α^t} , and construct each of L_t as a G_{α^t} -line bundle over X .

Definition 6.1 Chose an arbitrary G -invariant connection ∇ on L and let $i\omega \in \Omega(X; i\mathbb{R})$ denote the curvature of (L, ∇) . Then its first Chern class is equal to $c_1(L; \nabla) = \frac{1}{2\pi}\omega \in \Omega(X; \mathbb{R})$. Fix a base point $x_0 \in X$. For any $\gamma \in G$, we will construct a $\mathbb{R}$ -valued smooth function $\psi_\gamma \in C^\infty(X)$ which will be used to define a $U(1)$ -cocycle $\alpha \in C^2(G; U(1))$.

- Since $[c_1(L)] = 0 \in H^2(X; \mathbb{R})$, in particular $[\omega] = 0 \in H_{\text{dR}}^2(X; \mathbb{R})$, there exists $\eta \in \Omega^1(X; \mathbb{R})$ such that $d\eta = \omega$.
- Since ω is G -invariant, we have $d(\gamma^*\eta - \eta) = \gamma^*\omega - \omega = 0$. The assumption of $H_1(X; \mathbb{Z}) = 0$ implies that there exists $\psi_\gamma \in C^\infty(X; \mathbb{R})$ satisfying $d\psi_\gamma = \gamma^*\eta - \eta$ for each $\gamma \in G$. We may assume for every $\gamma \in G$, $\psi_\gamma(x_0) = 0$ for a fixed base point $x_0 \in X$.

Lemma 6.2 $\psi_e(x) = 0$ for every $x \in X$, where e denotes the identity element in G .

Moreover for any γ_1 and $\gamma_2 \in G$,

$$\psi_{\gamma_1}(\gamma_2 x) + \psi_{\gamma_2}(x) - \psi_{\gamma_1 \gamma_2}(x)$$

is independent of $x \in X$

Proof. Since $d\psi_e = 0$, it follows that $\psi_e(x) = \psi_e(x_0) = 0$ for every $x \in X$. In addition

$$\begin{aligned} d(\psi_{\gamma_1}(\gamma_2 x) + \psi_{\gamma_2}(x) - \psi_{\gamma_1 \gamma_2}(x)) &= [\gamma_1^* \eta - \eta](\gamma_2 x) + [\gamma_2^* \eta - \eta](x) - [(\gamma_1 \gamma_2)^* \eta - \eta](x) \\ &= [\gamma_2^* \gamma_1^* \eta - \gamma_2^* \eta](x) + [\gamma_2^* \eta - \eta](x) - [\gamma_2^* \gamma_1^* \eta - \eta](x) \\ &= 0. \end{aligned}$$

Then the second statement follows. □

Lemma 6.3 Set

$$\begin{aligned} \sigma(\gamma_1, \gamma_2) &:= \psi_{\gamma_1}(\gamma_2 x) + \psi_{\gamma_2}(x) - \psi_{\gamma_1 \gamma_2}(x) \\ &= \psi_{\gamma_1}(\gamma_2 x_0). \end{aligned}$$

Then σ defines a $\mathbb{R}$ -valued 2-cocycle $\sigma \in Z^2(G; \mathbb{R})$. In particular $\alpha := \exp[-i\sigma]$ is a $U(1)$ -valued 2-cocycle $\alpha \in Z^2(G; U(1))$.

Proof. We have to check

$$\begin{aligned} \sigma(\gamma, e) = \sigma(e, \gamma) &= 0 \quad \text{and} \\ \sigma(\gamma_1, \gamma_2) + \sigma(\gamma_1 \gamma_2, \gamma_3) &= \sigma(\gamma_1, \gamma_2 \gamma_3) + \sigma(\gamma_2, \gamma_3). \end{aligned}$$

The first equation follows directly from $\psi_e(x) = 0$ and $\psi_\gamma(x_0) = 0$ for every $x \in X$ and $\gamma \in G$. The second equation follows from direct calculation. Due to the previous lemma,

$$\begin{aligned} \psi_{\gamma_1}(\gamma_2 \gamma_3 x_0) + \psi_{\gamma_2}(\gamma_3 x_0) - \psi_{\gamma_1 \gamma_2}(\gamma_3 x_0) &= \psi_{\gamma_1}(\gamma_2 x_0) + \psi_{\gamma_2}(x_0) - \psi_{\gamma_1 \gamma_2}(x_0) \\ &= \psi_{\gamma_1}(\gamma_2 x_0). \end{aligned}$$

Then we obtain

$$\begin{aligned}
\sigma(\gamma_1, \gamma_2) + \sigma(\gamma_1\gamma_2, \gamma_3) &= \psi_{\gamma_1}(\gamma_2x_0) + \psi_{\gamma_1\gamma_2}(\gamma_3x_0) \\
&= \psi_{\gamma_1}(\gamma_2x_0) + \{\psi_{\gamma_1}(\gamma_2\gamma_3x_0) + \psi_{\gamma_2}(\gamma_3x_0) - \psi_{\gamma_1}(\gamma_2x_0)\} \\
&= \psi_{\gamma_1}(\gamma_2\gamma_3x_0) + \psi_{\gamma_2}(\gamma_3x_0) \\
&= \sigma(\gamma_1, \gamma_2\gamma_3) + \sigma(\gamma_2, \gamma_3).
\end{aligned}$$

□

Lemma 6.4 *If we choose another G -invariant connection ∇' on L , then the resulting $\alpha' \in Z^2(G; U(1))$ is cohomologous to α . In particular at the $U(1)$ -cohomology level, $\alpha \in H^2(G; U(1))$ is independent of the choice of G -invariant connection ∇ and η used in the definition of α .*

Proof. If we let $i\omega' \in \Omega^2(X; i\mathbb{R})$ denote the curvature of (L, ∇') . Then its first Chern class $c_1(L, \nabla')$ is equal to $\frac{-1}{2\pi}\omega' \in \Omega^2(X; \mathbb{R})$. According to the lemma 4.8, there exists a G -invariant 1-form $\theta \in \Omega^1(X)$ satisfying $\omega' - \omega = d\theta$. As in definition 6.1, since $[\omega'] \in H_{\text{dR}}^2(X; \mathbb{R})$, there exists $\eta' \in \Omega^1(X)$ such that $d\eta' = \omega'$. Since $d\eta' - d\eta = \omega' - \omega = d\theta$, there exists $h \in C^\infty(X)$ which satisfies $dh = \eta' - \eta - \theta$. We may assume $h(x_0) = 0$ for the fixed base point $x_0 \in X$. Similarly, define $\psi' \in C^\infty(X)$ for each $\gamma \in G$ by $d\psi'_\gamma = \gamma^*\eta' - \eta'$ and $\psi'_\gamma(x_0) = 0$. Now since $\gamma^*\theta = \theta$, we obtain

$$\begin{aligned}
d(\psi'_\gamma - \psi_\gamma) &= \gamma^*(\eta' - \eta) - (\eta' - \eta) \\
&= \gamma^*(\eta' - \eta - \theta) - (\eta' - \eta - \theta) \\
&= d(\gamma^*h - h) \\
\text{and } \psi'_\gamma(x_0) = \psi_\gamma(x_0) = h(x_0) &= 0, \\
\text{which imply } \psi'_\gamma - \psi_\gamma &= \gamma^*h - h - \gamma^*h(x_0).
\end{aligned}$$

Therefore the $U(1)$ -cocycle defined using ω' is

$$\begin{aligned}
\alpha'(\gamma_1, \gamma_2) &= \exp[-i\psi'_{\gamma_1}(\gamma_2x_0)] \\
&= \exp[-i\psi_{\gamma_1}(\gamma_2x_0)] \exp[-i(\gamma_1^*h(\gamma_2x_0) - h(\gamma_2x_0) - \gamma_1^*h(x_0))] \\
&= \alpha(\gamma_1, \gamma_2)\delta\beta(\gamma_1, \gamma_2)^{-1},
\end{aligned}$$

where $\beta \in C^1(G; U(1))$ is a $U(1)$ -cocycle defined by $\beta(\gamma) = \exp[-ih(\gamma x_0)]$ and recall that the derivative $\delta: C^1(G; U(1)) \rightarrow C^2(G; U(1))$ is given by the formula

$$(\delta\beta)(\gamma_1, \gamma_2) = \beta(\gamma_1)\beta(\gamma_2)\beta(\gamma_1\gamma_2)^{-1}.$$

To conclude, $\alpha' \in Z^2(G; U(1))$ is cohomologous to α .

□

Definition 6.5 For each $t \in [0, 1]$, take a $U(1)$ -cocycle $\alpha^t = \exp[-it\sigma]$. Let G_{α^t} be the central $U(1)$ -extension of G determined by the 2-cocycle $\alpha^t \in Z^2(G; U(1))$. In other words G_{α^t} is $G \times U(1)$ as a topological measure space and equipped with the following product;

$$(\gamma_1, u_1)(\gamma_2, u_2) = (\gamma_1\gamma_2, \alpha(\gamma_1, \gamma_2)^t u_1 u_2).$$

Suppose that the Haar measure of $U(1)$ is normalized so that $\int_{U(1)} du = 1$.

Since there is a one to one correspondence between the isomorphism classes of central $U(1)$ -extension of G and $H^2(G; U(1))$ when G acts on $U(1)$ trivially, G_{α^t} is determined uniquely up to isomorphism.

6.2 Almost Flat G_α -Line Bundles

Definition 6.6 For each $t \in [0, 1]$ we will construct a G_{α^t} -line bundle L_t over X whose curvature is equal to $it\omega$.

- L_t is topologically equal to a trivial bundle $X \times \mathbb{C}$.
- The connection of L_t is given by $\nabla^t = d + it\eta$.

Since L_t is a line bundle, its curvature is given by $d(it\eta) + (it\eta) \wedge (it\eta) = d(it\eta) = it\omega$.

Lemma 6.7

(1) G_{α^t} acts on L_t given by the formula;

$$(\gamma, u)(x, z) = (\gamma x, \exp[-it\psi_\gamma(x)]uz) \quad \text{for } (\gamma, u) \in G_{\alpha^t}, x \in X, z \in \mathbb{C} = (L_t)_x.$$

(2) ∇^t is a G_{α^t} -invariant connection, that is, the action of G_{α^t} and the connection ∇^t on L_t commute with each other. In particular let A_{L_t} denote the twisted Dirac operator acting on the sections of $\mathbb{V} \hat{\otimes} L_t$. Then A_{L_t} is G_{α^t} -equivariant.

Proof. This actually determines the action of G_{α^t} . In fact,

$$\begin{aligned} (\gamma_1, u_1) \{(\gamma_2, u_2)(x, z)\} &= (\gamma_1 \gamma_2 x, \exp[-it\psi_{\gamma_1}(\gamma_2 x)] \exp[-it\psi_{\gamma_2}(x)] u_1 u_2 z) \\ &= (\gamma_1 \gamma_2 x, \exp[-it\psi_{\gamma_1 \gamma_2}(x)] \alpha(\gamma_1, \gamma_2)^t u_1 u_2 z) \\ &= \{(\gamma_1, u_1)(\gamma_2, u_2)\}(x, z). \end{aligned}$$

Remark that the action of G_{α^t} on $C^\infty(L_t)$ is given by the formula;

$$(\gamma, u)[s](x) = (\gamma, u)(s(\gamma^{-1}x)) = \exp[-it\psi_\gamma(\gamma^{-1}x)] u \cdot \gamma[s](x)$$

As for the second statement, we will proof in the case of $t = 1$. The claim for general $t \in [0, 1]$ follows if we replace η and ψ by $t\eta$ and $t\psi$ respectively. Let $s \in C^\infty(L)$ be an arbitrary smooth section of L . To restate our claim, $(\gamma, u)[\nabla s] = \nabla((\gamma, u)[s])$.

The left hand side is equal to

$$(\gamma, u)[(d + i\eta)s] = u \exp[-i\psi_\gamma(\gamma^{-1}x)] \gamma[(d + i\eta)s]. \quad (6.1)$$

The right hand side is equal to

$$\begin{aligned} (d + i\eta)((\gamma, u)[s]) &= u(d + i\eta)(\exp[-i\psi_\gamma(\gamma^{-1}x)] \gamma[s]) \\ &= u \exp[-i\psi_\gamma(\gamma^{-1}x)] \{d(\gamma[s]) + id\psi_\gamma(\gamma^{-1}x) \cdot \gamma[s] + i\eta \cdot \gamma[s]\}. \end{aligned} \quad (6.2)$$

After deviding by $u \exp[-i\psi_\gamma(\gamma^{-1}x)]$,

$$\begin{aligned} \frac{(6.1)}{u \exp[-i\psi_\gamma(\gamma^{-1}x)]} &= d\gamma[s] - i(\gamma^{-1})^* \eta \cdot \gamma[s] \\ &= d\gamma[s] + i(\eta + d\psi_{\gamma^{-1}}(x)) \cdot \gamma[s], \end{aligned} \quad (6.3)$$

$$\frac{(6.2)}{u \exp[-i\psi_\gamma(\gamma^{-1}x)]} = d(\gamma[s]) - id\psi_\gamma(\gamma^{-1}x) \cdot \gamma[s] + i\eta \cdot \gamma[s]. \quad (6.4)$$

Then we conclude that

$$(6.3) - (6.4) = i \{d\psi_{\gamma^{-1}}(x) + d\psi_\gamma(\gamma^{-1}x)\} \gamma[s] = i \{d\psi_e(x)\} \gamma[s] = 0.$$

□

6.3 Indices of Almost Flat Bundles

Now we can apply the theorem 5.22 to each A_{L_t} to obtain

Proposition 6.8

$$\mathrm{tr}_{G_{\alpha t}} \mu_{G_{\alpha t}}([L_t] \hat{\otimes} [A]) = \mathrm{tr}_{G_{\alpha t}} \mathrm{ind}_{G_{\alpha t}}(A_{L_t}) = \int_{TX} c(x) \mathrm{Td}(T_{\mathbb{C}}X) \wedge \mathrm{ch}(\sigma_A) \wedge \exp\left(\frac{-t}{2\pi}\omega\right). \quad (6.5)$$

Later on we will regard it as a polynomial in t .

6.4 The Index Map $\mu_{G_{\alpha}}$

We will compare two index maps μ_G and $\mu_{G_{\alpha}}$. It is convenient to introduce the following natural homomorphisms.

Definition 6.9 Let B be a G -algebra, in particular $B = \mathbb{C}$ or $C_0(X)$. Define

- $\lambda: C^*(G; B) \rightarrow C^*(G_{\alpha}; B)$ given by

$$\lambda[f](\gamma, u) := f(\gamma) \quad \text{for } f \in C_c(G; B), \text{ and}$$

- $\theta: C^*(G_{\alpha}; B) \rightarrow C^*(G; B)$ is given by

$$\theta[b](\gamma) := \int_{U(1)} b(\gamma, u) du \quad \text{for } b \in C_c(G_{\alpha}; B).$$

Both of the operators are bounded with respect to the C^* -norms $\|\cdot\|_{C^*(G; B)}$ and $\|\cdot\|_{C^*(G_{\alpha}; B)}$, so they define the homomorphisms between $C^*(G; B)$ and $C^*(G_{\alpha}; B)$.

In fact, for any covariant representation (π, ρ) for (B, G) and for $f \in C_c(G)$,

$$\sigma_{\pi, \tilde{\rho}}[\lambda[f]] = \int_{G_{\alpha}} \pi(\lambda[f](\gamma, u)) \tilde{\rho}(\gamma, u) d\gamma du = \int_G \pi(f(\gamma)) \rho'(\gamma) d\gamma = \sigma_{\pi, \rho'}[f],$$

where $\rho'(\gamma)$ is a representation of G given by $\rho'(\gamma) := \int_{U(1)} \tilde{\rho}(\gamma, u) du$ is a representation on G .

Moreover for any covariant representation $(\pi, \tilde{\rho})$ for (B, G_{α}) and for $b \in C_c(G_{\alpha})$,

$$\sigma_{\pi, \rho}[\theta[b]] = \int_G \pi(\theta[b](\gamma)) \rho(\gamma) d\gamma = \int_{G_{\alpha}} \pi(b(\gamma, u)) \tilde{\rho}(\gamma, u) du d\gamma = \sigma_{\pi, \tilde{\rho}}[b],$$

where $\tilde{\rho}(\gamma, u)$ is a representation of G_{α} given by $\tilde{\rho}(\gamma, u) := \rho(\gamma)$.

Remark 6.10 It follows immediately that $\theta \circ \lambda = \mathrm{id}$. Note that $b \in C_c(G_{\alpha}; B)$ is constant in $(1, u) \in U(1) \subset G_{\alpha}$ if and only if $\lambda \circ \theta[b] = b$ holds.

The main purpose of this subsection is to verify the following proposition;

Proposition 6.11 *Let A be a properly supported G -invariant elliptic operator on $\mathbb{V}$ over X of order 0. Consider K -homology class $[A] \in KK^G(C_0(X), \mathbb{C})$ and apply $\mu_G: KK^G(C_0(X), \mathbb{C}) \rightarrow K_0(C^*(G))$ to it. One can regard also $[A]$ as a K -homology class in $KK^{G_{\alpha}}(C_0(X), \mathbb{C})$ and apply $\mu_{G_{\alpha}}: KK^{G_{\alpha}}(C_0(X), \mathbb{C}) \rightarrow K_0(C^*(G_{\alpha}))$ to it. Then,*

$$\lambda_* \mu_G([A]) = \mu_{G_{\alpha}}([A]), \quad \text{and} \quad \theta_* \mu_{G_{\alpha}}([A]) = \mu_G([A]).$$

Now we will use the expression of $\mu_{G'}([A])$ in the subsection 5.3; $\mu_{G'}([A]) = (\mathcal{E}^{G'}, \overline{A^{G'}})$ for $G' = G$ or G_α and compare them.

Lemma 6.12 *If we regard $\mathcal{E}^G$ as a module over $C^*(G_\alpha)$ through $\theta: C^*(G_\alpha) \rightarrow C^*(G)$ equipped with the $C^*(G_\alpha)$ -valued scalar product $\lambda \langle \cdot, \cdot \rangle_{\mathcal{E}^G}$, then it is isomorphic to $\mathcal{E}^{G_\alpha}$.*

Similarly, if we regard $\mathcal{E}^{G_\alpha}$ as a module over $C^(G)$ through $\lambda: C^*(G) \rightarrow C^*(G_\alpha)$ equipped with the $C^*(G)$ -valued scalar product $\theta \langle \cdot, \cdot \rangle_{\mathcal{E}^{G_\alpha}}$, then it is isomorphic to $\mathcal{E}^G$. In other words,*

$$\begin{aligned} (\mathcal{E}^{G_\alpha}; \langle \cdot, \cdot \rangle_{\mathcal{E}^{G_\alpha}}) &\simeq (\mathcal{E}^G; \lambda \langle \cdot, \cdot \rangle_{\mathcal{E}^G}) && \text{as } C^*(G_\alpha)\text{-modules} \\ \text{and } (\mathcal{E}^G; \langle \cdot, \cdot \rangle_{\mathcal{E}^G}) &\simeq (\mathcal{E}^{G_\alpha}; \theta \langle \cdot, \cdot \rangle_{\mathcal{E}^{G_\alpha}}) && \text{as } C^*(G)\text{-modules.} \end{aligned} \quad (6.6)$$

Proof. Consider the structure of Hilbert $C^*(G_\alpha)$ -module (5.4) and (5.5) on $\mathcal{E}^{G_\alpha}$. We will let the right action of $G' = G$ or G_α on $C_c(\mathbb{V})$ be denoted by $\cdot_{G'}$ when we have to distinguish them. An important observation is that the action of $U(1) \subset G_\alpha$ on $\mathbb{V}$ is trivial. Therefore the action ρ^{G_α} of $C_c(G_\alpha)$ on $C_c(\mathbb{V})$ factors through the action of $C_c(G)$, that is, for $s \in C_c(\mathbb{V})$ and $b \in C_c(G_\alpha)$,

$$\begin{aligned} s \cdot_{G_\alpha} b &= \int_{G_\alpha} (\gamma, u)[s] \cdot b((\gamma, u)^{-1}) d\gamma du \\ &= \int_G \gamma[s] \cdot \left(\int_{U(1)} b(\gamma, \alpha(\gamma, \gamma^{-1})^{-1} u^{-1}) du \right) d\gamma \\ &= \int_G \gamma[s] \cdot \theta[b](\gamma^{-1}) d\gamma \\ &= s \cdot_G \theta[b]. \end{aligned}$$

Replacing b by $\lambda[f]$ for an arbitrary $f \in C_c(G)$, we also obtain $s \cdot_{G_\alpha} \lambda[f] = s \cdot_G f$.

In addition the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{E}^{G_\alpha}}$ on $C_c(\mathbb{V})$ is constant in $(1, u) \in U(1) \subset G_\alpha$, in the other words,

$$\langle \cdot, \cdot \rangle_{\mathcal{E}^{G_\alpha}} = \lambda(\langle \cdot, \cdot \rangle_{\mathcal{E}^G}) \quad \text{and} \quad \theta(\langle \cdot, \cdot \rangle_{\mathcal{E}^{G_\alpha}}) = \langle \cdot, \cdot \rangle_{\mathcal{E}^G} \quad \text{on } C_c(\mathbb{V}).$$

They imply the two norms $\|s\|_{\mathcal{E}^G} := \sqrt{\|\langle s, s \rangle_{\mathcal{E}^G}\|_{C^*(G)}}$ and $\|s\|_{\mathcal{E}^{G_\alpha}} := \sqrt{\|\langle s, s \rangle_{\mathcal{E}^{G_\alpha}}\|_{C^*(G_\alpha)}}$ on $C_c(\mathbb{V})$ are equivalent because both λ and θ are bounded. In particular $\mathcal{E}^G \simeq \mathcal{E}^{G_\alpha}$. $\square$

Proof of the Proposition 6.11. $\lambda_* \mu_G([A])$ is represented by

$$(\mathcal{E}^G \widehat{\otimes}_\lambda C^*(G_\alpha), \overline{A^G \widehat{\otimes} 1}).$$

We will compare it with $(\mathcal{E}^{G_\alpha}, \overline{A^{G_\alpha}})$, which represents $\mu_{G_\alpha}([A])$.

Consider a map $\varphi: \mathcal{E}^G \widehat{\otimes}_\lambda C^*(G_\alpha) \rightarrow \mathcal{E}^{G_\alpha}$ given by $\varphi(s \widehat{\otimes} b) := s \cdot_{G_\alpha} b$. Notice that the right hand side is equal to $s \cdot_G \theta[b]$ and that $\mathcal{E}^G$ and $\mathcal{E}^{G_\alpha}$ are identified with each other.

The well-defined-ness follows from

$$\begin{aligned} \varphi\left((s \cdot_G f) \widehat{\otimes} b\right) &= (s \cdot_G f) \cdot_G \theta[b] = s \cdot_G (f *_G \theta[b]) \quad \text{and} \\ \varphi\left(s \widehat{\otimes} (\lambda[f] *_G b)\right) &= s \cdot_G \theta[\lambda[f] *_G b] = s \cdot_G (\theta \lambda[f] *_G \theta[b]) = s \cdot_G (f *_G \theta[b]), \end{aligned}$$

where $*_{G'}$ denotes the convolution product of $C^*(G')$. From the previous lemma, it is obvious that φ is compatible with the right action of $C^*(G_\alpha)$.

Next we will check that φ is isometric. For $s_1, s_2 \in \mathcal{E}^G$ and $b_1, b_2 \in C^*(G_\alpha)$,

$$\begin{aligned} \langle s_1 \otimes b_1, s_2 \otimes b_2 \rangle &= b_1^* * \lambda(\langle s_1, s_2 \rangle_{\mathcal{E}^G}) * b_2 \\ &= b_1^* * \langle s_1, s_2 \rangle_{\mathcal{E}^{G_\alpha}} * b_2 \\ &= \left\langle s_1 \cdot_{G_\alpha} b_1, s_2 \cdot_{G_\alpha} b_2 \right\rangle_{\mathcal{E}^{G_\alpha}} = \langle \varphi(s_1 \otimes b_1), \varphi(s_2 \otimes b_2) \rangle_{\mathcal{E}^{G_\alpha}}, \end{aligned}$$

which implies that φ preserves the scalar products and hence it is continuous and injective.

Using an approximating unit $\{e_m\}_{m \in \mathcal{M}}$ for $C^*(G_\alpha)$, the subset of all elements of the following form is dense in $\mathcal{E}^G$.

$$\lim_{m \in \mathcal{M}} \varphi(s \otimes e_m) = \lim_{m \in \mathcal{M}} s \cdot e_m = \lim_{m \in \mathcal{M}} q^{G_\alpha}(i^{G_\alpha}(s) * e_m) = q^{G_\alpha}(i^{G_\alpha}(s)) = s$$

for any $s \in C_c(\mathbb{V})$. Therefore φ is an isomorphism.

Finally, since the sub group $U(1) \hookrightarrow G_\alpha$ acts trivially on $\mathbb{V}$, we have that

$$\begin{aligned} \overline{A^{G_\alpha}} &= \int_{G_\alpha} (\gamma, u) [\sqrt{c} A \sqrt{c}] d\gamma du \\ &= \int_{G_\alpha} \gamma [\sqrt{c} A \sqrt{c}] d\gamma du \\ &= \int_G \gamma [\sqrt{c} A \sqrt{c}] d\gamma \int_{U(1)} du = \overline{A^G}. \end{aligned}$$

Now we have that $\varphi \circ (\overline{A^G} \widehat{\otimes} 1)(s \widehat{\otimes} b) = \overline{A^G} \circ \varphi(s \widehat{\otimes} b) = \overline{A^{G_\alpha}} \circ \varphi(s \widehat{\otimes} b)$ for $s \in C_c(\mathbb{V})$ and $b \in C_c(G_\alpha)$, and it means that φ is compatible with the operators.

As a conclusion, $(\mathcal{E}^G \widehat{\otimes}_\lambda C^*(G_\alpha), \overline{A^G} \widehat{\otimes} 1)$ and $(\mathcal{E}^{G_\alpha}, \overline{A^{G_\alpha}})$ are isomorphic by φ and hence $\lambda_* \mu_G([A]) = \mu_{G_\alpha}([A])$.

Since $\theta\lambda = \text{id}_{C^*(G)}$, the second statement of the proposition is implied from the first one. $\square$

7 The Index Twisted by Product Bundles

From now on, let G_{α^t} be denoted by G^t for $t \in [0, 1]$.

7.1 Infinite Product of C^* -algebras

Definition 7.1 Let B_k be a sequence of C^* -algebras.

- Define $\prod_{k \in \mathbb{N}} B_k$ as the C^* -algebra consisting of norm-bounded sequences

$$\prod_{k \in \mathbb{N}} B_k := \left\{ \{b_1, b_2, \dots\} \mid b_k \in B_k, \sup_k \{\|b_k\|_{B_k}\} < \infty \right\}.$$

The norm of B_k is given by $\|\{b_1, b_2, \dots\}\|_{\prod B_k} := \sup_k \{\|b_k\|_{B_k}\}$.

- Let $\bigoplus_{k \in \mathbb{N}} B_k$ a closed two-sided ideal in $\prod_{k \in \mathbb{N}} B_k$ consisting of sequences vanishing at infinity

$$\bigoplus_{k \in \mathbb{N}} B_k := \left\{ \{b_1, b_2, \dots\} \mid b_k \in B_k, \lim_{k \rightarrow \infty} \|b_k\| = 0 \right\}.$$

In other words, $\bigoplus_{k \in \mathbb{N}} B_k$ is a closure of the sub-space consisting of sequences $\{b_1, b_2, \dots, 0, 0, \dots\}$ whose entries are zero except for finitely many of them, in $\prod_{k \in \mathbb{N}} B_k$.

- Define $\mathcal{Q} B_k$ as the quotient algebra given by

$$\mathcal{Q} B_k := \left(\prod_{k \in \mathbb{N}} B_k \right) / \left(\bigoplus B_k \right).$$

The norm of $\mathcal{Q} B_k$ is given by $\|\{b_1, b_2, \dots\}\|_{\mathcal{Q} B_k} := \limsup_{k \rightarrow \infty} \|b_k\|_{B_k}$.

- If $\mathcal{E}_k$ are Hilbert B_k -modules, one can define, in the similar way, $\prod \mathcal{E}_k$ as a Hilbert $\prod B_k$ -module consisting of bounded sequences

$$\prod_{k \in \mathbb{N}} \mathcal{E}_k := \left\{ \{s_1, s_2, \dots\} \mid s_k \in \mathcal{E}_k, \sup_k \{\|s_k\|_{\mathcal{E}_k}\} < \infty \right\}.$$

The action of $\prod B_k$ and $\prod B_k$ -valued scalar product is defined as follows;

$$\begin{aligned} \{s_k\} \cdot \{b_k\} &:= \{s_k \cdot b_k\} \in \prod \mathcal{E}_k \quad \text{for } \{s_k\} \in \prod \mathcal{E}_k, \{b_k\} \in \prod B_k, \\ \langle \{s_k^1\}, \{s_k^2\} \rangle_{\prod \mathcal{E}_k} &:= \left\langle \{s_k^1, s_k^2\}_{\mathcal{E}_k} \right\rangle \in \prod B_k \quad \text{for } \{s_k^1\}, \{s_k^2\} \in \prod \mathcal{E}_k. \end{aligned}$$

One can define similraly

$$\bigoplus_{k \in \mathbb{N}} \mathcal{E}_k := \left\{ \{s_1, s_2, \dots\} \mid s_k \in \mathcal{E}_k, \lim_{k \rightarrow \infty} \|s_k\|_{\mathcal{E}_k} = 0 \right\}$$

as a Hilbert $\prod B_k$ -module, and define

$$\mathcal{Q} \mathcal{E}_k := \left(\prod \mathcal{E}_k \right) \widehat{\otimes}_{\pi} (\mathcal{Q} B_k) = \left(\prod \mathcal{E}_k \right) / \left(\bigoplus \mathcal{E}_k \right)$$

as a Hilbert $\mathcal{Q} B_k$ -module, where $\pi: \prod B_k \rightarrow \mathcal{Q} B_k$ denotes the projection.

Example 7.2 Let all of B_k be $\mathbb{C}$. Then, $\prod \mathbb{C} = \ell^\infty(\mathbb{N})$ and $\bigoplus \mathbb{C} = C_0(\mathbb{N})$.

7.2 Infinite Product of C^* -algebras with Uniformly Compact Supports

Definition 7.3 Let B_k be $\mathbb{C}$ or $C_0(X)$ regarded as $G^{1/k}$ -algebras.

- Define $\prod' C^*(G^{1/k}; B_k)$ be the sub C^* -algebra of $\prod C^*(G^{1/k}; B_k)$ given by the completion of the following sub-space;

$$\prod' C_c(G^{1/k}; B_k) := \left\{ \{b_k\} \mid \begin{array}{l} b_k \in C_c(G^{1/k}; B_k), \\ \text{there exists a compact subset } Z \subset G \times U(1) \\ \text{such that } \text{supp}(b_k) \subset Z \text{ for all } k, \\ \text{and } \|b_k(\gamma, u)\|_{B_k} \text{ is uniformly bounded in } k \\ \text{for all } (\gamma, u) \in G^{1/k}. \end{array} \right\}$$

Note that each $G^{1/k}$ is identified with $G \times U(1)$ as topological spaces.

Moreover if $B_k = C_0(X)$, every elements $\{b_k\} \in C_c(G^{1/k}; C_0(X))$ are required to satisfy that there exists a compact subset $Y \subset X$ such that $b_k(\gamma, u) \in C_c(X)$ and $\text{supp}(b_k(\gamma, u)) \subset Y$ for all $(\gamma, u) \in G \times U(1)$ and k as an additional condition.

- Let $\mathcal{E}_k$ be $L^2(\mathbb{V})$ regarded as $G^{1/k}$ -Hilbert $\mathbb{C}$ -modules, or $C_0(E_k)$ regarded as $G^{1/k}$ -Hilbert $C_0(X)$ -modules for some Hermitian $G^{1/k}$ -vector bundles E_k .

One can define, in a similar way, $\prod' C^*(G^{1/k}; \mathcal{E}_k)$ as a Hilbert $\prod' C^*(G^{1/k}; B_k)$ -module given by the closure of the following sub-space;

$$\prod' C_c(G^{1/k}; \mathcal{E}_k) := \left\{ \{s_k\} \left| \begin{array}{l} s_k \in C_c(G^{1/k}; \mathcal{E}_k), \\ \text{there exists a compact subset } Z \subset G \times U(1) \\ \text{and a compact subset } Y \subset X \\ \text{such that } \text{supp}(s_k) \subset Z \text{ for all } k, \\ \text{and } \text{supp}(s_k(\gamma, u)) \subset Y \text{ for all } (\gamma, u) \in G \times U(1), \\ \text{and } \|s_k(\gamma, u)\|_{\mathcal{E}_k} \text{ is uniformly bounded in } k \\ \text{for all } (\gamma, u) \in G^{1/k}. \end{array} \right. \right\}$$

Note that the subspace consisting of compactly supported sections of $\mathbb{V}$ or E_k are dense in $L^2(\mathbb{V})$ or $C_0(E_k)$.

The right $\prod' C_c(G^{1/k}; B_k)$ -action and $\prod' C_c(G^{1/k}; B_k)$ -valued scalar product are given by

$$\begin{aligned} \{s_k\} \cdot \{b_k\} &:= \left\{ s_k \cdot_{G^{1/k}} b_k \right\} = \left\{ g \mapsto \int_{G^{1/k}} s_k(g_1) \cdot g_1 [b_k(g_1^{-1}g)] dg_1 \right\} \\ \langle \{s_k^1\}, \{s_k^2\} \rangle &:= \left\{ \langle s_k^1, s_k^2 \rangle_{C^*(G^{1/k}; \mathcal{E}_k)} \right\} = \left\{ g \mapsto \int_{G^{1/k}} g_1^{-1} \langle s_k^1(g_1), s_k^2(g_1 g) \rangle_{\mathcal{E}_k} dg_1 \right\} \end{aligned}$$

for $\{s_k\}, \{s_k^1\}, \{s_k^2\} \in \prod' C_c(G^{1/k}; \mathcal{E}_k)$ and $\{b_k\} \in \prod' C_c(G^{1/k}; B_k)$.

Remarks 7.4

- (1) Let $\{b_k\} \in \prod' C_c(G^{1/k}; C_0(X))$. Since the supports of b_k are contained in a compact subset $Z \subset G \times U(1)$ independent of k , the condition that the support of $b_k(\gamma, u)$ is uniformly compact is equivalent to the condition that $\bigcup_{(\gamma, u) \in G \times U(1)} \text{supp}(b_k(\gamma, u)) \subset Y$ for some compact subset $Y \subset X$ independent of k . Similarly for $\{s_k\} \in \prod' C_c(G^{1/k}; \mathcal{E}_k)$, it is implied that there exists a compact subset $Y \subset X$ such that $\bigcup_{(\gamma, u) \in G \times U(1)} \text{supp}(s_k(\gamma, u)) \subset Y$.
- (2) For $b^1, b^2 \in C_c(G^{1/k}; B_k)$, we have that $\text{supp}(b^1 * b^2) \subset \text{supp}(b^1) \cdot \text{supp}(b^2)$ and that $\text{supp}(b^1 * b^2(g)) \subset \bigcup_{g_1 \in G \times U(1)} \text{supp}(b^1(g_1))$ when $B_k = C_c(X)$. These imply that $\prod' C_c(G^{1/k}; B_k)$ is closed under the product.
- (3) The fact that $\prod' C_c(G^{1/k}; \mathcal{E}_k)$ is closed under the right $\prod' C_c(G^{1/k}; B_k)$ -action can be verified in the similar way.
- (4) Since $\text{supp}(\langle s_k^1, s_k^2 \rangle) \subset \text{supp}(s_k^1)^{-1} \text{supp}(s_k^2)$, it follows that $\langle \{s_k^1\}, \{s_k^2\} \rangle \in \prod' C_c(G^{1/k}; B_k)$ when $B_k = \mathbb{C}$. When $B_k = C_0(X)$, it follows from $\text{supp}(\langle s_k^1, s_k^2 \rangle(g)) \subset \text{supp}(s_k^1)^{-1} \cdot \bigcup_{g_1 \in G \times U(1)} \text{supp}(s_k^1(g_1))$.

Remarks 7.5

- (1) $\bigoplus C^*(G^{1/k}; B_k)$ is still an ideal in $\prod' C^*(G^{1/k}; B_k)$. This is because elements in $\bigoplus C^*(G^{1/k}; B_k)$ is always approximated by elements of the form $\{b_k\}$ with $b_k \in C_c(G^{1/k}; B_k)$ and $b_k = 0$ for all but finitely many k .
- (2) $\prod' C^*(G; B_k)$ and $\prod' C^*(G; \mathcal{E}_k)$ are defined in the similar way.

Definition 7.6 Let $\mathcal{Q}' C^*(G^{1/k}; B_k)$ be the quotient algebra given by

$$\mathcal{Q}' C^*(G^{1/k}; B_k) := \left(\prod' C^*(G^{1/k}; B_k) \right) / \left(\bigoplus C^*(G^{1/k}; B_k) \right).$$

Similarly, one can define

$$\mathcal{Q}' \mathcal{E}_k := \left(\prod' C^*(G^{1/k}; \mathcal{E}_k) \right) \widehat{\otimes}_\pi \left(\mathcal{Q}' C^*(G^{1/k}; B_k) \right) = \left(\prod' C^*(G^{1/k}; \mathcal{E}_k) \right) / \left(\bigoplus C^*(G^{1/k}; \mathcal{E}_k) \right)$$

as a Hilbert $\mathcal{Q}' C^*(G^{1/k}; B_k)$ -module, where $\pi: \prod' C^*(G^{1/k}; B_k) \rightarrow \mathcal{Q}' C^*(G^{1/k}; B_k)$ denotes the projection.

Same as 6.9, we introduce the following maps

Definition 7.7 Let B be a G -algebra, in particular $B = \mathbb{C}$ or $C_0(X)$. Define

- $\lambda^t: C^*(G; B) \rightarrow C^*(G^t; B)$ given by $\lambda^t[b](\gamma, u) := b(\gamma)$ for $b \in C_c(G; B)$.
- $\theta^t: C^*(G^t; B) \rightarrow C^*(G; B)$ is given by $\theta^t[b](\gamma) := \int_{U(1)} b(\gamma, u) du$ for $b \in C_c(G^t; B)$.

Lemma 7.8 $\{\lambda^{1/k}\}$ and $\{\theta^{1/k}\}$ determine homomorphisms $\prod' C^*(G; B_k) \rightarrow \prod' C^*(G^{1/k}; B_k)$ and $\prod' C^*(G^{1/k}; B_k) \rightarrow \prod' C^*(G; B_k)$.

Proof. Since the operator norm of $\lambda^{1/k}$ and $\theta^{1/k}$ are uniformly bounded in k by 1, $\{\lambda^{1/k}\}$ and $\{\theta^{1/k}\}$ determine homomorphisms $\prod' C^*(G; B_k) \rightarrow \prod' C^*(G^{1/k}; B_k)$ and $\prod' C^*(G^{1/k}; B_k) \rightarrow \prod' C^*(G; B_k)$. Moreover, if $\{b_k\} \in \prod' C_c(G; B_k)$ and the support of $b_k \in C_c(G; B_k)$ are contained in a compact subset $Z \subset G$ uniformly in k , the support of $\lambda^{1/k}[b_k]$ are contained in $Z \times U(1) \subset G \times U(1)$ and hence, $\{\lambda^{1/k}[b_k]\} \in \prod' C^*(G^{1/k}; B_k)$. If $B_k = C_0(X)$, the additional condition follows from $\text{supp}(\lambda^{1/k}[b_k](\gamma, u)) = \text{supp}(b_k(\gamma))$. Similarly, one can verify that $\{\theta^{1/k}\}$ maps $\prod' C^*(G^{1/k}; B_k)$ to $\prod' C^*(G; B_k)$. $\square$

7.3 Assembling $\mu_{G^{1/k}}([A])$

In order to gather the information of index of A twisted by the bundle $L_{1/k}$, we will make use of the specific expression in the section 5.3.

Recall that $\mu_{G^t}([L_t] \widehat{\otimes} [A]) \in KK(\mathbb{C}, C^*(G^t))$ is represented by the class $(\mathcal{E}_{L_t}^{G^t}, \overline{A_{L_t}}^{G^t})$, where $\mathcal{E}_{L_t}^{G^t}$ is given by the completion of $C_c(L_t \otimes \mathbb{V})$ and the right action of $C_c(G^t)$ and the $C_c(G^t)$ -valued scalar product are given by

$$s \cdot b = \int_{G^t} g[s] \cdot b(g^{-1}) dg \in C_c(L_t \otimes \mathbb{V}) \quad \text{and} \quad \langle s_1, s_2 \rangle_{\mathcal{E}^{G^t}}(g) = \langle s_1, g[s_2] \rangle_{L^2(L_t \otimes \mathbb{V})}$$

for $s, s_1, s_2 \in C_c(L_t \otimes \mathbb{V})$ and $b \in C_c(G^t)$. The operator is defined by the formula;

$$\overline{A_{L_t}}^{G^t} = \int_{G^t} g[\sqrt{c} A_{L_t} \sqrt{c}] dg.$$

Firstly let us consider the class in $KK(\mathbb{C}, \prod' C^*(G^{1/k}))$ gathering the informations of $\mu_{G^{1/n}}([A])$ consisting of a constant family before considering the information coming from $L_{1/k}$.

Definition 7.9 Consider the space

$$C_c\left(\prod \mathbb{V}\right) := \left\{ \{s_k\}_{k \in \mathbb{N}} \left| \begin{array}{l} s_k \in C_c(\mathbb{V}), \text{ there exists a compact subset } Y \subset X \\ \text{such that } \text{supp}(s_k) \subset Y \text{ for every } k. \\ \text{and } \|s_k\|_{C_0(\mathbb{V})} \text{ are uniformly bounded in } k. \end{array} \right. \right\}$$

In the similar way to the subsection 5.3, we define the structure of a pre-Hilbert module over $\prod' C_c(G^{1/k})$ as follows;

- The action of $\prod' C_c(G^{1/k})$ on $C_c(\prod \mathbb{V})$ from the right is given by

$$\begin{aligned} \{s_k\} \cdot \{b_k\} &:= \left\{ s_k \cdot_{G^{1/k}} b_k \right\} \\ &= \left\{ \int_{G^{1/k}} g[s_k] \cdot b_k(g^{-1}) dg \right\} \in C_c(\prod \mathbb{V}) \end{aligned}$$

for $\{s_k\} \in C_c(\prod \mathbb{V})$ and $\{b_k\} \in \prod' C_c(G)$. Note that $\text{supp}(s_k \cdot b_k) \subset \text{supp}(b_k) (\text{supp}(s_k))$ implies that $\{s_k \cdot b_k\}$ belongs to $C_c(\prod \mathbb{V})$.

- The $\prod' C^*(G^{1/k})$ -valued scalar product is given by

$$\begin{aligned} \langle \{s_k^1\}, \{s_k^2\} \rangle_{\prod' \mathcal{E}^{G^{1/k}}} &:= \left\{ \langle s_k^1, s_k^2 \rangle_{\mathcal{E}^{G^{1/k}}} \right\} \\ &= \left\{ g_k \mapsto \langle s_k^1, g_k[s_k^2] \rangle_{L^2(\mathbb{V})} \right\} \quad \text{for } \{s_k^1\}, \{s_k^2\} \in C_c(\prod \mathbb{V}). \end{aligned}$$

Note that since the action of G on X is proper, each function $g \mapsto \langle s_k^1, g[s_k^2] \rangle_{L^2(\mathbb{V})}$ has a compact support in $G \times U(1)$ depending only on the support of s_k^1 and s_k^2 and hence this sequence belongs to $\prod' C^*(G^{1/k})$.

Then, $\prod' \mathcal{E}^{G^{1/k}}$ is defined as the completion of this pre-Hilbert module.

Similarly, we define $\prod' \mathcal{E}^G$ as a Hilbert $\prod' C^*(G)$ -module.

Lemma 7.10

(1) $\prod' \mathcal{E}^G$ is considered as a $\prod' C^*(G^{1/k})$ -module with the right action of $\prod' C^*(G^{1/k})$ via $\{\theta^{1/k}\} : \prod' C^*(G^{1/k}) \rightarrow \prod' C^*(G)$ and the scalar product $\{\lambda^{1/k}\} \langle -, - \rangle$. Under this isomorphism, $\prod' \mathcal{E}^G \simeq \prod' \mathcal{E}^{G^{1/k}}$.

(2) $(\prod' \mathcal{E}^G) \widehat{\otimes}_{\{\lambda^{1/k}\}} \prod' C^*(G^{1/k}) \cong \prod' \mathcal{E}^{G^{1/k}}$.

Proof. By the same calculation as the proof of Lemma 6.6, we have that $\{s_k\} \cdot \{b_k\} = \{s_k\} \cdot \{\theta^{1/k} b_k\}$, for $\{s_k\} \in C_c(\prod \mathbb{V})$ and $\{b_k\} \in \prod' C^*(G^{1/k})$.

Besides, the image of the $\prod' C^*(G^{1/k})$ -valued scalar product of $C_c(\prod \mathbb{V})$ is constant in all the $U(1)$ -component. In the other words,

$$\{\lambda^{1/k}\} \langle -, - \rangle_{\prod' \mathcal{E}^G} = \langle -, - \rangle_{\prod' \mathcal{E}^{G^{1/k}}} \quad \text{and} \quad \{\theta^{1/k}\} \langle -, - \rangle_{\prod' \mathcal{E}^{G^{1/k}}} = \langle -, - \rangle_{\prod' \mathcal{E}^G}$$

hold on $C_c(\prod \mathbb{V})$. Therefore $\prod' \mathcal{E}^{G^{1/k}}$ and $\prod' \mathcal{E}^G$ are isomorphic by the identity map on $C_c(\prod \mathbb{V})$, both as Hilbert $\prod' C^*(G)$ -modules and as $\prod' C^*(G^{1/k})$ -modules.

Now define a map

$$\begin{aligned} \varphi : (\prod' \mathcal{E}^G) \widehat{\otimes}_{\{\lambda^{1/k}\}} \prod' C^*(G^{1/k}) &\rightarrow \prod' \mathcal{E}^{G^{1/k}} \\ \text{by } \varphi(\{s_k\} \widehat{\otimes} \{b_k\}) &:= \left\{ s_k \cdot_{G^{1/k}} b_k \right\}. \end{aligned}$$

$\{s_k\} \in \prod' \mathcal{E}^G$ is also regarded as an element in $\prod' \mathcal{E}^{G^{1/k}}$ using the identification $\prod' \mathcal{E}^G \cong \prod' \mathcal{E}^{G^{1/k}}$. Let $\varphi_k : \mathcal{E}^G \widehat{\otimes}_{\lambda^{1/k}} C^*(G^{1/k}) \rightarrow \mathcal{E}^{G^{1/k}}$ be the isomorphisms used in the proof of Proposition 6.11 replacing α by $\alpha^{1/k}$. Then φ is written as $\{\varphi_k\}$. Since each of φ_k preserves the scalar products, so does φ . In addition, let $\{u_k^m \in C_c(G^{1/k})\}_{m \in \mathcal{M}}$ be approximating unit for $C^*(G^{1/k})$ converging to the delta function at the identity element. We may assume that for each m , $\text{supp}(u_k^m)$ is contained

in some compact subset $Z \subset G \times U(1)$ uniformly in k . Then, $\{u_k^m\}_{k \in \mathbb{N}}$ belong to $\prod' C^*(G^{1/k})$ for each m . For $\{s_k\} \in C_c(\prod \mathbb{V})$, we have that

$$\lim_{\overrightarrow{m}} \varphi(\{s_k\} \widehat{\otimes} \{u_k^m\}) = \lim_{\overrightarrow{m}} \{\varphi_k(s_k \widehat{\otimes} u_k^m)\} = \{s_k\},$$

which implies that the image of φ is dense in $\prod' \mathcal{E}^{G^{1/k}}$ and hence it is an isomorphism. $\square$

Definition 7.11 Let the operator $\{\overline{A}^{G^{1/k}}\}$ acting diagonally on $\prod \mathcal{E}^{G^{1/k}}$ be given by the family

$$\overline{A}^{G^{1/k}} = \int_{G^{1/k}} g[\sqrt{c}A\sqrt{c}]dg$$

acting on the k -th component $\mathcal{E}^{G^{1/k}}$.

Similarly, we define $\{\overline{A}^G\}$ acting on $\prod \mathcal{E}^G$.

Lemma 7.12 $\{\overline{A}^{G^{1/k}}\}$ determines the element of $\mathbb{L}(\prod' \mathcal{E}^{G^{1/k}})$. Similarly, $\{\overline{A}^G\}$ determines the element of $\mathbb{L}(\prod' \mathcal{E}^G)$.

Proof. We will prove only for $\{\overline{A}^{G^{1/k}}\}$. Since the operator norm $\|\overline{A}^{G^{1/k}}\|_{\mathcal{E}^{G^{1/k}}}$ is uniformly bounded in k , they determines an element in $\mathbb{L}(\prod' \mathcal{E}^{G^{1/k}}; \prod \mathcal{E}^{G^{1/k}})$ with the operator norm given by $\left(\sup_k \|\overline{A}^{G^{1/k}}\|_{\mathcal{E}^{G^{1/k}}}\right)$. Let $s_k \in C_c(\mathbb{V})$ with $\text{supp}(s_k)$ contained in a compact subset $Y \subset X$ uniformly in k .

$$\overline{A}^{G^{1/k}}(s_k) = \int_{G^{1/k}} g[\sqrt{c}A\sqrt{c}(g^{-1}[s_k])]dg$$

Remark that $g \mapsto \sqrt{c}g^{-1}[s_k] \in C_c(\mathbb{V})$ has a compact support $Z \subset G \times U(1)$ depending on the $\text{supp}(c)$ and Y . Therefore, the support of $\overline{A}^{G^{1/k}}(s_k)$ is contained in $Z \cdot \text{supp}(c)$ and it depends only on $\text{supp}(c)$ and Y and independent of k . As a conclusion, $\{\overline{A}^{G^{1/k}}(s_k)\} \in \prod' \mathcal{E}^{G^{1/k}}$ and $\{\overline{A}^{G^{1/k}}\} \in \mathbb{L}(\prod' \mathcal{E}^{G^{1/k}})$ with the operator norm given by $\left(\sup_k \|\overline{A}^{G^{1/k}}\|_{\mathcal{E}^{G^{1/k}}}\right)$. $\square$

Lemma 7.13 $\{\overline{A}^G\}^2 - 1 \in \mathbb{K}(\prod' \mathcal{E}^G)$ and $\{\overline{A}^{G^{1/k}}\}^2 - 1 \in \mathbb{K}(\prod' \mathcal{E}^{G^{1/k}})$.

Proof. Let $T := (\overline{A}^G)^2 - 1 \in \mathbb{L}(\mathcal{E}^G)$. From the fact that $(\mathcal{E}^G, \overline{A}^G)$ is an element of $\mathbb{E}(\mathbb{C}, C^*(G))$ which represents $\mu_G([A])$, for any $\varepsilon > 0$, there exists a finite rank operator $F^\varepsilon = \sum_{m:\text{finite}} s^m \langle s^m, - \rangle$ such that $\|T - F^\varepsilon\| < \varepsilon$, where $s^m \in \mathcal{E}^G$. Then, take an operator of constant family $\{F^\varepsilon\} = \sum_{m:\text{finite}} \{s^m\} \langle \{s^m\}, - \rangle$. Then this is an operator of finite rank and $\|\{T\} - \{F^\varepsilon\}\| < \varepsilon$, which implies $\{T\} = \{\overline{A}^G\}^2 - 1$ belongs to $\mathbb{L}(\prod' \mathcal{E}^G)$.

Besides, note that the $U(1) \subset G^{1/k}$ acts on the operator $\sqrt{c}A\sqrt{c}$ trivially, in the other words, $(\gamma, u)[\sqrt{c}A\sqrt{c}] = \gamma[\sqrt{c}A\sqrt{c}]$ for any $(\gamma, u) \in G^{1/k}$. Therefore it follows that $\{\overline{A}^{G^{1/k}}\}$ is, in fact, a

constant family $\{\int_G \gamma[\sqrt{c}A\sqrt{c}]d\gamma\}$, which is the same operatr as $\{\overline{A}^G\}$ under the identification $\prod' \mathcal{E}^{G^{1/k}} \simeq \prod' \mathcal{E}^G$. Then $\left\{\overline{A}^{G^{1/k}}\right\}^2 - 1$ is also a compact operator. $\square$

Definition 7.14

$$[[A]] := \left(\prod' \mathcal{E}^{G^{1/k}}, \left\{ \overline{A}^{G^{1/k}} \right\} \right) \in \mathbb{E} \left(\mathbb{C}, \prod' C^*(G^{1/k}) \right).$$

Similarly we also define

$$[[A]]_c := \left(\prod' \mathcal{E}^G, \left\{ \overline{A}^G \right\} \right) \in \mathbb{E} \left(\mathbb{C}, \prod' C^*(G) \right).$$

By the previous lemma, they actually determine elements in each Kasparov modules since the other conditions are obviously satisfied.

Lemma 7.15 $[[A]] = \{\lambda^{1/k}\}_* [[A]]_c \in KK(\mathbb{C}, \prod' C^*(G^{1/k}))$.

Proof. $\{\lambda^{1/k}\}_* [[A]]_c$ is represented by

$$\left(\left(\prod' \mathcal{E}^G \right) \widehat{\otimes}_{\{\lambda^{1/k}\}} \prod' C^*(G^{1/k}), \left\{ \overline{A}^G \right\} \widehat{\otimes} 1 \right).$$

By Lemma 7.10, $\left(\prod' \mathcal{E}^G \right) \widehat{\otimes}_{\{\lambda^{1/k}\}} \prod' C^*(G^{1/k}) \simeq \prod' \mathcal{E}^{G^{1/k}}$. In addition, under this isomorphism, $\left\{ \overline{A}^G \right\} \widehat{\otimes} 1 = \left\{ \int_G \gamma[\sqrt{c}A\sqrt{c}]d\gamma \right\} = \left\{ \overline{A}^{G^{1/k}} \right\}$. To conclude, $\{\lambda^{1/k}\}_* [[A]]_c$ is also represented by $\left(\prod' \mathcal{E}^{G^{1/k}}, \left\{ \overline{A}^{G^{1/k}} \right\} \right) = [[A]]$. $\square$

7.4 Assembling $\mu_{G^{1/k}}([L_{1/k}] \widehat{\otimes} [A])$

Similarly, we define a pair

$$\left[[A_{\prod L_{1/k}}] \right] := \left(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}}, \left\{ \overline{A_{\prod L_{1/k}}}^{G^{1/k}} \right\} \right).$$

$\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}}$ is obtained as the completion of

$$C_c \left(\prod L_{1/k} \otimes \mathbb{V} \right) := \left\{ \{s_k\}_{k \in \mathbb{N}} \left| \begin{array}{l} s_k \in C_c(L_{1/k} \otimes \mathbb{V}), \\ \text{there exists a compact subset } Y \subset X \\ \text{such that } \text{supp}(s_k) \subset Y \text{ for every } k \\ \text{and } \|s_k\|_{C_0(L_{1/k} \otimes \mathbb{V})} \text{ are uniformly bounded in } k. \end{array} \right. \right\},$$

We have that $\left\{ \overline{A_{\prod L_{1/k}}}^{G^{1/k}} \right\} \in \mathbb{L} \left(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}} \right)$ similarly to Lemma 7.12.

Note that $\left[[A_{\prod L_{1/k}}] \right]$ may not be contained in $\mathbb{E}(\mathbb{C}, \prod' C^*(G^{1/k}))$ for general family of vector bundles since the sequence of indices $\mu_{G^{1/k}}([L_{1/k}] \widehat{\otimes} [A])$ may not be bounded. However it follows from the almost flat-ness of $L_{1/k}$ that $\left[[A_{\prod L_{1/k}}] \right] \in \mathbb{E}(\mathbb{C}, \prod' C^*(G^{1/k}))$, as proved later.

7.5 The Limit of the Almost Flat Bundles

Remark 7.16 $\bigoplus \mathcal{E}_{L_{1/k}}^{G^{1/k}}$ is a submodule of $\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}}$ because it is given by the completion of the space consisting of the bounded sequences $\{s_k\} \in \prod C_c(L_{1/k} \widehat{\otimes} \mathbb{V})$ with $s_k = 0$ for all but finitely many k , which is contained in $\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}}$.

Definition 7.17 Let the quotient $\left(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}}\right) / \left(\bigoplus \mathcal{E}_{L_{1/k}}^{G^{1/k}}\right)$ be denoted by $\mathcal{Q}' \mathcal{E}_{L_{1/k}}^{G^{1/k}}$.

It is a Hilbert $\mathcal{Q}' C^*(G^{1/k})$ -module and naturally isomorphic to $\left(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}}\right) \widehat{\otimes}_\pi \mathcal{Q}' C^*(G^{1/k})$ where $\pi: \prod' C^*(G^{1/k}) \rightarrow \mathcal{Q}' C^*(G^{1/k})$ denotes the projection. $\left(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}}\right) \widehat{\otimes}_\pi \mathcal{Q}' C^*(G^{1/k}) \rightarrow \left(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}}\right) / \left(\bigoplus \mathcal{E}_{L_{1/k}}^{G^{1/k}}\right)$ is given by $\{s_k\} \widehat{\otimes} \{b_k\} \mapsto \left\{s_k \cdot_{G^{1/k}} b_k\right\}$. In addition, since the operators $\overline{A_{L_{1/k}}}^{G^{1/k}}$ have uniformly bounded norm in k , they determines an operator $\left\{\overline{A_{L_{1/k}}}^{G^{1/k}}\right\}$ acting on $\bigoplus \mathcal{E}_{L_{1/k}}^{G^{1/k}}$. Hence, it also determines the operator in $\mathbb{L}\left(\mathcal{Q}' \mathcal{E}_{L_{1/k}}^{G^{1/k}}\right)$.

Definition 7.18

$$\left[[A_{\mathcal{Q} L_{1/k}}]\right] := \left(\left(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}}\right) / \left(\bigoplus \mathcal{E}_{L_{1/k}}^{G^{1/k}}\right), \left\{\overline{A_{L_{1/k}}}^{G^{1/k}}\right\}\right).$$

In order to obtain the information we introduce the module given by the fiber of $L_{1/k}$ at the base point.

Definition 7.19 Let $x_0 \in X$ be the base point. Let W_t denote the fiber $L_t|_{x_0}$ on x_0 .

- For $w \in W_t$, define Φw as the bounded continuous section of L_t given by

$$\Phi w(x) := (x, w) \quad \text{for } x \in X$$

under the identification $L_t = (X \times \mathbb{C}, d + it\eta)$. In the other words, Φw is given by the parallel transport along the path from x_0 to x with respect to the flat connection d on $X \times \mathbb{C}$, that is, $\Phi w(x) = \Phi_{x_0; x}^d(w)$.

- Define the action ρ of G^t on W_t by the formula;

$$\rho[(\gamma, u)](w) := \exp[-it\psi_\gamma(\gamma x_0)]uw \quad \text{for } (\gamma, u) \in G^t, w \in W_t.$$

In other words, this action is given by the composition of the action on L_t and the parallel transport along the path from γx_0 to x_0 with respect to the flat connection d , that is, $\rho[(\gamma, u)]w = \Phi_{\gamma x_0; x_0}^d((\gamma, u)(x_0, w))$.

Since the flat connection d does not commute with the action of G^t , the map $w \mapsto \Phi w$ is not G^t -equivariant in general.

Lemma 7.20 *For any compact subset $Z \subset G^t$ and compact subset $Y \subset X$, there exists a constant M depending on Z and Y satisfying that*

$$\|g[\Phi w](x) - \Phi(\rho[g](w))(x)\| \leq t \cdot M \|w\|$$

for any $g \in Z$ and $x \in Y$.

Proof. Let $g = (\gamma, u)$ and let W_t and L_t be identified with $\mathbb{C}$ and $X \times \mathbb{C}$ respectively. Recall that the action of G^t on $C_0(L_t)$ is given by $(\gamma, u)[s](x) = \exp[-it\psi_\gamma(x)] u \cdot \gamma[s](\gamma^{-1}x)$.

$$\begin{aligned} \|g[\Phi w](x) - \Phi(\rho[g](w))(x)\| &\leq \|\exp[-it\psi_\gamma(\gamma^{-1}x)] u \cdot w - \exp[-it\psi_\gamma(\gamma x_0)] u \cdot w\| \\ &\leq (\|\exp[-it\psi_\gamma(\gamma^{-1}x)] u - 1\| + \|\exp[-it\psi_\gamma(\gamma x_0)] u - 1\|) \|w\| \\ &\leq t \left(e^{|\psi_\gamma(\gamma^{-1}x)|} + e^{|\psi_\gamma(\gamma x_0)|} \right) \|w\| \quad (t \in [0, 1]) \\ &\leq t \cdot M \|w\|, \end{aligned}$$

where M is the maximum of the continuous function $((\gamma, u), x) \mapsto e^{|\psi_\gamma(\gamma^{-1}x)|} + e^{|\psi_\gamma(\gamma x_0)|}$ on the compact subset $Z \times Y$. $\square$

Definition 7.21

$$[[W]] := \left(\mathcal{Q}' C^*(G^{1/k}; W_{1/k}), 0 \right) \in \mathbb{E} \left(\prod' C^*(G^{1/k}), \mathcal{Q}' C^*(G^{1/k}) \right).$$

$\prod' C^*(G^{1/k})$ acts from the left on $\mathcal{Q}' C^*(G^{1/k}; W_{1/k})$ as

$$\{b_k\} * \{w_k\} := \left\{ g_k \mapsto \int_{G^{1/k}} b_k(h) \cdot \rho[h]w_k(h^{-1}g_k) dh \right\}.$$

Remark 7.22 For $\{b_k\} \in \prod' C^*(G^{1/k})$, take $\{1b_k\} \in \mathcal{Q}' C^*(G^{1/k}; W_{1/k})$, here $1 \in \mathbb{C} \simeq W_{1/k}$. Then we have that $\{b_k\} * (1 - 0^2) = \{1b_k\} \cdot \langle \{1b_k\}, - \rangle$ and it is an operator of rank 1, which implies that $[[W]]$ is actually an element of $\mathbb{E}(\prod' C^*(G^{1/k}), \mathcal{Q}' C^*(G^{1/k}))$ since the other conditions are obviously satisfied.

Proposition 7.23

$$\left[\left[A_{\mathcal{Q} L_{1/k}} \right] \right] = [[A]]_{\prod' C^*(G^{1/k})} \hat{\otimes} [[W]].$$

as pairs of $\mathcal{Q}' C^*(G^{1/k})$ -modules and operators. More precisely, the pair $\left[\left[A_{\mathcal{Q} L_{1/k}} \right] \right]$ on the left hand side represents the Kasparov product given by the right hand side.

Proof. $[[A]]_{\prod' C^*(G^{1/k})} \hat{\otimes} [[W]]$ is represented by

$$\left(\prod' \mathcal{E}^{G^{1/k}}_{\prod' C^*(G^{1/k})} \hat{\otimes} \mathcal{Q}' C^*(G^{1/k}; W_{1/k}), \left\{ \overline{A}^{G^{1/k}} \right\} \hat{\otimes} 1 \right).$$

Consider the following map;

$$\begin{aligned} \varphi: \quad \left(C_c \left(\prod \mathbb{V} \right) \right) \hat{\otimes} \left(\prod' C_c(G^{1/k}; W_{1/k}) \right) &\rightarrow \left(\prod' \mathcal{E}^{G^{1/k}}_{L_{1/k}} \right) / \left(\bigoplus \mathcal{E}^{G^{1/k}}_{L_{1/k}} \right) \\ \{s_k\} \hat{\otimes} \{w_k\} &\mapsto \left\{ \int_{G^{1/k}} g [\Phi(w_k(g^{-1}))] \otimes g[s_k] dg \right\}. \end{aligned}$$

Even though $g [\Phi(w_k(g^{-1}))]$ is contained in $C_{\text{bdd}}(L_{1/k})$ but not in $C_c(L_{1/k})$, the element $g[s_k] \otimes g [\Phi(w_k(g^{-1}))]$ will be considered as an element of $C_c(L_{1/k} \hat{\otimes})$ in the sense of the section

$$x \mapsto g [\Phi(w_k(g^{-1}))] (x) \otimes g[s_k](x) \in (L_{1/k} \otimes \mathbb{V})_x.$$

First of all, we must check the well defined-ness. Let $\{b_k\} \in \prod' C^*(G^{1/k})$.

$$\begin{aligned} & \varphi((\{s_k\} \cdot \{b_k\}) \widehat{\otimes} \{w_k\}) \\ &= \left\{ \int_{G^{1/k}} \int_{G^{1/k}} g [\Phi(w_k(g^{-1}))] \otimes g [g_0[s_k] \cdot b_k(g_0^{-1})] dg_0 dg \right\} \\ &= \left\{ \int_{G^{1/k}} \int_{G^{1/k}} g [\Phi(w_k(g^{-1}))] \otimes gg_0[s_k] \cdot b_k(g_0^{-1}) dg_0 dg \right\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \varphi(\{s_k\} \widehat{\otimes} (\{b_k\} * \{w_k\})) \\ &= \left\{ \int_{G^{1/k}} \int_{G^{1/k}} g [\Phi(b_k(g_1) \cdot \rho[g_1]w_k(g_1^{-1}g^{-1}))] \otimes g[s_k] dg_1 dg \right\} \\ &= \left\{ \int_{G^{1/k}} \int_{G^{1/k}} hh_0 [\Phi(b_k(h_0^{-1}) \cdot \rho[h_0^{-1}]w_k(h_0h_0^{-1}h^{-1}))] \otimes hh_0[s_k] dh_0 dh \right\} \\ &= \left\{ \int_{G^{1/k}} \int_{G^{1/k}} hh_0 [\Phi(\rho[h_0^{-1}]w_k(h^{-1}))] \otimes hh_0[s_k] \cdot b_k(h_0^{-1}) dh_0 dh \right\}. \end{aligned}$$

Let $Z \subset G \times U(1)$ and $Y \subset X$ be compact subsets such that $\text{supp}(b_k), \text{supp}(w_k) \subset Z$ for all k and $\text{supp}(s_k) \subset Y$ for all k and $g \in G \times U(1)$. We may assume that $Z^{-1} = Z$. Due to Lemma 7.20,

$$\begin{aligned} & \|\varphi((\{s_k\} \cdot \{b_k\}) \widehat{\otimes} \{w_k\}) - \varphi(\{s_k\} \widehat{\otimes} (\{b_k\} * \{w_k\}))\| \\ &\leq \limsup_{k \rightarrow \infty} \left(\text{vol}(Z \cdot Z)^2 \sup_{g, g_0 \in Z} \sup_{x \in Z \cdot Z \cdot Y} \|\Phi(w_k(g^{-1}))(x) - g_0 [\Phi(\rho[g_0^{-1}]w_k(g^{-1}))](x)\| \right) \\ &\quad \cdot \sup_{k, g_0} \left\{ \|s_k\|_{C_0(\mathbb{V})} |b_k(g_0^{-1})| \right\} \\ &\leq C_{\{s_k\}, \{b_k\}} \cdot \limsup_{k \rightarrow \infty} \left(\sup_{g \in Z} \frac{1}{k} \|w_k(g^{-1})\| \right) \\ &\leq C_{\{s_k\}, \{b_k\}} \|\{w_k\}\| \cdot \limsup_{k \rightarrow \infty} \frac{1}{k} = 0, \end{aligned}$$

where $C_{\{s_k\}, \{b_k\}}$ is a constant depending on $\{s_k\}$ and $\{b_k\}$ but not on k . Besides, the image of $(C_c(\prod \mathbb{V})) \widehat{\otimes} (\bigoplus C_c(G^{1/k}; W_{1/k}))$ by φ is obviously contained in $\bigoplus \mathcal{E}_{L_{1/k}}^{G^{1/k}}$ and hence 0, this map induces a map from $(C_c(\prod \mathbb{V})) \widehat{\otimes} (\mathcal{Q}' C_c(G^{1/k}; W_{1/k}))$.

Compatibility of φ with the right action of $\mathcal{Q}' C^*(G^{1/k})$ can be verified similarly.

Next we will check that φ is isometric. Let $\{s_k^1\}, \{s_k^2\} \in C_c(\prod \mathbb{V})$ and $\{w_k^1\}, \{w_k^2\} \in \mathcal{Q}' C_c(G^{1/k}; W_{1/k})$

$$\begin{aligned}
& \langle \varphi(\{s_k^1\} \widehat{\otimes} \{w_k^1\}), \varphi(\{s_k^2\} \widehat{\otimes} \{w_k^2\}) \rangle_{\mathcal{Q}' \mathcal{E}_{L_{1/k}}^{G^{1/k}}(\{g_k\})} \\
&= \left\langle \left\langle \int_{G^{1/k}} g [\Phi(w_k^1(g^{-1}))] \otimes g[s_k^1] dg, g_k \left[\int_{G^{1/k}} h [\Phi(w_k^2(g^{-1}))] \otimes h[s_k^2] dh \right] \right\rangle_{L^2(L_{1/k} \otimes \mathbb{V})} \right\rangle \\
&= \left\{ \int_{G^{1/k}} \int_{G^{1/k}} \int_X \langle \Phi(w_k^1(g^{-1}))(x), g^{-1} g_k h [\Phi(w_k^2(h^{-1}))](x) \rangle_{(L_{1/k})_x} \right. \\
&\quad \left. \cdot \langle s_k^1(x), g^{-1} g_k h[s_k^2](x) \rangle_{\mathbb{V}_x} d\text{vol}(x) dg dh \right\} \\
&\stackrel{(\dagger)}{=} \left\{ \int_{G^{1/k}} \int_{G^{1/k}} \int_X \langle \Phi(w_k^1(g^{-1}))(x), \Phi(\rho[g^{-1} g_k h] w_k^2(h^{-1}))(x) \rangle_{(L_{1/k})_x} \right. \\
&\quad \left. \cdot \langle s_k^1(x), g^{-1} g_k h[s_k^2](x) \rangle_{\mathbb{V}_x} d\text{vol}(x) dg dh \right\} \\
&\stackrel{(\dagger\dagger)}{=} \left\{ \int_{G^{1/k}} \int_{G^{1/k}} \langle w_k^1(g^{-1}), \rho[g^{-1} g_k h] w_k^2(h^{-1}) \rangle_{W_{1/k}} \cdot \langle s_k^1, g^{-1} g_k h[s_k^2] \rangle_{L^2(\mathbb{V})} dg dh \right\}.
\end{aligned}$$

The equation $(\dagger)$ follows from that the supports of s_k^1 and s_k^2 are contained in some compact subspace uniformly and there exists a compact subset $Z \subset G \times U(1)$ such that $\langle s_k^1(x), g^{-1} g_k h[s_k^2](x) \rangle_{\mathbb{V}_x} = 0$ for $g^{-1} g_k h$ outside of Z . Then one can apply Lemma 7.20 and obtain that the difference is contained in $\bigoplus C^*(G^{1/k})$. The equation $(\dagger\dagger)$ follows from that Φ preserves the hermitian metric of each fiber by the definition. On the other hand,

$$\begin{aligned}
& \langle \{s_k^1\} \widehat{\otimes} \{w_k^1\}, \{s_k^2\} \widehat{\otimes} \{w_k^2\} \rangle(\{g_k\}) \\
&= \left\langle \{w_k^1\}, \left(\langle \{s_k^1\}, \{s_k^2\} \rangle_{\prod \mathcal{E}^{G^{1/k}}} * \{w_k^2\} \right) \right\rangle_{\mathcal{Q}' C^*(G^{1/k}; W_{1/k})}(\{g_k\}) \\
&= \left\langle \{w_k^1\}, \left\{ \int_{G^{1/k}} \langle s_k^1, h_0[s_k^2] \rangle_{L^2(\mathbb{V})} \rho[h_0] w_k^2(h_0^{-1} g_k) dh_0 \right\} \right\rangle \\
&= \left\langle \{w_k^1\}, \left\{ \int_{G^{1/k}} \langle s_k^1, g_k h[s_k^2] \rangle_{L^2(\mathbb{V})} \rho[g_k h] w_k^2(h^{-1}) dh \right\} \right\rangle \\
&= \left\{ \int_{G^{1/k}} \int_{G^{1/k}} \langle w_k^1(g^{-1}), \right. \\
&\quad \left. \langle s_k^1, g^{-1} g_k h[s_k^2] \rangle_{L^2(\mathbb{V})} \rho[g^{-1} g_k h] w_k^2(h^{-1}) \rangle_{W_{1/k}} dg dh \right\} \\
&= \left\{ \int_{G^{1/k}} \int_{G^{1/k}} \langle w_k^1(g^{-1}), \rho[g^{-1} g_k h] w_k^2(h^{-1}) \rangle_{W_{1/k}} \langle s_k^1, g^{-1} g_k h[s_k^2] \rangle_{L^2(\mathbb{V})} dg dh \right\}.
\end{aligned}$$

As a conclusion, φ preserves the scalar product and hence it is extended to the map from $\prod' \mathcal{E}^{G^{1/k}} \widehat{\otimes} \mathcal{Q}' C^*(G^{1/k}; W_{1/k})$ and it is injective.

As for the surjectivity, consider an approximating unit $\{u_k^m \in C_c(G^{1/k})\}_{m \in \mathcal{M}}$ for $C^*(G^{1/k})$, converging to the delta functions at the identity element, and set $w_k^m(g) := 1 u_k^m(g)$, here $1 \in \mathbb{C} \cong W_{1/k}$. Since for each m , we may assume that u_k^m is supported in some compact subset in $G \times U(1)$ uniformly in k , they determine the elements $\{w_k^m\} \in \mathcal{Q}' C^*(G^{1/k}; W_{1/k})$ for any $m \in \mathcal{M}$. Then

for $\{s_k\} \in C_c(\prod' \mathbb{V})$,

$$\begin{aligned}
\lim_{\overrightarrow{m}} \varphi(\{s_k\} \widehat{\otimes} \{w_k^m\}) &= \lim_{\overrightarrow{m}} \left\{ \int_{G^{1/k}} g [\Phi(w_k^m(g^{-1}))] \otimes g[s_k] dg \right\} \\
&= \lim_{\overrightarrow{m}} \left\{ \int_{G^{1/k}} g [\Phi(1)] u_k^m(g^{-1}) \otimes g[s_k] dg \right\} \\
&= \lim_{\overrightarrow{m}} \left\{ \int_{G^{1/k}} g [\Phi(1) \otimes s_k] u_k^m(g^{-1}) dg \right\} \\
&= \lim_{\overrightarrow{m}} \left\{ (\Phi(1) \otimes s_k) \cdot_{G^{1/k}} u_k^m \right\} \\
&= \{\Phi(1) \otimes s_k\}.
\end{aligned}$$

Since the sections of the form $\{\Phi(1) \otimes s_k\}$ for $\{s_k\} \in C_c(\prod' \mathbb{V})$ generate the space $C_c(\prod L_{1/k} \otimes \mathbb{V})$, it follows that the image of φ is dense and hence φ is an isomorphism.

Finally we will check the compatibility with the operators. Let $L_{1/k}$ be identified with $X \times \mathbb{C}$. Then $\varphi \circ \left(\left\{ \overline{A}^{G^{1/k}} \right\} \widehat{\otimes} 1 \right) = \left\{ \overline{A_{L_0}}^{G^{1/k}} \right\} \circ \varphi$. Let $\{r_k\} \in C_c^\infty(\prod L_{1/k} \otimes \mathbb{V})$ with $\text{supp}(r_k)$ contained in a compact subset $Y \subset X$ uniformly in k and let $c \in C_c^\infty(X)$ be a cut-off function. We may assume that Y contains the support of c . There exists a compact subset $Z = Z^{-1} \subset G \times U(1)$ such that $\sqrt{c} \cdot g^{-1}[r_k] = 0$ for g outside of Z .

$$\begin{aligned}
&\left\| \overline{A_{L_{1/k}}}^{G^{1/k}}(r_k) - \overline{A_{L_0}}^{G^{1/k}}(r_k) \right\| \\
&= \left\| \int_{G^{1/k}} g \left[\sqrt{c} (A_{L_{1/k}} - A_{L_0}) \sqrt{c} (g^{-1}[r_k]) \right] dg \right\| \\
&\leq \text{vol}(Z) \text{vol}(Z \cdot Y) \cdot \sup_{g \in Z} \sup_{x \in Z \cdot Y} \left\| (A_{L_{1/k}} - A_{L_0}) \sqrt{c} (g^{-1}[r_k])(x) \right\| \\
&\leq \text{vol}(Z) \text{vol}(Z \cdot Y) \cdot \sup_{g \in Z} \sup_{x \in Z \cdot Y} \left\| \sqrt{c} (g^{-1}[r_k])(x) \right\| \left\| \frac{1}{k} \eta(x) \right\| \rightarrow 0 \quad (k \rightarrow \infty).
\end{aligned}$$

This implies that $\left\{ \overline{A_{L_{1/k}}}^{G^{1/k}} \right\} \{r_k\} = \left\{ \overline{A_{L_0}}^{G^{1/k}} \right\} \{r_k\}$ as elements of $\mathcal{Q}' \mathcal{E}_{L_{1/k}}^{G^{1/k}}$ for any $\{r_k\} \in \mathcal{Q}' \mathcal{E}_{L_{1/k}}^{G^{1/k}}$. Therefore $\varphi \circ \left(\left\{ \overline{A}^{G^{1/k}} \right\} \widehat{\otimes} 1 \right) = \left\{ \overline{A_{L_{1/k}}}^{G^{1/k}} \right\} \circ \varphi$. $\square$

Corollary 7.24

$$\left[[A_{\mathcal{Q} L_{1/k}}] \right] \in \mathbb{E}(\mathbb{C}, \mathcal{Q}' C^*(G^{1/k})).$$

Proposition 7.25

$$\left[[A_{\prod L_{1/k}}] \right] \in \mathbb{E}(\mathbb{C}, \prod' C^*(G^{1/k})).$$

Proof. Put $T := \left\{ \overline{A_{L_{1/k}}}^{G^{1/k}} \right\} \in \mathbb{L}(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}})$. What is non-trivial is that $T^2 - 1 \in \mathbb{K}(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}})$. Since each of $\left(\mathcal{E}_{L_{1/k}}^{G^{1/k}}, \overline{A_{L_{1/k}}}^{G^{1/k}} \right)$ is in $\mathbb{E}(\mathbb{C}, C^*(G^{1/k}))$, for any ε , there exists a sequence of operators of finite rank $F_k^\varepsilon \in \mathbb{K}(\mathcal{E}_{L_{1/k}}^{G^{1/k}})$ such that $\left\| \left(\overline{A_{L_{1/k}}}^{G^{1/k}} \right)^2 - 1 - F_k^\varepsilon \right\| < \varepsilon$. Note that $\{F_k^\varepsilon\} \in \mathbb{L}(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}})$ may not have finite rank since the rank of F_k^ε may not be uniformly

bounded in k . By Corollary 7.24, there exists a finite rank operator $F^{\mathcal{Q}} \in \mathbb{K}(\mathcal{Q}' \mathcal{E}_{L_{1/k}}^{G^{1/k}})$ such that $\|T^2 - 1 - F\|_{\mathbb{L}(\mathcal{Q}' \mathcal{E}_{L_{1/k}}^{G^{1/k}})} < \varepsilon$. Take some lift $\tilde{F}$ of $F^{\mathcal{Q}}$ of finite rank. Let $P_n \in \mathbb{L}(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}})$ be the projection onto $\prod_{k \geq n}' \mathcal{E}_{L_{1/k}}^{G^{1/k}}$. Then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$,

$$\|P_n (T^2 - 1 - \tilde{F}) P_n\|_{\mathbb{L}(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}})} < 2\varepsilon.$$

Then put $F := \{F_k^\varepsilon\}_{k < n_0} + P_{n_0} \tilde{F} P_{n_0}$. Then its rank does not exceed $\max_{k < n_0} \{\text{rk}(F_k^\varepsilon)\} + \text{rk}(\tilde{F})$. Moreover

$$\begin{aligned} \|T^2 - 1 - F\| &\leq \|(P_{n_0} - 1)(T^2 - 1 - \tilde{F})(P_{n_0} - 1)\| + \|P_{n_0}(T^2 - 1 - \tilde{F})P_{n_0}\| \\ &\leq \max_{k < n_0} \|\overline{A_{L_{1/k}}^{G^{1/k}}} - F_k^\varepsilon\| + \|P_{n_0}(T^2 - 1 - \tilde{F})P_{n_0}\| < 3\varepsilon, \end{aligned}$$

which implies that $T^2 - 1$ is a compact operator. $\square$

7.6 Calculation of the Index

Lemma 7.26

- (1) Let $n \in \mathbb{N}$ and $p_n: \prod' C^*(G^{1/k}) \rightarrow C^*(G^{1/n})$ be the projection. Then, $(p_n)_* \left[[A_{\prod L_{1/k}}] \right] = \mu_{G^{1/n}}([L_{1/n}] \hat{\otimes} [A]) \in KK(\mathbb{C}, C^*(G^{1/n}))$.
- (2) Let $\pi: \prod' C^*(G^{1/k}) \rightarrow \mathcal{Q}' C^*(G^{1/k})$ be the projection. Then, $\pi_* \left[[A_{\prod L_{1/k}}] \right] = \left[[A_{\mathcal{Q} L_{1/k}}] \right] \in KK(\mathbb{C}, \mathcal{Q}' C^*(G^{1/k}))$.

They follow from the constructions. $(p_n)_* \left[[A_{\prod L_{1/k}}] \right]$ is represented by

$$\left(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}} \hat{\otimes}_{p_n} C^*(G^{1/n}), \left\{ \overline{A_{L_{1/k}}^{G^{1/k}}} \right\} \hat{\otimes} \text{id} \right) \simeq \left(\mathcal{E}_{L_{1/n}}^{G^{1/n}}, \overline{A_{L_{1/n}}^{G^{1/n}}} \right),$$

which represents $\mu_{G^{1/n}}([L_{1/n}] \hat{\otimes} [A])$. Next, $\pi_* \left[[A_{\prod L_{1/k}}] \right]$ is represented by

$$\begin{aligned} &\left(\left(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}} \right) \hat{\otimes}_\pi (\mathcal{Q}' C^*(G^{1/k})), \left\{ \overline{A_{L_{1/k}}^{G^{1/k}}} \right\} \hat{\otimes} \text{id} \right) \\ &\simeq \left(\left(\prod' \mathcal{E}_{L_{1/k}}^{G^{1/k}} \right) / \left(\bigoplus \mathcal{E}_{L_{1/k}}^{G^{1/k}} \right), \left\{ \overline{A_{L_{1/k}}^{G^{1/k}}} \right\} \right), \end{aligned}$$

which represents $\left[[A_{\mathcal{Q} L_{1/k}}] \right]$.

Lemma 7.27 Assume that $\mu_G([A]) = 0 \in KK(\mathbb{C}, C^*(G))$. Then, $\left[[A_{\mathcal{Q} L_{1/k}}] \right] = 0 \in KK(\mathbb{C}, \mathcal{Q}' C^*(G^{1/k}))$.

Proof. Due to Proposition 7.23 and $[A] = \{\lambda^{1/k}\}([A]_c)$, it is sufficient to prove that $[A]_c = 0 \in KK(\mathbb{C}, \prod' C^*(G))$.

Since we assume that $\mu_G([A]) = (\mathcal{E}^G, \overline{A}^G) = 0 \in KK(\mathbb{C}, C^*(G))$, by the definition, it is homotopic to $(\{0\}, 0) \in \mathbb{E}(\mathbb{C}, C^*(G))$. Precisely, there exists a Kasparov module $(\mathcal{E}, T) \in \mathbb{E}(\mathbb{C}, IC^*(G))$ satisfying

$$\mathcal{E} \hat{\otimes}_{e_1} C^*(G) \cong \mathcal{E}^G, \quad T \hat{\otimes}_{e_1} \text{id} = \overline{A}^G \quad \text{and} \quad \mathcal{E} \hat{\otimes}_{e_0} C^*(G) = \{0\},$$

where $IC^*(G) = C([0, 1]; C^*(G))$ and $e_t: IC^*(G) \rightarrow C^*(G)$ denotes the evaluation map at $t \in [0, 1]$.

Let $\mathring{\mathcal{E}}$ be a dense submodule in $\mathcal{E}$ given by

$$\mathring{\mathcal{E}} := \left\{ \xi \in \mathcal{E} \mid \begin{array}{l} \xi \widehat{\otimes}_{e_1} b \in C_c(\mathbb{V}) \text{ for any } b \in C_c(G), \\ \text{under the isomorphism } \mathcal{E} \widehat{\otimes}_{e_1} C^*(G) \cong \mathcal{E}^G. \end{array} \right\}.$$

Consider

$$\prod' \mathring{\mathcal{E}} := \left\{ \{\xi_k\} \in \prod' \mathring{\mathcal{E}} \mid \begin{array}{l} \{\xi_k \widehat{\otimes}_{e_1} b_k\} \in C_c(\prod' \mathbb{V}) \text{ for any } \{b_k\} \in \prod' C_c(G), \\ \{\langle \xi_k, \xi_k \rangle_{\mathcal{E}}(t)\} \in \prod' C^*(G) \text{ for any } t \in [0, 1] \text{ and} \\ t \mapsto \{\langle \xi_k, \xi_k \rangle_{\mathcal{E}}(t)\} \text{ is continuous.} \end{array} \right\}.$$

Remark that for $\{\xi_k\}, \{\nu_k\} \in \prod' \mathring{\mathcal{E}}$, then $\{\langle \xi_k, \nu_k \rangle(t)\} \in \prod' C^*(G)$ and $t \mapsto \{\langle \xi_k, \nu_k \rangle(t)\}$ is continuous since

$$\langle \xi, \nu \rangle = \frac{1}{4} (\langle \xi + \nu, \xi + \nu \rangle - \langle \xi - \nu, \xi - \nu \rangle + i \langle \xi + i\nu, \xi + i\nu \rangle - i \langle \xi - i\nu, \xi - i\nu \rangle).$$

Define the right action of $C([0, 1]; \prod' C_c(G))$ and the scalar product on $\prod' \mathring{\mathcal{E}}$ by

$$\begin{aligned} \{\xi_k\} \cdot \{f_k\} &:= \{\xi_k \cdot f_k\} \quad \text{for } \{f_k\} \in C([0, 1]; \prod' C_c(G)) \\ \langle \{\xi_k\}, \{\nu_k\} \rangle &:= \langle \{\xi_k, \nu_k\} \rangle \in C([0, 1]; \prod' C^*(G)). \end{aligned}$$

Let $\{\xi_k\} \in \prod' \mathring{\mathcal{E}}$ and $\{f_k\} \in C([0, 1]; \prod' C_c(G))$. Then we have that

$$\langle \xi_k \cdot f_k, \xi_k \cdot f_k \rangle(t) = \{f_k^*(t) * \langle \xi_k, \xi_k \rangle(t) * f_k(t)\}$$

and it is contained in $\prod' C^*(G)$ for any $t \in [0, 1]$ and continuous in t . Therefore it follows that $\{\xi_k\} \cdot \{f_k\} \in \prod' \mathring{\mathcal{E}}$ and hence that $\prod' \mathring{\mathcal{E}}$ is a pre-Hilbert $C([0, 1]; \prod' C_c(G))$ -module.

Let $\prod' \mathcal{E}$ be the Hilbert $C([0, 1]; \prod' C^*(G))$ -module given by the completion of $\prod' \mathring{\mathcal{E}}$. By the construction, $\prod' \mathcal{E} \widehat{\otimes}_{e_1} \prod' C^*(G) \cong \prod' \mathcal{E}^G$ and $\prod' \mathcal{E} \widehat{\otimes}_{e_1} \prod' C^*(G) = \{0\}$.

Note that $\prod' \mathcal{E}$ can be regarded as a Hilbert $\prod' IC^*(G)$ -module. Let $i: \prod' \mathcal{E} \hookrightarrow \prod \mathcal{E}$ be the inclusion map and $p: \prod \mathcal{E} \rightarrow \prod' \mathcal{E}$ be the orthogonal projection. Let $\{T\} \in \mathbb{L}(\prod \mathcal{E})$ be an operator given by the constant family and put $T' := p\{T\}i \in \mathbb{L}(\prod' \mathcal{E})$. Then $T' \widehat{\otimes}_{e_1} \text{id} = \{\overline{A}^G\}$.

As a conclusion, $[[A]]_c = 0 \in KK(\mathbb{C}, \prod' C^*(G))$, and hence $\left[[A]_{\mathcal{Q}_{L_{1/k}}} \right] = [[A]] \widehat{\otimes} [[W]] = \{\lambda^{1/k}\} ([A]_c) \widehat{\otimes} [[W]] = 0$. $\square$

8 Proof of the Main Theorem

Let us restate the settings and the claim.

Let X be a complete Riemannian manifold with $H_1(X; \mathbb{R})$ and let G be a second countable locally compact Hausdorff unimodular group which is acting on X properly and isometrically. Assume X is G -compact. Let $\mu_G: KK^G(C_0(X), \mathbb{C}) \rightarrow K_0(C^*(G))$ be the index map, and $E = L^1 \oplus \dots \oplus L^N$ be a finite direct sum of smooth Hermitian G -line bundles each of which has the vanishing first Chern class; $c_1(L^k) = 0 \in H^2(X; \mathbb{R})$. Let A be a G -invariant elliptic operator of order 0 with proper support or a Dirac type operator.

Theorem A *If $[A] \in KK^G(C_0(X), \mathbb{C})$ satisfies that*

$$\langle [E], [A] \rangle_G := \text{tr}_G \mu_G([E] \widehat{\otimes} [A]) \neq 0 \in \mathbb{R},$$

then $\mu_G([A]) \neq 0 \in K_0(C^(G))$.*

Corollary B *Let X and G be as above and additionally we assume that X is a spin manifold and the scalar curvature of X is positive. Then for any Hermitian G -vectorbundle E satisfying the conditions above, the following higher G - $\widehat{A}$ -genus vanishes.*

$$\widehat{A}_G(X; E) := \int_X c(x) \widehat{A}(TX) \wedge \text{ch}(E) = 0,$$

where $c \in C_c(X)$ denotes an arbitrary cut-off function.

Proof of the Theorem A. As subsection 5.2, we may assume that A is a Dirac type operator. In this case,

$$\langle [E], [A] \rangle_G = \int_{TX} c(x) \text{Td}(T_{\mathbb{C}}X) \wedge \text{ch}(\sigma_A) \wedge \text{ch}(E).$$

Since $\text{ch}(L^1 \oplus \dots \oplus L^N) = \text{ch}(L^1) + \dots + \text{ch}(L^N)$, it is sufficient to prove when E is a single Hermitian G -line bundle L . Fix a G -invariant connection and let $i\omega$ denote its curvature. Assume the converse; $\mu_G[A] = 0$.

Firstly consider the following exact sequence;

$$0 \rightarrow \bigoplus C^*(G^{1/k}) \xrightarrow{\iota} \prod' C^*(G^{1/k}) \xrightarrow{\pi} \mathcal{Q}' C^*(G^{1/k}) \rightarrow 0.$$

Now we have the exact sequence of K -groups

$$K_0 \left(\bigoplus C^*(G^{1/k}) \right) \xrightarrow{\iota_*} K_0 \left(\prod' C^*(G^{1/k}) \right) \xrightarrow{\pi_*} K_0 \left(\mathcal{Q}' C^*(G^{1/k}) \right).$$

Let us start with $\left[\left[A_{\prod L_{1/k}} \right] \right] \in K_0 \left(C^*(\widetilde{G}_\alpha) \otimes (\prod \mathbb{C}) \right)$.

Due to Lemma 7.26 and Lemma 7.27,

$$\pi_* \left[\left[A_{\prod L_{1/k}} \right] \right] = \left[\left[A_{\mathcal{Q} L_{1/k}} \right] \right] = 0.$$

The exactness implies that there exists $\zeta \in K_0 \left(\bigoplus C^*(G^{1/k}) \right)$ such that

$$\iota_*(\zeta) = \left[\left[A_{\prod L_{1/k}} \right] \right].$$

In addition, there is a natural isomorphism $K_0 \left(\bigoplus C^*(G^{1/k}) \right) \simeq \bigoplus K_0 \left(C^*(G^{1/k}) \right)$ [HR, 4.1.15 Proposition, 4.2.3 Remark], with the last $\bigoplus$ meaning the algebraic direct sum. Then we can consider the following diagram;

$$\begin{array}{ccccc} K_0 \left(\bigoplus C^*(G^{1/k}) \right) & \xrightarrow{\iota_*} & K_0 \left(\prod' C^*(G^{1/k}) \right) & \xrightarrow{\pi_*} & K_0 \left(\mathcal{Q}' C^*(G^{1/k}) \right) \\ \{(p_k)_*\} \downarrow \cong & & \{(p_k)_*\} \downarrow & & \\ \bigoplus K_0 \left(C^*(G^{1/k}) \right) & \xrightarrow{\text{inclusion}} & \prod K_0 \left(C^*(G^{1/k}) \right) & & \end{array}$$

Since $p_k = \iota p_k$, this diagram commutes. Note that both $\bigoplus$ and $\prod$ in the bottom row are in the algebraic sense. By Lemma 7.26, we obtain that

$$\begin{aligned} \{ \mu_{G^{1/k}} \left([L_{1/k}] \widehat{\otimes} [A] \right) \}_{k \in \mathbb{N}} &= \{ (p_k)_* \} \left(\left[\left[A_{\prod L_{1/k}} \right] \right] \right) \\ &= \{ (p_k)_* \} (\iota_*(\zeta)) \\ &= \{ (p_k)_* \}(\zeta) \in \bigoplus K_0 \left(C^*(G^{1/k}) \right). \end{aligned}$$

This implies that all of $\mu_{G^{1/n}} \left([L_{1/n}] \widehat{\otimes} [A] \right) \in K_0 \left(C^*(G^{1/n}) \right)$ are equal to zero except for finitely many $n \in \mathbb{N}$.

Then apply the trace $\mathrm{tr}_{G^{1/n}}$ to each n -th component to obtain

$$\mathrm{tr}_{G^{1/n}} \mu_{G^{1/n}} ([L_{1/n}] \hat{\otimes} [A]) = \int_{TX} c(x) \mathrm{Td}(T_{\mathbb{C}}X) \wedge \mathrm{ch} \wedge (\sigma_A) \exp \left(\frac{-1}{2\pi n} \omega \right).$$

Since all but finitely many these values are zero,

$$\int_{TX} c(x) \mathrm{Td}(T_{\mathbb{C}}X) \wedge \mathrm{ch}(\sigma_A) \wedge \exp \left(\frac{-t}{2\pi} \omega \right) = 0$$

as a polynomial in t in particular

$$\int_{TX} c(x) \mathrm{Td}(T_{\mathbb{C}}X) \wedge \mathrm{ch}(\sigma_A) \wedge \exp \left(\frac{-1}{2\pi} \omega \right) = \int_{TX} c(x) \mathrm{Td}(T_{\mathbb{C}}X) \wedge \mathrm{ch}(\sigma_A) \wedge (\mathrm{ch}(L)) = 0.$$

This contradicts to the assumption. $\square$

Proof of the Corollary B. Let the scalar curvature on X be denoted by Sc , which is assumed to be positive everywhere. Note that by the bounded geometry of X (corollary 2.8), it is uniformly positive, namely, $\inf_{x \in X} \mathrm{Sc}(x) > 0$. Let $\mathcal{D}$ be the canonical Dirac operator on the spinor bundle $\mathbb{S}$ over X . Then we have

$$\mathcal{D}^* \mathcal{D} = \nabla^* \nabla + \frac{1}{4} \mathrm{Sc}$$

and since $\inf_{x \in X} \mathrm{Sc}(x) > 0$, the operator $A := \frac{\mathcal{D}}{\sqrt{1 + \mathcal{D}^2}}$ is a positive operator. In particular $\mu_G([A]) = 0$. Then due to the theorem A, we obtain

$$0 = \langle [E], [A] \rangle_G = \int_X c(x) \hat{\mathcal{A}}(TX) \wedge \mathrm{ch}(E) = \hat{\mathcal{A}}_G(X; E).$$

$\square$

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References

- [At] M. F. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras*, Astérisque **32-3**(1976), 43-72.
- [Ab] H. Abels, *A universal proper G -space*, Math. Z. **159**(1978), 143-158.
- [An] S. A. Antonyan, *Orbit spaces and unions of equivariant absolute extensors*, Topology Appl. **146-147**(2005), 289-315.
- [BCH] P. Baum, A. Connes and N. Higson, *Classifying space for proper actions and K -theory of group C^* -algebras*, Contemp. Math. **167**(1994), 241-291.

- [BHS] P. Baum, N. Higson, and T. Schick, *On the equivalence of geometric and analytic K -homology*, Pure Appl. Math. Quart. **3**(2007), 1-24.
- [CM] A. Connes and H. Moscovici, *The L^2 -Index theorem for homogeneous spaces of Lie groups*, Annals of Math. **115-2**(1982), 291-330.
- [HS] B. Hanke and T. Schick, *The strong Novikov conjecture for low degree cohomology*, Geom. Dedicata. **135**(2008), 119-127.
- [HR] N. Higson and J. Roe, *Analytic K -Homology*, Oxford University Press (2000).
- [Hö71] L. Hörmander, *Fourier integral operators. I.*, Acta Mathematica, **127.1**(1971), 79-183.
- [K88] G. Kasparov, *Equivariant KK -theory and the Novikov conjecture*, Invent. Math. **91**(1988), 147-201.
- [K] G. Kasparov, *K -theoretic index theorems for elliptic and transversal elliptic operators*, J. K -theory, to appear.
- [Ma99] V. Mathai, *K -theory of twisted group C^* -algebras and positive scalar curvature*, Contemp. Math., Amer. Math. Soc. Providence, **231**(1999), 203-225.
- [Ma03] V. Mathai, *The Novikov conjecture for low degree cohomology classes*, Geom. Dedicata. **99**(2003), 1-15.
- [Wa] H. Wang, *L^2 -index formula for proper cocompact group actions*, J. Noncommut. Geom. **8**(2014), 393-432.
- [We] N. E. Wegge-Olsen, *K -Theory and C^* -Algebras*, Oxford University Press (1993).