

POLAR DECOMPOSITION FOR ISOMORPHISMS OF C^* -ALGEBRAS

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We say that an automorphism ρ of a C^* -algebra is positive if it is self-adjoint, that is, $\rho^* = \rho$ and its spectrum is contained in the positive half axis, where the adjoint isomorphism ρ^* of an isomorphism ρ of a C^* -algebra A onto another B means an isomorphism of B onto A defined by the relation

$$\rho(y)^* = \rho^{-1}(y^*)$$

for $y \in B$. Several facts which assure the propriety of these terms we are met by are referred to [2].

It was proved in [2] (See also [1]) that if A and B are C^* -algebras, the former has property (D) and if ρ is an isomorphism of A onto B , then there are a $*$ -isomorphism π of A onto B and a positive automorphism φ of A , in the unique way, which satisfy the relation

$$\rho = \pi \varphi,$$

and that the mapping $\rho \rightarrow (\pi, \varphi)$ is bi-continuous under the norm

topology, where that a C^* -algebra A has property (D) means that any derivation of A is inner.

The main purpose of this note is to report that in the above statement a part of the assumption "A has property (D)" can be taken off, leaving the conclusion invariant:

Polar decomposition theorem for isomorphisms of C^* -algebras. Let A and B be C^* -algebras, ρ an isomorphism of A onto B . Then, there are a $*$ -isomorphism π of A onto B and a positive automorphism φ of A , in the unique way, which satisfy the relation

$$\rho = \pi \varphi ,$$

and that the mapping $\rho \rightarrow (\pi, \varphi)$ is bi-continuous under the norm topology.

We will only point out here a key to reduce this theorem to the preceding statement. It is so simple and fundamental.

Lemma. If the spectrum of a bounded linear operator ζ on a Banach space X is simply connected and if a closed subspace Y of X is invariant under ζ , then the spectrum of the restriction of ζ on Y is contained in the spectrum of ζ .

Thus, we know that if the restriction of a positive automorphism of a C^* -algebra B on a sub- C^* -algebra A of B becomes an automorphism of A , then it is also positive.

Positive automorphisms are so significant that we can prove the following theorems on them.

Theorem. Let A be a sub- C^* -algebra which has property (D) of a C^* -algebra B with identity. An automorphism ρ of A is positive if and only if there is a regular positive element h in B such that ρ is the restriction on A of the automorphism $\text{Ad}h$ of B defined by the formula

$$\text{Ad}h(x) = hxh^{-1}$$

for $x \in B$.

Theorem. Let A be a C^* -algebra, Φ a faithful $*$ -representation of A . An automorphism ρ of A is positive if and only if there is a positive automorphism $\tilde{\rho}$ of the von Neumann algebra $\overline{\Phi(A)}$ generated by $\Phi(A)$ which satisfies the relation

$$\tilde{\rho} \Phi = \Phi \rho.$$

Detailed arguments on this subject shall be published somewhere.

REFERENCES

- [1]. T. Okayasu, A structure theorem of automorphisms of von Neumann algebras, Tôhoku Math. Journ. 20(1968), 199-206.
- [2]. 岡本隆照, C^* -代数の同型写像について, 数研講究録166(1972), 8-17.