

Continuation of Real Analytic Solutions of Partial Differential Equations up to Convex Conical Singularities

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In his talk at the RIMS Seminar in December 1985, Kaneko gave the following conjecture (cf. [Kn3]):

Kaneko's Conjecture. Let $P = D_t^2 - \Delta$ be the wave operator on the Euclidean n space $\mathbf{R}^n$. Let Γ be a closed convex proper cone of $\mathbf{R}^n$ with vertex at the origin, sharp enough in a certain direction; i.e., Γ is contained in $\{x_1 \geq C|x_2|\}$ for a Euclidean coordinate $(x_1, \dots, x_n)$ of $\mathbf{R}^n$, for a large $C > 0$. Let $R > 0$ and set $K = \{(x, t) \in \mathbf{R}^n \times \mathbf{R} \mid x \in \Gamma, |t| \leq R|x|\}$. Then any real analytic solution to the wave equation $Pu = 0$ defined outside K can be analytically continued up to the origin $(0, 0)$ of $\mathbf{R}^n \times \mathbf{R}$.

We give an answer to this conjecture in a general context.

Definition. Let K be a closed subset of a real analytic manifold M of dimension n . K is said to be C^α -convex at $x \in M$ ($1 \leq \alpha \leq \omega$) if there exist a neighborhood U of x and an open C^α -immersion $\phi : U \rightarrow \mathbf{R}^n$ such that $\phi(U \cap K)$ is convex in $\mathbf{R}^n$. K is said to have a conical singularity at x if $x \in K$ and the tangent cone $C_x(K)$ is a closed proper cone of $T_x M$.

Theorem 0.1. Let K be a C^1 -convex closed subset of a real analytic manifold M , having a conical singularity at x . Let $P = P(x, D)$ be a second order differential operator with analytic coefficients defined in a neighborhood of x . Assume that P is of real principal type and is not elliptic. Then any real analytic solution to the equation $Pu = 0$ defined outside K is analytically continued up to x .

In order to state a similar result for overdetermined systems of differential equations, we first recall the notion of a virtual bicharacteristic manifold of a system $\mathcal{M}$ of differential equations.

Let $V = \text{Char}(\mathcal{M})$; V^c denotes the complex conjugate of V with respect to $T_M^* X$. Let $p \in V \cap (T_M^* X \setminus M)$. Assume the following:

- (b.1) V is nonsingular at p .
- (b.2) V and V^c intersect cleanly at p ; i.e., $V \cap V^c$ is a smooth manifold and

$$T_p V \cap T_p V^c = T_p (V \cap V^c).$$

- (b.3) $V \cap V^c$ is regular; i.e., $\omega|_{V \cap V^c} \neq 0$, with ω being the fundamental 1-form on $T^* X$.
- (b.4) The generalized Levi form of V has constant rank in a neighborhood of p .

Then one can define the virtual bicharacteristic manifold Λ_p of $\mathcal{M}$ passing through p (cf. [SKK, Ch.III, Sect.2.4]). we assume

$$(b.5) \quad d\pi(T_p\Lambda_p) \neq \{0\}.$$

Theorem 0.2. *Let (K, x) be as in Theorem 0.1. Let $\mathcal{M}$ be a system of differential equations defined in a neighborhood of x . Assume that $\text{Char}(\mathcal{M}) \cap \pi^{-1}(x)$ has codimension ≥ 2 in $\pi^{-1}(x)$ and that $V = \text{Char}(\mathcal{M})$ satisfies conditions (b.1)—(b.5) at each point p of $V \cap (T_M^*X \setminus M) \cap \pi^{-1}(x)$. Then any real analytic solution to $\mathcal{M}$ defined outside K is analytically continued up to x .*

Corollary. *Let (K, x) be as in Theorem 0.2. Let $\mathcal{M}$ be an elliptic system of differential equations and assume that $\text{Char}(\mathcal{M}) \cap \pi^{-1}(x)$ has codimension ≥ 2 in $\pi^{-1}(x)$. Then any solution u of $\mathcal{M}$ defined outside K can be analytically continued up to x .*

Remark. Cf. [Kw], theorems 4 and 5, for general results on analytic continuation of the solutions of overdetermined systems of differential equations.

The following theorem is a generalization of Theorem 0.1 to higher order differential equations for $K = \{x_0\}$. Cf. Theorem 17 and Corollary 22 of [Kn2].

Theorem 0.3. *Let $P = P(x, D)$ be a differential operator of real principal type. Assume that the polynomial $f(x_0; \zeta)$ in ζ has no elliptic factors. Then any real analytic solution to the equation $Pu = 0$ defined in a neighborhood of x_0 except x_0 can be analytically continued on the whole of a neighborhood of x_0 .*

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