

Fundamental Problems on Douady space

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Let X be a compact complex space. Then Douady^{*)} [1] proved the existence of a complex space $\mathcal{D}$ together with a commutative diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{j} & \mathcal{D} \times X \\ \pi \searrow & & \swarrow \text{projection} \\ & \mathcal{D} & \end{array}$$

where

(1) j is a closed imbedding and π is a flat morphism (morphism = complex-analytic map), and

(2) given any diagram (having the same property as in (1))

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i} & T \times X \\ p \searrow & & \swarrow \text{projection} \\ & T & \end{array}$$

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there exists a unique morphism $h: T \rightarrow \mathcal{D}$
 such that

$$\begin{array}{ccc} \mathcal{F} & = & T \times_{\mathcal{D}} \mathcal{S} \\ \downarrow i & & \downarrow j \end{array}$$

$$T \times X = T \times_{\mathcal{D}} (\mathcal{D} \times X)$$

We call $\mathcal{D}$ the Duady space of X . For each integer $m \geq 0$, we have an open and closed complex subspace $\mathcal{D}_m$ of $\mathcal{D}$ such that

$$d \in \mathcal{D}_m \iff d \in \mathcal{D} \text{ and } \dim \pi^+(d) = m$$

Thus

$$\mathcal{D} = \bigsqcup_{m \geq 0} \mathcal{D}_m \quad (\text{disjoint})$$

Remark 1. Duady space in complex-analytic geometry as above is a generalization of what Grothendieck called Hilbert scheme in algebraic geometry. Namely, if an imbedding $X \hookrightarrow \mathbb{P}_{\mathbb{C}}^N$ is given, then

$$\mathcal{D} = \bigsqcup_H \mathcal{D}_H$$

where $d \in \mathcal{D}_H \iff d \in \mathcal{D}$ and H is the

Hilbert polynomial of $\pi^{-1}(d)$ with respect to the projective imbedding $\pi^{-1}(d) \subset X \subset \mathbb{P}_{\mathbb{C}}^N$. (Here $H(v) = \chi(\mathcal{O}_{\pi^{-1}(d)}(v))$, $\forall v \in \mathbb{Z}$, where (v) denotes the twisting v -times by the tautological line bundle on $\mathbb{P}_{\mathbb{C}}^N$.) In this case, moreover, $\mathcal{D}_m$ is the disjoint union of those $\mathcal{D}_H$ with $\deg H = m$, and for every H , $\mathcal{D}_H$ is projective (and hence compact).

As Duady extended the notion of Hilbert scheme to complex-analytic geometry, Barlet [2] did the notion of Chow variety. The group of cycles (complex-analytic), denoted by $C(X)$, for a complex-analytic space X (not necessarily compact), is the additive group freely generated by reduced irreducible closed complex subspaces of X . Each cycle $c \in C(X)$ is uniquely written as

$$c = \sum_i m_i Y_i$$

where Y_i are different complex subspaces and m_i are integers $\neq 0$. We say that C is a q -cycle if $\dim Y_i = q$ for all i . The union of those Y_i (or rather their point sets) is called the support of C and is denoted by $|C|$. We say that C is positive if $m_i > 0$ for all i . We denote by $C_+^q(X)$ the set of all the positive q -cycles of X .

If f is a map (set-theoretical) from a complex space V (reduced) into $C_+^q(X)$, we ask whether f is complex-analytic or not. This complex-analyticity is defined as follows.

First of all, if $\mathbb{D}$ is a complex manifold (manifold = complex space which is smooth), then $C_+^0(\mathbb{D})$ is naturally identified with the point-set of the complex space

$$\text{Sym}(\mathbb{D}) = \coprod_{v \geq 0} \text{Sym}^v(\mathbb{D})$$

where $\text{Sym}^v(\mathbb{D})$ denotes the v -times symmetric product of $\mathbb{D}$, i.e., the quotient space of $\mathbb{D} \times \cdots \times \mathbb{D}$ (v -times) by the per-

mutations of the factors. This identification $C_+^0(\mathbb{D}) = \text{Sym}(\mathbb{D})$ will play the key role.

For $c \in C_+^g(X)$, we consider various closed imbeddings $i: V \hookrightarrow \mathbb{D}^g \times \mathbb{D}^{N-g} \subset \mathbb{C}^N$, where V is an open subset of X and $\mathbb{D}^a$ denotes the unit polydisc in $\mathbb{C}^a$. Say $c = \sum_i m_i \gamma_i$ as before. We say that i is an well-adjusted chart for c if the projection $\gamma_i \cap V \longrightarrow \mathbb{D}^g$ is proper and finite-to-one everywhere for all i . If i is such, the intersection cycle $c \cdot (b \times \mathbb{D}^{N-g})$ is well defined in $\mathbb{D}^g \times \mathbb{D}^{N-g}$, which projects to a 0-cycle in $\mathbb{D}^{N-g}$, where b is any point in $\mathbb{D}^g$. This 0-cycle will be denoted by $s(c, b) \in C_+^0(\mathbb{D}^{N-g}) = \text{Sym}(\mathbb{D}^{N-g})$.

We say that $f: V \longrightarrow C_+^g(X)$ is complex-analytic if for every $w \in V$ the following conditions are satisfied:

(a) for every open neighborhood A of $|f(w)|$ in X , there exists an open neighborhood B of

v in V such that if $u \in B$ then $|f(u)| \subset A$.

(b) for every $x \in |f(v)|$, there exists an open neighborhood B' of v in V and a closed imbedding $i: U \hookrightarrow \mathbb{D}^q \times \mathbb{D}^{N-q}$ which is well-adjusted for all $f(u)$ with $u \in B'$ and such that the map $B' \times \mathbb{D}^q \rightarrow \text{Sym}(\mathbb{D}^{N-q})$ by $(u, b) \mapsto s(f(u), b)$ is a complex-analytic morphism.

Barlet proved the existence of a complex-analytic map $\beta: \mathcal{B}_q \rightarrow C_+^q(X)$ where $\mathcal{B}_q$ is a reduced complex space and β is bijective (point-set-theoretically). We call $\mathcal{B}_q$ the Barlet space (in dimension q) of X .

PROBLEM I. For a compact complex space X , if m is an integer ≥ 0 and if every connected component of $\mathcal{B}_q$ for every $q \leq m$ is compact, then is it true that every connected component of $\mathcal{D}_m$ is compact?

It is known by Fujiki [3] and others

that if X is a compact Kähler manifold, then every connected component of B_g for every $g \geq 0$ is compact. Hence PROBLEM I leads us to:

PROBLEM I* Assume that X is a ^{Kähler} compact complex manifold. Is it then true that every connected component of D_m for every $m \geq 0$ should be compact?

In order to give a good sense to the next problem, we need to extend the notion of "Kähler" to a general complex space (with singularities). Let Z be a complex space. We then consider various pairs (j, ω) where j is a closed imbedding of the restriction of Z to an open subset, say U , into a complex manifold W (eg. an open polydisc in some $\mathbb{C}^N$) and ω is a Kählerian form on W . For each point $u \in U$, ω defines a hermitian form on the tangent space of W at $j(u)$. Hence

by $\tilde{f}$ it induces a hermitian form on the Zariski tangent space of Z at u . This induced form will be denoted by $(\tilde{f}, \omega)_u$. We shall say that Z is Kähler if there exists an open covering $\{U_\alpha\}_{\alpha \in \Lambda}$ of Z and a pair $(\tilde{f}_\alpha, \omega_\alpha)$ with $\tilde{f}_\alpha: U_\alpha \rightarrow W_\alpha$ for every $\alpha \in \Lambda$ such that for every $\alpha, \beta \in \Lambda$ and for every $u \in U_\alpha \cap U_\beta$ we have $(\tilde{f}_\alpha, \omega_\alpha)_u = (\tilde{f}_\beta, \omega_\beta)_u$.

PROBLEM II When X is a Kähler compact complex space, is it true that $\mathcal{D}$ is Kähler?

As a special case of this problem, we ask

PROBLEM II* Let $f: M \rightarrow N$ be a flat surjective morphism of compact complex manifolds. Is it true that if M is Kähler, so is N ?

References

- [1] Duady, A., "Le problème de modules pour les sous-espaces analytiques compacts d'un espace analytique donné", Ann. Inst. Fourier, Grenoble 16 (1966), pp. 1-95
- [2] Barlet, D., "Espace analytique réduit des cycles analytiques complexes de dimension finie", Sem. F. Norguet, Lecture note in Math. 482 Springer (1975), pp. 1-158
- [3] Fujiki, A., "Closedness of the Duady Spaces of Compact Kähler Spaces", Mimeographed note (RIMS, Kyoto Univ.).