

Holonomic deformation of linear differential equations of the A_g type

LIU DEMING

Department of Mathematical Sciences University of Tokyo

0 Introduction.

In this paper, we consider linear differential equations of the form:

$$(0.1) \quad \frac{d^2 y}{dx^2} + p_1(x, t) \frac{dy}{dx} + p_2(x, t)y = 0,$$

defined on the Riemann sphere $\mathbf{P}^1$, with the coefficients:

$$(0.2) \quad \begin{aligned} p_1(x, t) &= -2x^{g+1} - \sum_{j=1}^g j t_j x^{j-1} - \sum_{k=1}^g \frac{1}{x - \lambda_k}, \\ p_2(x, t) &= -(2\alpha + 1)x^g - 2 \sum_{j=1}^g H_j x^{g-j} + \sum_{k=1}^g \frac{\mu_k}{x - \lambda_k}. \end{aligned}$$

The Riemann scheme of this equation reads:

$$(0.3) \quad \left(\begin{array}{c|ccccccccc} x = \lambda_k & & & & & & & & x = \infty \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha + \frac{1}{2} \\ 2 & \frac{2}{g+2} & 0 & t_g & t_{g-1} & \cdots & t_1 & -\alpha + \frac{1}{2} \end{array} \right).$$

Here the symbol in (0.3) means that, at the irregular point $x = \infty$, the equation (0.1)-(0.2) admits a system of formal solutions of the form:

$$(0.4) \quad \begin{aligned} \hat{y}_1 &= x^{-\alpha-\frac{1}{2}} \left(1 + \sum_{i \geq 1} h_i^1 x^{-i} \right), \\ \hat{y}_2 &= x^{\alpha-\frac{1}{2}} \exp \left[\frac{2}{g+2} x^{g+2} + t_g x^g + \cdots + t_1 x \right] \left(1 + \sum_{i \geq 1} h_i^2 x^{-i} \right). \end{aligned}$$

Note that the Poincaré rank at $x = \infty$ of the linear equation (0.1)-(0.2) is $g+2$. The principal parts of these formal solutions are given by the primitive

function of the polynomials representing the versal deformation of the simple singularity of the A_g type, so we call the linear equation (0.1)-(0.2) as the equation of the A_g -type.

When considering the holonomic deformation of equations of the A_1 -type, we obtain the Hamiltonian structure:

$$(\lambda_1, \mu_1, H_1, t_1),$$

which determines the Hamiltonian system, equivalent to the second Painlevé equation, see [6]. C. H. LIN and Y. SIBUYA studied on the holonomic deformation of linear equations; having the irregular singularity at $x = \infty$ with higher Poincaré rank and admitting a non-logarithmic singular point $x = \lambda$, see [4].

When $g = 2$, it is known ([3]) that the holonomic deformation is governed by the Hamiltonian system with respect to the canonical variables:

$$(\lambda_1, \lambda_2, \mu_1, \mu_2, H_1, H_2, t_1, t_2).$$

On the other hand, in the case $g \geq 3$, the quantities $H = (H_1, \dots, H_g)$ and $t = (t_1, \dots, t_g)$ do not compose the Hamiltonian structure. In fact, in the case of $g = 3$, we have to determine the variables $s = (s_1, s_2, s_3)$ such that

$$(0.5) \quad s_1 = t_1 - \frac{3}{4}t_3^2, \quad s_2 = t_2, \quad s_3 = t_3,$$

and then obtain the Hamiltonian structure: (see [5])

$$(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, H_1, H_2, H_3, s_1, s_2, s_3).$$

As to general natural number g , in order to determine the Hamiltonian structure, we have tried to find the transformation of qualities $t = (t_1, \dots, t_g)$ such as (0.5), but we didn't succeed. So we determine the following new qualities instead.

$$(0.6) \quad \overline{H}_j = 2 \sum_{i=0}^{j-1} a_{i+1}^{(j)}(t) (H_{j-i} + T_{j-i}^*), \quad (j = 1, \dots, g),$$

where

$$(0.7) \quad \begin{aligned} T_j^* &= \frac{1}{4}(j-1)T_{g+2-j} + \frac{1}{8} \sum_{l=1}^g T_l T_{g+2-j-l} \quad (1 \leq j \leq g), \\ T_j &= jt_j \quad (1 \leq j \leq g), \quad T_j = 0 \quad (j < 1) \quad \text{or} \quad (j > g), \end{aligned}$$

$a_{i+1}(t)$ is given by

$$(0.8) \quad a_1(t) = \frac{1}{2}, \quad a_2(t) = 0, \quad a_{i+1}(t) = \sum_{m=1}^{[\frac{i}{2}]} M^{(m, 2m-i)} \quad (i \geq 2).$$

$M^{(m, q)}$ are defined as the coefficients of the expansion of function,

$$(0.9) \quad \sum_{q=n(1-g)}^0 M^{(n, q)} x^{-q} = \frac{1}{2} \left(-\frac{1}{2} \sum_{l=1}^g T_l x^{g-l} \right)^n$$

Such that

$$(\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g, \overline{H}_1, \dots, \overline{H}_g, t_1, \dots, t_g)$$

is the Hamiltonian Structure of the equation of the A_g type.

We suppose throughout this paper that $2\alpha + 1$ is not an integer. This equation has an irregular singularity at $x = \infty$ of the Poincaré rank $g + 2$ and g regular singular points $x = \lambda_k$ ($k = 1, \dots, g$). We also make the following assumption:

(A) *none of $x = \lambda_k$ ($k = 1, \dots, g$) is logarithmic singularity.*

Since exponents at each regular singular point, $x = \lambda_k$, are 0 and 2, we deduce from the assumption (A) that H_i ($i = 1, \dots, g$) are rational functions of $t = (t_1, \dots, t_g)$, $\lambda = (\lambda_1, \dots, \lambda_g)$ and $\mu = (\mu_1, \dots, \mu_g)$. The explicit forms of H_i will be given in Section 2, they play important roles in our studies.

Now we state the Main Theorem:

Main Theorem. *The holonomic deformation of the linear ordinary differential equation (0.1)-(0.2) is governed by the completely integrable Hamiltonian system:*

$$(\overline{H}) \quad \frac{\partial \lambda_k}{\partial t_j} = \frac{\partial \overline{H}_j}{\partial \mu_k}, \quad \frac{\partial \mu_k}{\partial t_j} = -\frac{\partial \overline{H}_j}{\partial \lambda_k} \quad (k, j = 1, \dots, g),$$

with the Hamiltonian functions $\overline{H}_j$ defined by (0.6).

Since the completely integrable Hamiltonian system $(\overline{H})$ determines the holonomic deformation of linear equations of the A_g type, we call $(\overline{H})$ the A_g -system.

1 Holonomic deformation of linear equation of the second order.

In this section, we recall the theory of the holonomic deformation of linear differential equation of the form:

$$(1.1) \quad \frac{d^2 y}{dx^2} + p_1(x, t) \frac{dy}{dx} + p_2(x, t)y = 0,$$

We make in the following of this section a review of known results, which are available for us to study the holonomic deformation of linear equation of the A_g -type.

Proposition 1.1. *The equation (1.1) has a fundamental system of solutions whose monodromy and Stokes multiplier are independent of t , if and only if there exist rational functions of x , $A_j(x)$, $B_j(x)$, such that the following system of partial differential equations is completely integrable:*

$$(1.2) \quad \begin{aligned} \frac{\partial^2 y}{\partial x^2} + p_1(x, t) \frac{\partial y}{\partial x} + p_2(x, t)y &= 0, \\ \frac{\partial y}{\partial t_j} &= B_j(x)y + A_j(x) \frac{\partial y}{\partial x}, \end{aligned} \quad (j = 1, \dots, g).$$

Proposition 1.2. *The conditions of the complete integrability of (1.2) are given by:*

$$(1.3) \quad \frac{\partial A_j}{\partial t_i} + A_j \frac{\partial A_i}{\partial x} = \frac{\partial A_i}{\partial t_j} + A_i \frac{\partial A_j}{\partial x}, \quad (i, j = 1, \dots, g),$$

$$(1.4) \quad \frac{\partial^3}{\partial x^3} A_j - 4P \frac{\partial}{\partial x} A_j - 2A_j \frac{\partial}{\partial x} P + 2 \frac{\partial}{\partial t_j} P = 0 \quad (j = 1, \dots, g).$$

where

$$(1.5) \quad P(x, t) = -p_2(x, t) + \frac{1}{4}p_1^2(x, t) + \frac{1}{2} \frac{\partial}{\partial x} p_1(x, t).$$

If we make the change of the unknown function:

$$(1.6) \quad y = \Phi(x)z, \quad \Phi(x) = \exp\left(-\frac{1}{2} \int^x p_1(x, t) dx\right),$$

then (1.1) is transformed into an equation of the form:

$$(1.7) \quad \frac{d^2 z}{dx^2} = P(x, t)z,$$

where $P(x, t)$ is the function given by (1.5). It follows that:

Proposition 1.3. *The holonomic deformation of (1.1) is reduced to that of (1.7). Finally, the holonomic deformation of (1.1) is reduced to the existence of rational functions $A_j(x)$ ($j = 1, \dots, g$), satisfying the system (1.3)-(1.4) of partial differential equations. We will call (1.2) the extended system of (1.1) and the functions $A_j(x)$ the deformation functions.*

2 Deformation functions $A_j(x)$.

In the following of this paper, we consider the holonomic deformation of linear equations of the form:

$$(2.1) \quad \frac{d^2 y}{dx^2} + p_1(x, t) \frac{dy}{dx} + p_2(x, t)y = 0,$$

$$(2.2) \quad \begin{aligned} p_1(x, t) &= -2x^{g+1} - \sum_{(j)} j t_j x^{j-1} - \sum_{(k)} \frac{1}{x - \lambda_k}, \\ p_2(x, t) &= -(2\alpha + 1)x^g - 2 \sum_{(j)} H_j x^{g-j} + \sum_{(k)} \frac{\mu_k}{x - \lambda_k}, \end{aligned}$$

For the limiting cases, here we only can give the results, omit their proofs.

Firstly we determine the deformation functions.

Proposition 2.1. *For $j = 1, \dots, g$, the deformation functions $A_j(x)$ are given as follows:*

$$(2.3) \quad A_j(x) = \frac{\overline{Q}_j(x)}{\Lambda(x)},$$

where $\Lambda(x) = \prod_{j=1}^g (x - \lambda_j)$, and $\overline{Q}_j(x)$ is a polynomial of degree $j - 1$.

The explicit form of $\overline{Q}_j(x)$ will be given by proposition 3.2. In order to prove this proposition, we need following lemmata.

Lemma 2.1. *As function of x , $A_j(x)$ is holomorphic on $\mathbb{C} \setminus \{\lambda_1, \dots, \lambda_g\}$.*

Lemma 2.2. For $k = 1, \dots, g$; $x = \lambda_k$ is a pole of the first order of $A_j(x)$.

Lemma 2.3. $A_j(x)$ admits a zero of order $g + 1 - j$ at $x = \infty$.

proposition 2.1 is an immediate consequence of lemmata 2.1, 2.2 and 2.3.

To give the explicit form of $\bar{Q}_j(x)$, we prove following lemma:

Lemma 2.4. For $i \geq 3$, $a_i(t)$ defined by (0.8) satisfies

$$(2.4) \quad a_1(t) = \frac{1}{2}, \quad a_2(t) = 0, \quad a_i(t) = -\frac{1}{2} \sum_{m=1}^{i-2} T_{g+m+2-i} a_m \quad (i \geq 3).$$

Proposition 2.2. If differential equation (0.1)-(0.2) admits the holonomic deformation, then the deformation functions $A_j(x) = \frac{\bar{Q}_j(x)}{\Lambda(x)}$ ($j = 1, \dots, g$) are determined as

$$(2.5) \quad \bar{Q}_j(x) = 2 \sum_{i=0}^{j-1} a_{i+1}(t) Q_{j-i}(x), \quad (j = 1, \dots, g)$$

where

$$Q_j(x) = -\frac{1}{2} \sum_{n=0}^{j-1} \sigma_n x^{j-1-n}.$$

$$(2.6) \quad \sigma_n = (-1)^{n+1} e_n, \quad (n = 0, 1, \dots, g),$$

and e_n denotes the n -th elementary polynomial of the g variables, $\lambda_1, \dots, \lambda_g$, in particular we define $e_0 = 1$.

Remark 2.1. From the proof of proposition 2.2, we know when $\bar{Q}_j(x)$ is of the form of (2.5), then the degree of $\Delta_j(x)$ is at most $g - 1$.

Proposition 2.3. $Q_j(x)$ and $\bar{Q}_j(x)$ have the following properties:

$$(2.7) \quad Q_j(\lambda_k) = \frac{1}{2} N^{j,k}, \quad \bar{Q}_j(\lambda_k) = \frac{1}{2} \bar{N}^{j,k}.$$

3 Equation of the SL-type.

In this section, we will investigate the equation of the SL-type:

$$(3.1) \quad \begin{aligned} \frac{d^2 z}{dx^2} &= P(x, t) z, \\ P(x, t) &= -p_2(x, t) + \frac{1}{4} p_1^2(x, t) + \frac{1}{2} \frac{\partial}{\partial x} p_1(x, t). \end{aligned}$$

By using (0.2), we see that $P(x, t)$ can be written in the following form:

$$(3.2) \quad P(x, t) = x^{2g+2} + \sum_{i=0}^g F_i x^{g+i} + 2 \sum_{(j)} K_j x^{g-j} - \sum_{(k)} \frac{\nu_k}{x - \lambda_k} + \frac{3}{4} \sum_{(k)} \frac{1}{(x - \lambda_k)^2},$$

where we denote by $\Sigma_{(k)}$ the sum for $k = 1, \dots, g$. And we have:

$$(3.3) \quad F_j = \frac{1}{4} \sum_{i=2+j}^g T_i T_{g+2+j-i} + T_j + 2\alpha \delta_{j0} \quad (0 \leq j \leq g).$$

$$(3.4) \quad K_j = H_j + T_j^* + \frac{1}{2} \sum_{(k)} \lambda_k^j + \frac{1}{4} \sum_{(k)} \sum_{m=1}^{j-2} T_{m+g+2-j} \lambda_k^m \quad (1 \leq j \leq g).$$

$$(3.5) \quad \nu_k = \mu_k - \frac{1}{2} \left(\sum_{(l)}^{(k)} \frac{1}{\lambda_k - \lambda_l} + \sum_{(i)} T_i \lambda_k^{i-1} + 2\lambda_k^{g+1} \right) \quad (1 \leq k \leq g).$$

Here we denote by $\sum_{(l)}^{(k)}$ the sum for $l = 1, \dots, g$ except for $l = k$. Let $e_j^{(k)}$ be the j -th elementary symmetric polynomial of $g-1$ variables, λ_l ($l = 1, \dots, g, \neq k$), in particular, we put $e_0^{(k)} = 1$. Moreover, we define $\sigma_j^{(k)} = (-1)^{j+1} e_j^{(k)}$. For the simplicity of presentation, we put:

$$(3.6) \quad N_k = \frac{1}{\Lambda'(\lambda_k)}, \quad N^{j,K} = -\sigma_{j-1}^{(k)}, \quad k, j = 1, \dots, g,$$

where $\Lambda(x) = \prod_{i=1}^g (x - \lambda_i)$, and $\Lambda'(x) = \frac{d}{dx} \Lambda(x)$.

We have following two propositions. Since the proofs are almost same, we omit them (see [5]).

Proposition 3.1. *In the linear equation (0.1)-(0.2) H_j ($j = 1, \dots, g$) are given by:*

$$(3.7) \quad H_j = \frac{1}{2} \sum_{(k)} [N_k N^{j,K} \mu_k^2 - U_{jk} \mu_k - N_k N^{j,k} (2\alpha + 1) \lambda_k^g],$$

where

$$U_{jk} = N_k N^{j,k} (2\lambda_k^{g+1} + \sum_{(l)} T_l \lambda_k^{l-1}) - \sum_{(l)}^{(k)} \frac{N_k N^{j,k} + N_l N^{j,l}}{\lambda_l - \lambda_k}.$$

Proposition 3.2. *In the linear equation (3.1)-(3.2), K_j ($j = 1, \dots, g$) are written as follows:*

$$(3.8) \quad K_j = \frac{1}{2} \sum_{(k)} \left(N_k N^{j,k} \nu_k^2 - \sum_{(l)}^{(k)} \frac{N_l N^{j,l}}{\lambda_k - \lambda_l} \nu_k - N_k N^{j,k} V_k \right),$$

where $V_k = \lambda_k^{2g+2} + \lambda_k^g \sum_{i=0}^g F_i \lambda_k^i + \frac{3}{4} \sum_{(l)}^{(k)} \frac{1}{(\lambda_k - \lambda_l)^2}$.

For the $M^{(m,q)}$ given by (0.9), we have following lemma.

Lemma 3.1. *For arbitrary natural numbers m and n ; , nonnegative integer q satisfying $1 - g \leq -q \leq 0$, we have:*

$$(3.9) \quad M^{(m+n,q)} = \sum_{r=q}^0 M^{(m,r)} M^{(n,q-r)}.$$

4 the canonical transformation.

Using H_j and K_j given by (2.7) and (2.8) respectively, we define $\overline{H}_j$ and $\overline{K}_j$ $j = 1, \dots, g$ as follows:

$$(4.1) \quad \overline{H}_j = 2 \sum_{i=0}^{j-1} a_{i+1}(t) (H_{j-i} + T_{j-i}^*) \quad (j = 1, \dots, g),$$

$$(4.2) \quad \overline{K}_j = 2 \sum_{i=0}^{j-1} a_{i+1}(t) K_{j-i} \quad (j = 1, \dots, g),$$

Note that (4.1) is nothing but (0.6). Combining (2.7) with (4.1) and (2.8) with (4.2), we obtain

$$(4.3) \quad \overline{H}_j = \frac{1}{2} \sum_{(k)} [N_k \overline{N}^{j,k} \mu_k^2 - \overline{U}_{jk} \mu_k - N_k \overline{N}^{j,k} (2\alpha + 1) \lambda_k^g + \overline{T}_j^*],$$

$$(4.4) \quad \overline{K}_j = \frac{1}{2} \sum_{(k)} (N_k \overline{N}^{j,k} \nu_k^2 - \sum_{(l)}^{(k)} \frac{N_l \overline{N}^{j,l}}{\lambda_k - \lambda_l} \nu_k - N_k \overline{N}^{j,k} V_k),$$

where

$$(4.5) \quad \overline{N}^{j,k} = 2 \sum_{i=0}^{j-1} a_{i+1}(t) N^{j-i,k},$$

$$\overline{U}_{j,k} = 2 \sum_{i=0}^{j-1} a_{i+1}(t) U_{j-i,k}, \quad \overline{T}_j^* = 2 \sum_{i=0}^{j-1} a_{i+1}(t) T_{j-i}^*.$$

We have

Lemma 4.1. $\overline{K}_j$ and $\overline{H}_j$ have the following relation:

$$(4.6) \quad \overline{K}_j = \overline{H}_j + \frac{1}{2} \sum_{(k)} \lambda_k^j.$$

Proposition 4.1. The transformation defined by (2.5) and (4.6)

$$(\lambda, \mu, \overline{H}, t) \rightarrow (\lambda, \nu, \overline{K}, t)$$

is canonical, where $\lambda = (\lambda_1, \dots, \lambda_g)$, $\mu = (\mu_1, \dots, \mu_g)$, $\overline{H} = (\overline{H}_1, \dots, \overline{H}_g)$, $\nu = (\nu_1, \dots, \nu_g)$, $\overline{K} = (\overline{K}_1, \dots, \overline{K}_g)$ and $t = (t_1, \dots, t_g)$.

5 the A_g -system.

In this section, we will prove Main Theorem. By means of propositions 1.2, 1.3 and 2.3, it suffices to establish the following theorem:

Theorem 5.1. The conditions (1.3), (1.4) of the complete integrability are equivalent to the following completely integrable Hamiltonian system:

$$(\overline{K}) \quad \frac{\partial \overline{K}_j}{\partial \nu_k} = \frac{\partial \lambda_k}{\partial t_j}, \quad \frac{\partial \overline{K}_j}{\partial \lambda_k} = -\frac{\partial \nu_k}{\partial t_j}, \quad (j, k = 1, \dots, g).$$

Lemma 5.1. The equation (1.4) induces the system $(\overline{K})$.

Lemma 5.2. The equations (1.4) is derived from the system $(\overline{K})$.

Lemma 5.3. The equation (1.3) is derived from the system $(\overline{K})$.

Lemma 5.4. system $(\overline{K})$ is complete integrable.

Acknowledgement. The author wishes to express his deepest appreciation to Professor K. Okamoto who gave him much valuable advice.

References

- [1] Iwasaki, K., Kimura, H., Shimomura, S. and M. Yoshida: *From Gauss to Painlevé*, Vieweg Verlag, Wiesbanten (1991).
- [2] Jimbo, M., Miwa, T. and K. Ueno: *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, I*, General theory and τ -function-, *Physica* **2D** (1981), 306—352.
- [3] Kimura, H.: *The degeneration of the two dimensional Garnier system and the polynomial Hamiltonian structure*, *Ann. Mat. Pura. Appl.*, **CLV**(1989), P. 25-74.
- [4] Lin, C. H. & Y. Sibuya: *Some applications of isomonodromic deformations to the study of Stokes multipliers*, *J. Fac. Sci. Univ. Tokyo Sect. IA, Math.* **36** (1989), 649-663.
- [5] Liu, Deming: *Holonomic deformation of linear equations of the A_3 type*, submitted.
- [6] Okamoto, K.: *Isomonodromic deformation and Painlevé equations and the Garnier system*, *J. Fac. Sci. Univ. Tokyo Sec. IA, Math.* **33**(1986), 575-618.
- [7] Okamoto, K.: *Déformation d'une équation différentielle linéaire avec une singularité irrégulière sur un tore*, *J. Fac. Univ. Tokyo sect. IA Math.* **26** (1979), 501—508.
- [8] Ueno, K.: *Monodromy preserving deformation of linear differential equations with irregular points*, *Proc. Japan Acad. Ser. A Math. Sci.* **56** (1980), 97—102.