

SOME MICRODIFFERENTIAL EQUATIONS FOR MICROFUNCTIONS WITH A HOLOMORPHIC PARAMETER AND THEIR FORMAL SYMBOL TYPE SOLUTIONS

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§1. Introduction

Let X be a complex manifold $\mathbb{C}_z \times \mathbb{C}_x^n$ and M be its submanifold

$$M = \{(z, x) \in X; \operatorname{Im} x = 0\} \simeq M^{\mathbb{R}},$$

where $M^{\mathbb{R}}$ is the underlying real structure of M . We denote by $(z, x; \zeta, \xi)$ the coordinates of T^*X . We use the notation $D_z = \frac{\partial}{\partial z}$ and $D_x = \frac{\partial}{\partial x}$.

Around a point $(0, x^0; 0, i\eta^0) \in T^*(\mathbb{C} \times \mathbb{C}^n)$ with real x^0 and $\eta^0 \neq 0$, we construct a microfunction solution $v(z, x)$ with a holomorphic parameter z of

$$P(z, x, D_z, D_x)v(z, x) := \left(\sum_{k=0}^m A_k(z, x, D_z, D_x) D_z^{m-k} \right) v(z, x) = 0 \quad (1.1)$$

with ramified singularities along $\{z - \varphi(x, \xi) = 0\}$. Here $\varphi(x, \xi)$ is a holomorphic function of homogenous degree 0 with respect to ξ defined in a neighbourhood of $(0, x^0; 0, i\eta^0)$ with

$$\varphi(x^0, i\eta^0) = 0.$$

We suppose that $P(z, x, D_z, D_x)$ has Fuchsian singularities along $\{z = \varphi(x, \xi)\}$; that is each $A_k(z, x, D_z, D_x)$ is a microdifferential operator with $\operatorname{ord}(A_k) \leq 0$ and satisfies

$$\sigma_0(A_0)(z, x, 0, \xi) = z - \varphi(x, \xi) \quad \text{and} \quad \sigma_0(A_1)(0, x^0, 0, i\eta^0) \notin \{0, -1, -2, \dots\}. \quad (1.2)$$

Definition 1.1

A $Q(z, x, D_z, D_x)$ is called an m -th order microdifferential operator if there exists a formal symbol $\{Q_j(z, x, \zeta, \xi)\}_{j=-\infty}^m$ such that

$$Q(z, x, D_z, D_x) = \sum_{j=-\infty}^m Q_j(z, x, D_z, D_x). \quad (1.3)$$

Here, there exists a neighbourhood W of $(z^0, x^0; \zeta^0, \xi^0)$ in T^*X and a positive constant C such that each $Q_j(z, x, \zeta, \xi)$ is holomorphic on W , and homogeneous of degree j with respect to $(\zeta, \xi) \in \mathbb{C} \times \mathbb{C}^n$, and that we have

$$\sup_{(z, x; \zeta, \xi) \in W} |Q_j(z, x, \zeta, \xi)| \leq (-j)! C^{-j} \quad (-j \gg 1). \quad (1.4)$$

We denote by $\mathcal{E}_X$ the sheaf on T^*X of microdifferential operators above.

Definition 1.2

We denote by $\mathcal{CO}_M$ a subsheaf of $\mathcal{C}_{M^{\mathbb{R}}}$:

$$\mathcal{CO}_M = \{v(z, x) \in \mathcal{C}_{M^{\mathbb{R}}}; \bar{\partial}_z v(z, x) = 0\}. \quad (1.5)$$

We call a section of $\mathcal{CO}_M$ a microfunction in (z, x) with a holomorphic parameter z .

Before constructing the solutions of (1.1) we reduce P to a simpler microdifferential operator by using some quantized contact transformation preserving sheaf $\mathcal{CO}_M$. By the implicit function theorem $\sigma_0(A_0)(z, x, \zeta, \xi)$ is written as follows:

$$\sigma_0(A_0)(z, x, \zeta, \xi) = \alpha(z, x, \zeta, \xi)(z - \Phi(x, \zeta, \xi)),$$

where α and Φ are homogenous of degree 0 with respect to (ζ, ξ) and satisfying

$$\alpha(0, x^0, 0, i\eta^0) = 1 \quad \text{and} \quad \Phi(x, 0, \xi) = \varphi(x, \xi).$$

Therefore by applying $\alpha(z, x, D_z, D_x)^{-1}$ to both sides of (1.1) we can reduce P to the case that

$$\sigma_0(A_0)(z, x, \zeta, \xi) = z - \Phi(x, \zeta, \xi)$$

with the same condition (1.2).

Proposition 1.3.

There exists a holomorphic contact transformation

$$S: \begin{cases} z^* &= z - \Phi(x, \zeta, \xi) \\ x^* &= x^*(z, x, \zeta, \xi) \\ \zeta^* &= \zeta \\ \xi^* &= \xi^*(z, x, \zeta, \xi) \end{cases}$$

satisfying

$$x^*(z, x, 0, \xi) = x, \quad \xi^*(z, x, 0, \xi) = \xi.$$

[Proof]

Solve the following Cauchy problem for $\psi = \psi(x, \zeta^*, \xi^*)$

$$\begin{cases} \frac{\partial \psi}{\partial \zeta^*} + \Phi(x, \zeta^*, \xi^* + \frac{\partial \psi}{\partial x}) = 0, \\ \psi|_{\zeta^*=0} = 0. \end{cases}$$

Then a function

$$\chi(z, x, \zeta^*, \xi^*) = z\zeta^* + x \cdot \xi^* + \psi(x, \zeta^*, \xi^*)$$

generates the desired contact transformation S . $\square$

We note here that S preserves

$$T_M^* X = \{(z, x; \zeta, \xi) | \zeta = 0, \operatorname{Im} x = 0, \operatorname{Re} \xi = 0\}.$$

Hence there exists a quantized contact transformation

$$S : S^{-1} \mathcal{CO}_M \xrightarrow{\sim} \mathcal{CO}_M$$

such that

$$\begin{aligned} S \circ D_{z^*} \circ S^{-1} &= D_z, \\ S \circ z^* \circ S^{-1} &= z - \Phi(x, D_z, D_x). \end{aligned}$$

Therefore $S^{-1} \circ P \circ S$ gives a desired reduction of P . That is, we have

$$A_0(z, x, D_z, D_x) = z \tag{1.6}$$

under the same condition (1.2) with $\varphi = 0$. Hereafter we suppose this form (1.6) of A_0 .

We construct a solution $v(z, x) \in \mathcal{CO}_M$ around $\{z = 0\}$ of

$$P(z, x, D_z, D_x)v(z, x) = 0 \tag{1.7}$$

of the form

$$v(z, x) = U(z, x, D_x)f(x). \tag{1.8}$$

Here, $f(x)$ is any microfunction in x , and

$$U(z, x, D_x) = \sum_{j=-\infty}^0 u_j(z, x, D_x) \tag{1.9}$$

is a microdifferential operator commuting with z with ramified singularities along $\{z = 0\}$ and satisfying the following equation as a microdifferential operator :

$$P(z, x, D_z, D_x)U(z, x, D_x) = 0 \mod \mathcal{E}_X \cdot D_z. \tag{1.10}$$

Indeed, (1.10) is equivalent to some system of equations for formal symbols. However, here we use the method of successive approximation.

Let us introduce a fundamental Fuchsian ordinary differential operator by

$$L := \sum_{k=0}^m a_k(z, x, \xi) \partial_z^{m-k}, \tag{1.11}$$

where $a_k(z, x, \xi) = a_{k,0}(z, x, 0, \xi)$ for the homogeous expansion

$$A_k(z, x, D_z, D_x) = \sum_{j=-\infty}^0 a_{k,j}(z, x, D_z, D_x) \quad (1.12)$$

of microdifferential operator $A_k(z, x, D_z, D_x)$ in (1.1). Further we define an operation L and $\mathcal{L}$ on formal symbols

$$U(z, x, \xi) = \sum_{j=-\infty}^0 u_j(z, x, \xi) \quad (1.13)$$

by

$$LU(z, x, \xi) = \sum_{j=-\infty}^0 (Lu_j)(z, x, \xi) \quad (1.14)$$

and

$$\mathcal{L}U(z, x, \xi) = \sum_{j=-\infty}^0 \left(\sum_{0 \leq k \leq m, |r|+q=-j} \frac{1}{r!} \partial_\xi^r a_k(z, x, \xi) \partial_z^{m-k} \partial_x^r u_{-q}(z, x, \xi) \right). \quad (1.15)$$

Then, our successive approximation process is formulated as follows:

$$\begin{cases} LU_0 = 0 \\ LU_{k+1} = \{(L - \mathcal{L}) - R \circ\} U_k \quad (k = 0, 1, 2, \dots). \end{cases} \quad (1.16)$$

Here each U_k is a formal symbol of the form

$$U_k = \sum_{j=-\infty}^0 u_j^k(z, x, \xi), \quad (1.17)$$

($u_j^k(z, x, \xi)$ is the j -th degree homogeous part of U_k) and R is a microdifferential operator given by

$$R = \sum_{k=1}^m A'_k(z, x, D_z, D_x) D_z^{m-k}, \quad (1.18)$$

where

$$A'_k(z, x, D_z, D_x) \equiv \sum_{j=-\infty}^0 a'_{k,j}(z, x, D_z, D_x)$$

with

$$a'_{k,j}(z, x, \zeta, \xi) = a_{k,j}(z, x, \zeta, \xi) - \delta_{j0} \cdot a_{k,0}(z, x, 0, \xi).$$

Further $R \circ$ denotes the usual operator product mod $\mathcal{E}_x \cdot D_z$; that is,

$$R \circ U \equiv S(z, x, 0, D_x) \quad \text{when} \quad R(z, x, D_z, D_x)U(z, x, D_x) = S(z, x, D_z, D_x).$$

It is easy to see that the sum

$$U(z, x, D_x) = \sum_{k=0}^{\infty} U_k(z, x, D_x) \quad (1.19)$$

formally satisfies (1.9).

Therefore our problem is reduced to the following:

- (1) Can we find formal symbols U_k around $\{z = 0\}$ successively?
- (2) Does $\sum_{k=0}^{\infty} U_k(z, x, D_x)$ converge around $\{z = 0\}$ as a series of microdifferential operators?

In §2, we get suitable estimations along $\{z = 0\}$ for regular and ramified solutions of L , which are important for the successive construction of formal symbols $\{U_k\}$.

In §3, we introduce some formal norms with weight around $\{z = 0\}$, and obtain some a' pri-o'ri estimations for these formal norms.

In §4, we solve our reduced problems (1), (2) above. Therefore we succeed in constructing one ramified and $m - 1$ regular independent solutions around $\{z = 0\}$.

§2. Preliminaries

Let L be an m -th order ordinary differential operator of the form

$$L = \sum_{k=0}^m a_k(z) \partial_z^{m-k},$$

where $a_0(z) = z$ and each $a_k(z)$ is holomorphic in a neighbourhood of

$$D = \{z \in \mathbb{C}; |z| \leq 1\}.$$

For an $\varepsilon > 0$ we set

$$\Omega = \{z \in \mathbb{C}; 0 < |z| \leq 1, |\arg z| \leq \pi - \varepsilon\}.$$

We obtain estimations for solutions of

$$Lu = f \quad (2.1)$$

for two cases: Holomorphic functions $f(z)$ on D and also on Ω .

Notation

For a holomorphic function u in a neighbourhood of D , we define two norms as follows:

$$\|u\| = \sup_{|z| \leq 1} |u(z)|$$

$$\|u\|' = \sup_{|z| \leq 1, j=0, \dots, m} |u^{(j)}(z)|$$

and define another two norms with weight $\mu \in \mathbb{R}$

$$\|u\|_\mu = \sup_{z \in \Omega} |z|^\mu |u(z)|$$

$$\|u\|'_\mu = \sup_{z \in \Omega, j=0, \dots, m} |z|^{\mu-m+1+j} |u^{(j)}(z)|$$

for a holomorphic function $u(z)$ defined in a neighbourhood of Ω .

Theorem 2.1.

We suppose that $a_1(0) \neq 0, -1, -2, \dots$. Set

$$M = \max\{1, \sup_{z \in D} \sum_{k=1}^m |a_k(z)|\} < +\infty \quad (2.2)$$

and

$$\delta = \min\{|p + a_1(0)|; p = 0, 1, 2, \dots\} > 0. \quad (2.3)$$

Then we have a positive constant C depending only on M and δ , which satisfies the following estimations:

(1) *Regular case:* For a holomorphic function $f(z)$ in a neighbourhood of D any holomorphic solution $u(z)$ in a neighbourhood of D of (2.1) satisfies

$$\|u\|' \leq C\{\|f\| + |u(0)| + \dots + |u^{(m-2)}(0)|\}. \quad (2.4)$$

(2) *Non-regular case:* For a holomorphic function $f(z)$ in a neighbourhood of Ω any holomorphic solution $u(z)$ in a neighbourhood of Ω of (2.1) satisfies

$$\|u\|'_\mu \leq C\{\|f\|_\mu + |u(1)| + \dots + |u^{(m-1)}(1)|\} \quad (2.5)$$

with $\forall \mu \geq M + m + 1$.

Remark. It is well known by the theory of Fuchsian differential equations that under the assumption $a_1(0) \neq 0, -1, -2, \dots$ there exists a unique solution for any given $(u(0), \dots, u^{(m-2)}(0))$ or $(u(1), \dots, u^{(m-1)}(1))$ for both cases.

[Proof]

Put an $m \times m$ -matrix

$$A(z) = \begin{pmatrix} 0 & z & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & z \\ -a_m(z) & \dots & \dots & \dots & -a_1(z) \end{pmatrix},$$

and two m -dimensional vectors

$$\overrightarrow{x(z)} = \begin{pmatrix} u(z) \\ u'(z) \\ \vdots \\ u^{(m-1)}(z) \end{pmatrix}, \quad \overrightarrow{b(z)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(z) \end{pmatrix}.$$

Then, equation (2.1) reduces to

$$\frac{d\overrightarrow{x(z)}}{dz} = \frac{1}{z}A(z)\overrightarrow{x(z)} + \frac{1}{z}\overrightarrow{b(z)}. \quad (2.6)$$

Hence,

$$\overrightarrow{x(z)} = \overrightarrow{x(z_0)} + \int_{z_0}^z \frac{1}{s}A(s)\overrightarrow{x(s)}ds + \int_{z_0}^z \frac{1}{s}\overrightarrow{b(s)}ds. \quad (2.7)$$

Here we introduce the following norms for $m \times m$ matrix $X = (x_{i,j})_{i,j=1}^m$ and m -vector $\overrightarrow{x} = (x_i)_{i=1}^m$:

$$|X| \equiv \max_{i=1,\dots,m} \left(\sum_{j=1}^m |x_{i,j}| \right), \quad |\overrightarrow{x}| \equiv \max_{i=1,\dots,m} |x_i|.$$

Then we have an estimation

$$|A(z)| \leq M \quad \text{on } D.$$

We shall prove (1) after [Proof of (2)].

[Proof of (2)]

Firstly we put $z = e^{i\theta}$ and $z_0 = 1$ in (2.7) and we get the following integral inequality for $\theta \in [0, \pi - \varepsilon]$:

$$\begin{aligned} |\overrightarrow{x(e^{i\theta})}| &\leq |\overrightarrow{x(1)}| + \left| \int_1^{e^{i\theta}} \frac{1}{s}A(s)\overrightarrow{x(s)}ds \right| + \left| \int_1^{e^{i\theta}} \frac{1}{s}\overrightarrow{b(s)}ds \right| \\ &= |\overrightarrow{x(1)}| + \left| \int_0^\theta \frac{1}{e^{i\varphi}}A(e^{i\varphi})\overrightarrow{x(e^{i\varphi})}ie^{i\varphi}d\varphi \right| + \left| \int_0^\theta \frac{1}{e^{i\varphi}}\overrightarrow{b(e^{i\varphi})}ie^{i\varphi}d\varphi \right| \\ &\leq |\overrightarrow{x(1)}| + \int_0^\theta |A(e^{i\varphi})||\overrightarrow{x(e^{i\varphi})}|d\varphi + \int_0^\theta |\overrightarrow{b(e^{i\varphi})}|d\varphi \\ &\leq |\overrightarrow{x(1)}| + \int_0^\theta M|\overrightarrow{x(e^{i\varphi})}|d\varphi + \int_0^\theta |f(e^{i\varphi})|d\varphi \\ &\leq |\overrightarrow{x(1)}| + \pi\|f\|_\mu + \int_0^\theta M|\overrightarrow{x(e^{i\varphi})}|d\varphi. \end{aligned} \quad (2.8)$$

Secondly we put $z = re^{i\theta}$ and $z_0 = e^{i\theta}$ and we get the following integral inequality for $|\theta| \leq \pi - \varepsilon$ and $r \in (0, 1]$:

$$\begin{aligned}
|\overrightarrow{x(re^{i\theta})}| &\leq |\overrightarrow{x(e^{i\theta})}| + \left| \int_{e^{i\theta}}^{re^{i\theta}} \frac{1}{s} A(s) \overrightarrow{x(s)} ds \right| + \left| \int_{e^{i\theta}}^{re^{i\theta}} \frac{1}{s} \overrightarrow{b(s)} ds \right| \\
&= |\overrightarrow{x(e^{i\theta})}| + \left| \int_1^r \frac{1}{se^{i\theta}} A(se^{i\theta}) \overrightarrow{x(se^{i\theta})} e^{i\theta} ds \right| + \left| \int_1^r \frac{1}{se^{i\theta}} \overrightarrow{b(se^{i\theta})} e^{i\theta} ds \right| \\
&\leq |\overrightarrow{x(e^{i\theta})}| + \int_r^1 \frac{1}{s} |A(se^{i\theta})| |\overrightarrow{x(se^{i\theta})}| ds + \int_r^1 \frac{1}{s} |\overrightarrow{b(se^{i\theta})}| ds \\
&\leq |\overrightarrow{x(e^{i\theta})}| + \int_r^1 \frac{M}{s} |\overrightarrow{x(se^{i\theta})}| ds + \int_r^1 \frac{|f(se^{i\theta})|}{s} ds \\
&\leq |\overrightarrow{x(e^{i\theta})}| + \int_r^1 \frac{M}{s} |\overrightarrow{x(se^{i\theta})}| ds + \int_r^1 \frac{\|f\|_\mu}{s^{\mu+1}} ds \\
&\leq |\overrightarrow{x(e^{i\theta})}| + \frac{r^{-\mu} - 1}{\mu} \|f\|_\mu + \int_r^1 \frac{M}{s} |\overrightarrow{x(se^{i\theta})}| ds. \tag{2.9}
\end{aligned}$$

We prepare next Lemma:

Lemma 2.2 (Gronwall).

Let $f(t), g(t), h(t)$ be non-negative valued continuous functions defined on $[a, b]$. If they satisfy

$$f(t) \leq g(t) + \int_a^t h(s) f(s) ds \quad \text{for } \forall t \in [a, b],$$

then we have

$$f(t) \leq g(t) + \int_a^t g(s) h(s) \exp\left(\int_s^t h(r) dr\right) ds \quad \text{for } \forall t \in [a, b].$$

[Proof of Lemma]

We put

$$H(t) = \int_a^t h(s) f(s) ds,$$

then we get

$$\frac{dH(t)}{dt} = h(t) f(t) \leq h(t) \{g(t) + H(t)\} = h(t) g(t) + h(t) H(t).$$

That is,

$$\frac{dH(t)}{dt} - h(t) H(t) \leq h(t) g(t).$$

Multiplying both sides by $\exp\left(-\int_a^t h(s) ds\right)$, we obtain

$$\frac{d}{dt} \left[H(t) \exp\left(-\int_a^t h(s) ds\right) \right] \leq h(t) g(t) \exp\left(-\int_a^t h(s) ds\right).$$

Integrating both sides from a to t , we have

$$H(t) \exp \left(- \int_a^t h(s) ds \right) \leq \int_a^t h(s) g(s) \exp \left(- \int_a^s h(r) dr \right) ds.$$

Therefore,

$$H(t) \leq \int_a^t h(s) g(s) \exp \left(\int_s^t h(r) dr \right) ds.$$

Combining these inequalities, we get

$$f(t) \leq g(t) + \int_a^t h(s) g(s) \exp \left(\int_s^t h(r) dr \right) ds. \quad \square$$

Applying Lemma 2.2 to (2.8), we obtain

$$\begin{aligned} |\overrightarrow{x(e^{i\theta})}| &\leq |\overrightarrow{x(1)}| + \pi \|f\|_\mu + \int_0^\theta \{|\overrightarrow{x(1)}| + \pi \|f\|_\mu\} M \exp \left(\int_\varphi^\theta M dr \right) d\varphi \\ &= |\overrightarrow{x(1)}| + \pi \|f\|_\mu + \{|\overrightarrow{x(1)}| + \pi \|f\|_\mu\} \int_0^\theta M e^{M(\theta-\varphi)} d\varphi \\ &= |\overrightarrow{x(1)}| + \pi \|f\|_\mu + \{|\overrightarrow{x(1)}| + \pi \|f\|_\mu\} e^{M\theta} (-e^{-M\theta} + 1) \\ &\leq e^{M\pi} \{|\overrightarrow{x(1)}| + \pi \|f\|_\mu\}. \end{aligned} \quad (2.10)$$

It is easy to see that the conclusion of (2.10) is valid also for $\theta \in [-\pi + \varepsilon, 0]$.

Applying Lemma 2.2 to (2.9) for $\mu \geq M + m + 1$, we obtain

$$\begin{aligned} |\overrightarrow{x(re^{i\theta})}| &\leq |\overrightarrow{x(e^{i\theta})}| + \frac{r^{-\mu} - 1}{\mu} \|f\|_\mu + \int_r^1 \left\{ |\overrightarrow{x(e^{i\theta})}| + \frac{t^{-\mu} - 1}{\mu} \|f\|_\mu \right\} \frac{M}{t} \exp \left(\int_r^t \frac{M}{s} ds \right) dt \\ &\leq |\overrightarrow{x(e^{i\theta})}| + \frac{r^{-\mu}}{\mu} \|f\|_\mu + \int_r^1 \frac{M}{r^M} \left\{ |\overrightarrow{x(e^{i\theta})}| t^{M-1} + \frac{\|f\|_\mu}{\mu} t^{M-\mu-1} \right\} dt \\ &\leq r^{-M} |\overrightarrow{x(e^{i\theta})}| + \frac{r^{-\mu}}{\mu - M} \|f\|_\mu \leq r^{-M} |\overrightarrow{x(e^{i\theta})}| + r^{-\mu} \|f\|_\mu. \end{aligned} \quad (2.11)$$

Combining (2.11) with (2.10), we have

$$\begin{aligned} |u^{(m)}(z)| &= |z|^{-1} | -a_1(z) u^{(m-1)}(z) - \dots - a_m(z) u(z) + f(z) | \\ &\leq |z|^{-\mu-1} M (1 + \pi e^{M\pi}) (\|f\|_\mu + |\overrightarrow{x(1)}|). \end{aligned}$$

Further

$$|u^{(m-1)}(z)| \leq |\overrightarrow{x(z)}| \leq |z|^{-\mu} (1 + \pi e^{M\pi}) (\|f\|_\mu + |\overrightarrow{x(1)}|),$$

and so

$$\begin{aligned}
|u^{(m-2)}(z)| &\leq |u^{(m-1)}(e^{i\theta})| + \int_r^1 |u^{(m-1)}(se^{i\theta})| ds \\
&\leq (1 + \pi e^{M\pi}) (\|f\|_\mu + |\overrightarrow{x(1)}|) \left(1 + \frac{r^{1-\mu} - 1}{\mu - 1}\right) \\
&\leq r^{1-\mu} (1 + \pi e^{M\pi}) (\|f\|_\mu + |\overrightarrow{x(1)}|).
\end{aligned}$$

Since $\mu \geq m + 1$, we can repeat this process $m - 1$ times. Therefore we have

$$|u^{(j)}(z)| \leq |z|^{-\mu+m-j-1} M (1 + \pi e^{M\pi}) (\|f\|_\mu + |\overrightarrow{x(1)}|) \quad \text{for } j = 0, \dots, m.$$

Hence the inequality (2.5) holds for $C = M(1 + \pi e^{M\pi})$.

[Proof of (1)]

In equation (2.6), we expand all the functions into power series with center 0:

$$A(z) = \sum_{p=0}^{\infty} A_p z^p, \quad \overrightarrow{b(z)} = \sum_{p=0}^{\infty} \overrightarrow{b_p} z^p, \quad \overrightarrow{x(z)} = \sum_{p=0}^{\infty} \overrightarrow{x_p} z^p.$$

Hence we have the following equations for the coefficients:

$$(p - A_0) \overrightarrow{x_p} = \sum_{q=1}^p A_q \overrightarrow{x_{p-q}} + \overrightarrow{b_p} \quad (2.12)$$

for $\forall p = 0, 1, 2, \dots$. Here we note that

$$\det(p - A_0) = p^{m-1} (p + a_1(0)) \neq 0$$

for $\forall p \geq 1$. Therefore we get for $\forall p \geq 1$ that

$$|\overrightarrow{x_p}| = \left| (p - A_0)^{-1} \left(\sum_{q=1}^p A_q \overrightarrow{x_{p-q}} + \overrightarrow{b_p} \right) \right| \leq |(p - A_0)^{-1}| \left(\sum_{q=1}^p |A_q| |\overrightarrow{x_{p-q}}| + |\overrightarrow{b_p}| \right).$$

Since A_q is written as integration of $z^{-q-1} A(z)$ on the unit circle, we have estimations

$$|A_q| \leq \sup_{z \in D} |A(z)| \leq M, \quad |\overrightarrow{b_q}| \leq \|f\|$$

for every q . Therefore we obtain

$$|\overrightarrow{x_p}| \leq |(p - A_0)^{-1}| \left(\sum_{q=1}^p M |\overrightarrow{x_{p-q}}| + \|f\| \right) \quad (\forall p \geq 1). \quad (2.13)$$

On the other hand $(p - A_0)^{-1}$ is given by

$$(p - A_0)^{-1} = \frac{1}{p(p + a_1(0))} \begin{pmatrix} p + a_1(0) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & p + a_1(0) & 0 \\ -a_m(0) & \cdots & \cdots & -a_2(0) & p \end{pmatrix}$$

for $p \geq 1$, and so

$$\begin{aligned} |(p - A_0)^{-1}| &\leq \max \left\{ \frac{1}{p}, \frac{p + |a_2(0)| + \cdots + |a_m(0)|}{p|p + a_1(0)|} \right\} \\ &\leq \max \left\{ 1, \frac{p + M}{|p + a_1(0)|} \right\} \\ &\leq \max \left\{ 1, \frac{3M}{\delta}, \sup_{p \geq 2M} \frac{p + M}{p - M} \right\} \\ &\leq \max \left\{ \frac{3M}{\delta}, 3 \right\} \leq 3 \left(1 + \frac{M}{\delta} \right) =: K. \end{aligned}$$

Hence,

$$|\vec{x}_p| \leq K \left(M \sum_{q=0}^{p-1} |\vec{x}_q| + \|f\| \right) \quad (\forall p \geq 1). \quad (2.14)$$

Therefore, putting

$$y_p = \sum_{q=0}^p |\vec{x}_q|,$$

we have an estimation

$$y_p \leq (KM + 1)y_{p-1} + K\|f\| \leq \frac{(KM + 1)^p - 1}{M} \|f\| + (KM + 1)^p |\vec{x}_0|$$

for $\forall p \geq 1$, and so

$$|\vec{x}_p| \leq y_p \leq (KM + 1)^p \left(\frac{\|f\|}{M} + |\vec{x}_0| \right) \quad \text{for } \forall p \geq 1. \quad (2.15)$$

Further from

$$-A_0 \vec{x}_0 = \vec{b}_0$$

we obtain that

$$|u^{(m-1)}(0)| \leq \frac{1}{\delta} \left\{ |f(0)| + M(|u(0)| + \cdots + |u^{(m-2)}(0)|) \right\}.$$

Hence

$$|\vec{x}_0| \leq \frac{1}{\delta} \|f\| + K(|u(0)| + \cdots + |u^{(m-2)}(0)|).$$

Consequently

$$|\overrightarrow{x_p}| \leq (KM + 1)^p \left\{ \left(\frac{1}{M} + \frac{1}{\delta} \right) \|f\| + K(|u(0)| + \dots + |u^{(m-2)}(0)|) \right\}$$

for $\forall p \geq 0$, and so we have

$$\sup \left\{ |\overrightarrow{x(z)}|; |z| \leq \frac{1}{2(KM + 1)} \right\} \leq 2 \left(K + \frac{1}{M} + \frac{1}{\delta} \right) \left(\|f\| + |u(0)| + \dots + |u^{(m-2)}(0)| \right). \quad (2.16)$$

Putting $\sigma = 1/\{2(MK + 1)\} < 1$, we get an integral inequality similar to (2.9):

$$|\overrightarrow{x(re^{i\theta})}| \leq |\overrightarrow{x(\sigma e^{i\theta})}| + \int_{\sigma}^r \frac{M}{s} |\overrightarrow{x(se^{i\theta})}| ds + \int_{\sigma}^r \frac{\|f\|}{s} ds$$

for any $r \in [\sigma, 1]$. By Gronwall's inequality and (2.16) we get

$$\begin{aligned} |\overrightarrow{x(re^{i\theta})}| &\leq \left(\frac{r}{\sigma} \right)^M \left\{ 2 \left(K + \frac{1}{M} + \frac{1}{\delta} \right) + \log \frac{1}{\sigma} \right\} \\ &\quad \times \left(\|f\| + |u(0)| + \dots + |u^{(m-2)}(0)| \right) \end{aligned}$$

for any $r \in [\sigma, 1]$. Therefore

$$\sup_{z \in D} |\overrightarrow{x(z)}| \leq \left(\frac{1}{\sigma} \right)^M \left\{ 2 \left(K + \frac{1}{M} + \frac{1}{\delta} \right) + \log \frac{1}{\sigma} \right\} \left(\|f\| + |u(0)| + \dots + |u^{(m-2)}(0)| \right).$$

Note that

$$\sup_{z \in D} |u^{(m)}(z)| = \sup_{|z|=1} \left| \frac{-a_1(z)u^{(m-1)}(z) - \dots - a_m(z)u(z)}{z} \right| \leq M \sup_{z \in D} |\overrightarrow{x(z)}|.$$

Therefore since $M \geq 1$,

$$\|u\|' \leq M \sup_{z \in D} |\overrightarrow{x(z)}| \leq C \left(\|f\| + |u(0)| + \dots + |u^{(m-2)}(0)| \right)$$

with

$$C = M \left\{ 2(KM + 1) \right\}^M \left[2 \left(K + \frac{1}{M} + \frac{1}{\delta} \right) + \log \{2(KM + 1)\} \right] \quad (2.17)$$

and

$$K = 3 \left(1 + \frac{M}{\delta} \right).$$

This completes the proof of **Theorem 2.1**.

§3. Estimations of Formal Symbols

We take $U, L, \mathcal{L}, R$ defined in §1. Hereafter considering a suitable scale transformation in z , we may assume that each $A_k(z, x, D_z, D_x)$ is defined in a conic neighbourhood of

$$\{z \in \mathbb{C}; |z| \leq 1\} \times (x^0, i\eta^0).$$

To show the convergence of series of formal symbols $\sum_{k=0}^{\infty} U_k(z, x, \xi)$, we introduce 2 types of formal norms, which are similar to Boutet-de-Monvel and Kree's one.

(1) Regular type: When each component $u_j(z, x, \xi)$ of U is holomorphic in a neighbourhood of $\{|z| \leq 1\}$, we define a formal power series $N_m(U; X)$ in X with parameters x, ξ by

$$N_m(U; X) \equiv \sum_{p, \alpha, \beta, l} \frac{p! C^{p+l+|\alpha+\beta|} X^{2p+l+|\alpha+\beta|}}{(p+l+|\alpha|)!(p+|\beta|)!} \max_{0 \leq j \leq m} \|\partial_z^{j+l} \partial_x^\alpha \partial_\xi^\beta u_{-p}\|. \quad (3.1)$$

(2) Non-regular type: When each component $u_j(z, x, \xi)$ of U is holomorphic in a neighbourhood of

$$\Omega = \{z \in \mathbb{C}; 0 < |z| \leq 1, |\arg z| \leq \pi - \varepsilon\},$$

we define a formal power series $N_m^\mu(U; X)$ in X with parameters x, ξ by

$$N_m^\mu(U; X) \equiv \begin{cases} \sum_{p, \alpha, \beta, l} \frac{p! C^{p+l+|\alpha+\beta|} X^{2p+l+|\alpha+\beta|}}{(p+l+|\alpha|)!(p+|\beta|)!} \max_{0 \leq j \leq m} \|\partial_z^{j+l} \partial_x^\alpha \partial_\xi^\beta u_{-p}\|_{\mu+j+l+|\alpha+\beta|+p-m+1} & (m \geq 1) \\ \sum_{p, \alpha, \beta, l} \frac{p! C^{p+l+|\alpha+\beta|} X^{2p+l+|\alpha+\beta|}}{(p+l+|\alpha|)!(p+|\beta|)!} \|\partial_z^l \partial_x^\alpha \partial_\xi^\beta u_{-p}\|_{\mu+l+|\alpha+\beta|+p} & (m = 0). \end{cases} \quad (3.2)$$

Further, when each component $u_j(x, \xi)$ is not depending on z , we define

$$K(U; X) \equiv \sum_{p, \alpha, \beta} \frac{p! C^{p+|\alpha+\beta|} X^{2p+|\alpha+\beta|}}{(p+|\alpha|)!(p+|\beta|)!} |\partial_x^\alpha \partial_\xi^\beta u_{-p}|. \quad (3.3)$$

In the approximation process (1.15), we need an a priori estimation for $N_m(U_k; X)$ or $N_m^\mu(U_k; X)$. For this purpose, in the symbol equation

$$LU = F \equiv \sum_{p=0}^{\infty} f_{-p} \quad (3.4)$$

we estimate $N_m(U; X)$ by $N_0(F; X)$ and $\sum_{j=0}^{m-2} K(\partial_z^j U(0, x, \xi); X)$; or we estimate

$N_m^\mu(U; X)$ by $N_0^\mu(F; X)$ and $\sum_{j=0}^{m-1} K(\partial_z^j U(1, x, \xi); X)$.

To derive such estimations we apply $\partial_z^l \partial_x^\alpha \partial_\xi^\beta$ to both sides of $Lu_{-p} = f_{-p}$. Then we obtain

$$\begin{aligned} L(\partial_z^l \partial_x^\alpha \partial_\xi^\beta u_{-p}) &= \partial_z^l \partial_x^\alpha \partial_\xi^\beta f_{-p} \\ &- \sum_{l', l'', \alpha', \alpha'', \beta', \beta''} \sum_{k=0}^m \binom{l}{l'} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \partial_z^{l'} \partial_x^{\alpha'} \partial_\xi^{\beta'} a_k \cdot \partial_z^{l''+k} \partial_x^{\alpha''} \partial_\xi^{\beta''} u_{-p} \\ &\quad (l = l' + l'', \alpha = \alpha' + \alpha'', \beta = \beta' + \beta'', (l', \alpha', \beta') \neq 0). \end{aligned}$$

Here we employ **Theorem 2.1**. For a sufficiently small $\varepsilon > 0$ we set

$$M_\varepsilon = \max \left\{ 1, \sup_{|z| \leq 1+\varepsilon, (x, \xi) \in V_\varepsilon} \sum_{k=1}^m |a_k(z, x, \xi)| \right\} < +\infty$$

and

$$\delta_\varepsilon = \inf \{ |p + a_1(0, x, \xi)|; p = 0, 1, 2, \dots, (x, \xi) \in V_\varepsilon \} > 0$$

with

$$V_\varepsilon = \{(x, \xi) \in \mathbb{C}^n \times \mathbb{C}^n; |x - x^0| \leq \varepsilon, |\xi/|\xi| - i\eta^0/|\eta^0|| \leq \varepsilon\}.$$

Then there exists a positive constant C_0 depending only on M_ε and δ_ε , which satisfies some estimations (2.4), (2.5) for

$$L = \sum_{k=0}^m a_k(z, x, \xi) \partial_z^{m-k}.$$

In particular we have the following estimation on $|\xi| = 1$:

$$|\partial_z^l \partial_x^\alpha \partial_\xi^\beta a_k(z, x, \xi)| \leq l! \alpha! \beta! \left(\frac{2}{\varepsilon} \right)^{l+|\alpha|+|\beta|} M_\varepsilon \quad (|z| \leq 1, (x, \xi) \in V_{\varepsilon/2}).$$

Hereafter we fix a $(x, \xi) \in V_{\varepsilon/2}$ and set

$$C_1 = \max \left\{ M_\varepsilon, \frac{2}{\varepsilon} \right\}.$$

(1) Regular type case:

$$\begin{aligned} \max_{0 \leq j \leq m} \|\partial_z^{j+l} \partial_x^\alpha \partial_\xi^\beta u_{-p}\| &\leq C_0 \left(\|\partial_z^l \partial_x^\alpha \partial_\xi^\beta f_{-p}\| + \sum_{j=0}^{m-2} |\partial_z^{j+l} \partial_x^\alpha \partial_\xi^\beta u_{-p}(0, x, \xi)| \right. \\ &\quad \left. + (m+1) \sum_{(l', \alpha', \beta') \neq 0} \binom{l}{l'} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} l'! \alpha'! \beta'! C_1^{l'+|\alpha'|+|\beta'|+1} \max_{0 \leq j \leq m} \|\partial_z^{j+l''} \partial_x^{\alpha''} \partial_\xi^{\beta''} u_{-p}\| \right). \end{aligned}$$

Then, we obtain

$$\begin{aligned} N_m(U; X) &\ll C_0 \left\{ N_0(F; X) + \sum_{j=0}^{m-2} K(\partial_z^j U(0, x, \xi); X) + C(m-1)X \cdot N_m(U; X) \right. \\ &\quad \left. + (m+1)C_1 N_m(U; X) \sum_{(l', \alpha', \beta') \neq 0} (C_1 C X)^{l'+|\alpha'+\beta'|} \right\}. \end{aligned}$$

That is, letting

$$\psi(X) \equiv \sum_{(l', \alpha', \beta') \neq 0} (C_1 C X)^{l' + |\alpha' + \beta'|} \quad (3.5)$$

and

$$\Phi(X) \equiv \frac{C_0}{1 - (m+1)C_0 C_1 \psi(X) - (m-1)C_0 C X}, \quad (3.6)$$

we get the following proposition:

Proposition 3.1. *If each component of F and U is holomorphic on a neighbourhood of $\{|z| \leq 1\}$, we have on $|\xi| = 1$*

$$N_m(U; X) \ll \Phi(X) \left\{ N_0(F; X) + \sum_{j=0}^{m-2} K(\partial_z^j U(0, x, \xi); X) \right\}. \quad (3.7)$$

(2) Non-regular type case:

For $m \geq 1$, we obtain

$$\begin{aligned} & \max_{0 \leq j \leq m} \|\partial_z^{j+l} \partial_x^\alpha \partial_\xi^\beta u_{-p}\|_{\mu+j+l+p+|\alpha+\beta|-m+1} \\ & \leq C_0 \left\{ \|\partial_z^l \partial_x^\alpha \partial_\xi^\beta f_{-p}\|_{\mu+l+|\alpha+\beta|+p} + \sum_{j=0}^{m-1} |\partial_z^{j+l} \partial_x^\alpha \partial_\xi^\beta u_{-p}(1, x, \xi)| \right. \\ & \quad + (m+1) \sum_{(l', \alpha', \beta') \neq 0} \binom{l}{l'} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} l'! \alpha'! \beta'! C_1^{l'+|\alpha'+\beta'|+1} \\ & \quad \left. \times \max_{0 \leq j \leq m} \|\partial_z^{j+l''} \partial_x^{\alpha''} \partial_\xi^{\beta''} u_{-p}\|_{\mu+l+|\alpha+\beta|+p} \right\}. \end{aligned}$$

Since $\mu + l + |\alpha + \beta| + p \geq \mu + j + l'' + |\alpha'' + \beta''| + p - m + 1$, we obtain

$$\max_{j=0, \dots, m} \|\partial_z^{j+l''} \partial_x^{\alpha''} \partial_\xi^{\beta''} u_{-p}\|_{\mu+l+|\alpha+\beta|+p} \leq \max_{j=0, \dots, m} \|\partial_z^{j+l''} \partial_x^{\alpha''} \partial_\xi^{\beta''} u_{-p}\|_{\mu+j+l''+|\alpha''+\beta''|+p-m+1}.$$

In the same way as the regular type case, we obtain the following proposition:

Proposition 3.2. *If each component of F and U is holomorphic on a neighbourhood of Ω , we have on $|\xi| = 1$*

$$N_m^\mu(U; X) \ll \Phi(X) \left\{ N_0^\mu(F; X) + \sum_{j=0}^{m-1} K(\partial_z^j U(1, x, \xi); X) \right\} \quad (3.8)$$

with $\forall \mu \geq M_\varepsilon + m + 1$.

In the last part of this section we estimate the formal norms of the remaining terms

$$(\mathcal{L} - L)U \quad \text{and} \quad R \circ U$$

by those of U .

Proposition 3.3.

Set

$$\psi_1(X) \equiv \sum_{l'=0}^{\infty} (CC_1X)^{l'} \sum_{\alpha'} (CC_1X)^{|\alpha'|} \sum_{\beta'} (2CC_1X)^{|\beta'|} \sum_{|r| \geq 1} C_1(2C_1X)^{|r|}. \quad (3.9)$$

(1) *Regular type case:* If each component of U is holomorphic on a neighbourhood of $\{|z| \leq 1\}$, we have on $|\xi| = 1$

$$N_0((\mathcal{L} - L)U; X) \ll \psi_1(X) N_m(U; X). \quad (3.10)$$

(2) *Non-regular type case:* If each component of U is holomorphic on a neighbourhood of Ω , we have on $|\xi| = 1$

$$N_0^\mu((\mathcal{L} - L)U; X) \ll \psi_1(X) N_m^\mu(U; X) \quad (3.11)$$

with $\forall \mu \geq M_\varepsilon + m + 1$.

[Proof]

(1) Regular type case:

$$\begin{aligned} N_0((\mathcal{L} - L)U; X) &= \sum_{p, \alpha, \beta, l} \frac{p! C^{p+l+|\alpha+\beta|} X^{2p+l+|\alpha+\beta|}}{(p+l+|\alpha|)!(p+|\beta|)!} \\ &\quad \times \left\| \partial_z^l \partial_x^\alpha \partial_\xi^\beta \left(\sum_{p=|r|+q, |r|>0, k=0, \dots, m} \frac{1}{r!} \partial_\xi^r a_k \cdot \partial_z^{m-k} \partial_x^r u_{-q} \right) \right\| \\ &\ll \sum_{l', l'', \alpha', \alpha'', \beta', \beta'', q, |r|>0} \frac{p! C^{p+l+|\alpha+\beta|} X^{2p+l+|\alpha+\beta|}}{(p+l+|\alpha|)!(p+|\beta|)!} \frac{1}{r!} \binom{l}{l'} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} l'! \alpha'! (\beta' + r)! \\ &\quad \times C_1^{|\alpha'+\beta'+r|+l'+1} \max_{j=0, \dots, m} \left\| \partial_z^{j+l''} \partial_x^{\alpha''+r} \partial_\xi^{\beta''} u_{-q} \right\| \\ &\ll \dots \ll N_m(U; X) \left\{ \sum_{l'=0}^{\infty} (CC_1X)^{l'} \sum_{\alpha'} (CC_1x)^{|\alpha'|} \sum_{\beta'} (2CC_1X)^{|\beta'|} \sum_{|r| \geq 1} C_1(2C_1X)^{|r|} \right\} \\ &= N_m(U; X) \psi_1(X). \end{aligned}$$

(2) Non-regular type case:

$$\begin{aligned} N_0^\mu((\mathcal{L} - L)U; X) &\ll \sum_{l', l'', \alpha', \alpha'', \beta', \beta'', q, |r|>0} \frac{p! C^{p+l+|\alpha+\beta|} X^{2p+l+|\alpha+\beta|}}{(p+l+|\alpha|)!(p+|\beta|)!} \frac{1}{r!} \binom{l}{l'} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \\ &\quad \times l'! \alpha'! (\beta' + r)! C_1^{1+l'+|\alpha'+\beta'+r|} \max_{j=0, \dots, m-1} \left\| \partial_z^{j+l''} \partial_x^{\alpha''+r} \partial_\xi^{\beta''} u_{-q} \right\|_{\mu+p+l+|\alpha+\beta|}, \end{aligned}$$

where we use the fact that $\partial_\xi^r a_0 \equiv 0$ for any $|r| > 0$.

Since $\mu + p + l + |\alpha + \beta| \geq \mu + j + l'' + |\alpha'' + r| + |\beta''| + q - m + 1$, we have

$$\max_{j=0,\dots,m-1} \|\cdot\|_{\mu+p+l+|\alpha+\beta|} \leq \max_{j=0,\dots,m} \|\cdot\|_{\mu+j+l''+|\alpha''+r|+|\beta''|+q-(m-1)}.$$

Therefore the same argument as in the regular type case leads to the conclusion

$$N_0^\mu((\mathcal{L} - L)U; X) \ll \psi_1(X) N_m^\mu(U; X). \quad \square$$

Note that

$$R \circ U = \sum_{k=1}^m A'_k \circ (\partial_z^{m-k} U), \quad (3.12)$$

where

$$A'_k = \sum_{p=0}^{\infty} a'_{k,-p}(z, x, \zeta, \xi) \quad (3.13)$$

are microdifferential operators of $\text{ord}(A'_k) \leq 0$ defined in a neighbourhood of $\{|z| \leq 1\} \times (x^0; i\eta^0)$ satisfying

$$a'_{k,0}(z, x, 0, \xi) = 0 \quad \forall k.$$

Moreover, there exists a constant $C_2 > 0$ such that on

$$\left\{ |z| \leq 1, |x - x^0| \leq \varepsilon, |\zeta| \leq \varepsilon|\xi|, |\xi/|\xi| - i\eta^0/|\eta^0|| \leq \varepsilon \right\} \cap \{|\xi| = 1\}$$

we have

$$|\partial_z^l \partial_x^\alpha \partial_\zeta^s \partial_\xi^\beta a'_{k,-t}(z, x, \zeta, \xi)| \leq t!l!\alpha!\beta!s!C_2^{l+s+|\alpha+\beta|+t}. \quad (3.14)$$

By the similar argument due to Boutet-de-Monvel and Kree we get the following estimation

Lemma 3.4.

There exists a convergent majorant series $\psi_2(X)$ with $\psi_2(0) = 0$ depending only on C, C_2 and n such that on $|\xi| = 1$

(1) *Regular type case:*

$$N_0(A'_k \circ U; X) \ll \psi_2(X) N_0(U; X), \quad (3.15)$$

(2) *Non-regular type case:*

$$N_0^\mu(A'_k \circ U; X) \ll \psi_2(X) N_0^\mu(U; X) \quad (3.16)$$

with $\forall \mu \geq M_\varepsilon + m + 1$.

Proposition 3.5.

We have the following estimation on $|\xi| = 1$:

(1) *Regular type case:*

$$N_0(R \circ U; X) \ll m\psi_2(X) N_m(U; X), \quad (3.17)$$

(2) *Non-regular type case:*

$$N_0^\mu(R \circ U; X) \ll m\psi_2(X) N_m^\mu(U; X) \quad (3.18)$$

with $\forall \mu \geq M_\varepsilon + m + 1$.

§4. Construction of Solutions

We consider the following relation:

$$\begin{cases} LU_0 = 0 \\ LU_{k+1} = (L - \mathcal{L})U_k - R \circ U_k \quad (k = 0, 1, 2, \dots). \end{cases}$$

(1) Regular solution

We obtain

$$\begin{aligned} N_m(U_{k+1}; X) &\ll \Phi(X) \left\{ N_0((L - \mathcal{L})U_k - R \circ U_k; X) + \sum_{j=0}^{m-2} K(\partial_z^j U_{k+1}(0, x, \xi); X) \right\} \\ &\ll \Phi(X) \left\{ N_0(R \circ U_k; X) + N_0((\mathcal{L} - L)U_k; X) + \sum_{j=0}^{m-2} K(\partial_z^j U_{k+1}(0, x, \xi); X) \right\} \\ &\ll \Phi(X) \left\{ m\psi_2(X)N_m(U_k; X) + \psi_1(X)N_m(U_k; X) + \sum_{j=0}^{m-2} K(\partial_z^j U_{k+1}(0, x, \xi); X) \right\} \\ &= \Phi(X) \left\{ (\psi_1(X) + m\psi_2(X))N_m(U_k; X) + \sum_{j=0}^{m-2} K(\partial_z^j U_{k+1}(0, x, \xi); X) \right\}. \end{aligned}$$

With the condition $U_{k+1}(0) = \dots = U_{k+1}^{(m-2)}(0) = 0$, we obtain

$$\begin{aligned} N_m(U_{k+1}; X) &\ll \{\Phi(X)(\psi_1(X) + m\psi_2(X))\}N_m(U_k; X) \\ &\ll \dots \ll \{\Phi(X)(\psi_1(X) + m\psi_2(X))\}^{k+1}N_m(U_0; X). \end{aligned}$$

That is,

$$N_m\left(\sum_{k=0}^{\infty} U_k; X\right) \ll \sum_{k=0}^{\infty} N_m(U_k; X) \ll \sum_{k=0}^{\infty} \{\Phi(X)(\psi_1(X) + m\psi_2(X))\}^{k+1} N_m(U_0; X).$$

Since $LU_0 = 0$, we obtain

$$\begin{aligned} N_m(U_0; X) &\ll \Phi(X) \left\{ N_0(0; X) + \sum_{j=0}^{m-2} K(\partial_z^j U_0(0, x, \xi); X) \right\} \\ &= \Phi(X) \left\{ 0 + \sum_{j=0}^{m-2} K(\partial_z^j U_0(0, x, \xi); X) \right\} < +\infty. \end{aligned}$$

Therefore, we get

$$N_m\left(\sum_{k=0}^{\infty} U_k; X\right) \ll \sum_{k=0}^{\infty} \{\Phi(X)(\psi_1(X) + m\psi_2(X))\}^k N_m(U_0; X) < +\infty,$$

that is, $\sum_{k=0}^{\infty} U_k$ is convergent in N_m norm.

(2) Non-regular solution

As the same as regular's case, we obtain

$$N_m^\mu(U_{k+1}; X) \ll \Phi(X) \left\{ (\psi_1(X) + m\psi_2(X)) N_m^\mu(U_k; X) + \sum_{j=0}^{m-1} K(\partial_z^j U_{k+1}(1, x, \xi); X) \right\}.$$

Let U_0 be non-regular function and homogeneous of degree 0 with respect to ξ . From now, we solve a Cauchy problem;

$$\begin{cases} LU_{k+1} = (L - \mathcal{L})U_k - R \circ U_k \\ \partial_z^j U_{k+1}(1, x, \xi) = 0 \quad (j = 0, 1, \dots, m-1). \end{cases}$$

Then, we obtain

$$\begin{aligned} N_m^\mu(U_{k+1}; X) &\ll \{ \Phi(X)(\psi_1(X) + m\psi_2(X)) \} N_m^\mu(U_k; X) \\ &\ll \dots \ll \{ \Phi(X)(\psi_1(X) + m\psi_2(X)) \}^{k+1} N_m^\mu(U_0; X). \end{aligned}$$

As same as the regular case, we obtain

$$N_m^\mu\left(\sum_{k=0}^{\infty} U_k; X\right) \ll \sum_{k=0}^{\infty} \{ \Phi(X)(\psi_1(X) + m\psi_2(X)) \}^k N_m^\mu(U_0; X) < +\infty,$$

that is, $\sum_{k=0}^{\infty} U_k$ is convergent in N_m^μ norm.

Thus we obtained $m-1$ regular solutions and a non-regular solution, which span the full solutions of $(\mathcal{L} + R \circ)U = 0$.

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