

CAP automorphic representations of low rank groups *

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Abstract

In this talk, I report my recent joint work with K. Konno on non-tempered automorphic representations on low rank groups [KK]. We obtain a fairly complete classification of such automorphic representations for the quasisplit unitary groups in four variables.

1 CAP forms

The term CAP in the title is a short hand for the phrase “Cuspidal but Associated to Parabolic subgroups”. This is the name given by Piatetski-Shapiro [PS83] to those cuspidal automorphic representations which apparently contradict the generalized Ramanujan conjecture. More precisely, let G be a connected reductive group defined over a number field F , and G^* be its quasisplit inner form. We write $\mathbb{A} = \mathbb{A}_F$ for the adèle ring of F . An irreducible cuspidal representation $\pi = \bigotimes_v \pi_v$ is a *CAP form* if there exists a residual discrete automorphic representation $\pi^* = \bigotimes_v \pi_v^*$ such that, at all but finite number of v , π_v and π_v^* share the same absolute values of Hecke eigenvalues.

It is a consequence of the result of Jacquet-Shalika [JS81a], [JS81b] and Mœglin-Waldspurger [MW89] that there are no CAP forms on the general linear groups. On the other hand, for a central division algebra D of dimension n^2 over $F^\times$, the trivial representation of $D^\times(\mathbb{A})$ is clearly a CAP form which shares the same local component, at any place v where D is unramified, with the residual representation $1_{GL(n, \mathbb{A})}$. On the other hand, a quasisplit unitary group $U_{E/F}(3)$ of 3-variables already have non-trivial CAP forms, which can be obtained as θ -lifts of some automorphic characters of $U_{E/F}(1)$ [GR90], [GR91]. But the first and the most well-known example of CAP forms are the analogues of the θ_{10} representation by Howe-Piatetski-Shapiro [Sou88] and the Saito-Kurokawa representations of Sp_4 [PS83]. Also Gan-Gurevich-Jiang obtained very interesting example

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of CAP forms on the split group of type G_2 [GGJ02] (see also the article by Gan in this volume).

In any case, the local components of CAP forms at almost all places are non-trivial Langlands quotients by definition, and hence non-tempered in an apparent way. To put such forms into the framework of Langlands' conjecture, J. Arthur proposed a series of conjectures [Art89]. The conjectural description is through the so-called *A-parameters*, homomorphisms ψ from the direct product of the hypothetical Langlands group $\mathcal{L}_F$ of F with $SL(2, \mathbb{C})$ to the L -group ${}^L G$ of G [Bor79]:

$$\psi : \mathcal{L}_F \times SL(2, \mathbb{C}) \longrightarrow {}^L G,$$

considered modulo $\widehat{G}$ -conjugation. We write $\Psi(G)$ for the set of $\widehat{G}$ -conjugacy classes of A -parameters for G . By restriction, we obtain the local component

$$\psi_v : \mathcal{L}_{F_v} \times SL(2, \mathbb{C}) \rightarrow {}^L G_v$$

of ψ at each place v . Here the local Langlands group $\mathcal{L}_{F_v}$ is defined in [Kot84, §12], and ${}^L G_v$ is the L -group of the scalar extension $G_v = G \otimes_F F_v$. The local conjecture, among other things, associates to each ψ_v a finite set $\Pi_{\psi_v}(G_v)$ of isomorphism classes of irreducible unitarizable representations of $G(F_v)$, called an *A-packet*. At all but finite number of v , $\Pi_{\psi_v}(G_v)$ is expected to contain a unique unramified element π_v^1 . Using such elements, we can form the global A -packet associated to ψ

$$\Pi_{\psi}(G) := \left\{ \bigotimes_v \pi_v \mid \begin{array}{l} \text{(i)} \quad \pi_v \in \Pi_{\psi_v}(G_v), \forall v; \\ \text{(ii)} \quad \pi_v = \pi_v^1, \forall' v \end{array} \right\}.$$

Arthur's conjecture predicts the multiplicity of each element in $\Pi_{\psi}(G)$ in the discrete spectrum of the right regular representation of $G(\mathbb{A})$ on $L^2(G(F)\mathfrak{A}_G \backslash G(\mathbb{A}))$. Here $\mathfrak{A}_G$ is the maximal $\mathbb{R}$ -vector subgroup in the center of the infinite component $G(\mathbb{A}_{\infty})$ of $G(\mathbb{A})$.

We say an A -parameter ψ is of *CAP type* if

- (i) ψ is *elliptic*. This is the condition for $\Pi_{\psi}(G)$ to contain an element which occurs in the discrete spectrum.
- (ii) $\psi|_{SL(2, \mathbb{C})}$ is non-trivial.

According to the conjecture, the CAP automorphic representations of $G(\mathbb{A})$ is contained in some of the global A -packets associated to such A -parameters. In this talk, we shall classify the CAP forms by such parameters along the line of Arthur's conjecture, in the case of the quasisplit unitary group $U_{E/F}(4)$ of four variables. Although our description of such forms tells nothing about the character relations conjectured in [Art89], it is quite explicit and fairly complete. We hope to apply this to certain analysis of the cohomology of the Shimura variety attached to $GU_{E/F}(4)$.

2 Parameter consideration

Global case Take a quadratic extension E/F of number fields and write σ for the generator of the Galois group of this extension. Let $G = G_n := U_{E/F}(n)$ be the quasisplit

unitary groups in n variables associated to E/F . Later we shall mainly be concerned with the case $n = 4$. The L -group ${}^L G$ is the semi-direct product of $\widehat{G} = GL(n, \mathbb{C})$ by the absolute Weil group W_F of F , where W_F acts through $W_F/W_E \simeq \text{Gal}(E/F)$ by

$$\rho_G(\sigma)g = \text{Ad}(I_n)^t g^{-1}, \quad I_n := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \ddots & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

Thus an A -parameter ψ for G is determined by its restriction to $\mathcal{L}_E \times SL(2, \mathbb{C})$, which is just a completely reducible representation:

$$\psi|_{\mathcal{L}_E \times SL(2, \mathbb{C})} = \bigoplus_{i=1}^r \varphi_{\Pi_i} \otimes \rho_{d_i}.$$

Here Π_i is an irreducible cuspidal representation of $GL(m_i, \mathbb{A}_E)$ enjoying the following properties:

- $\sigma(\Pi_i) := \Pi_i \circ \sigma$ is isomorphic to the contragredient $\Pi_i^\vee$.
- Its central character ω_{Π_i} restricted to $\mathbb{A}^\times$ equals $\omega_{E/F}^{n-d_i-m_i+1}$, where $\omega_{E/F}$ is the quadratic character associated to E/F by the classfield theory.
- Some condition on the order of its twisted Asai L -functions at $s = 1$.

ρ_d is the d -dimensional irreducible representation of $SL(2, \mathbb{C})$. We note that ψ is elliptic if and only if its irreducible components $\varphi_{\Pi_i} \otimes \rho_{d_i}$ are distinct to each other. The S -group

$$\mathcal{S}_\psi(G) := \pi_0(\text{Cent}(\psi, \widehat{G})/Z(\widehat{G}))$$

is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{r-1}$, where $\pi_0(\bullet)$ stands for the group of connected components. This plays a central role in the conjectural multiplicity formula.

Local case Similar description for the A -packets of the unitary group $G = G_n$ associated to a quadratic extension E/F of local fields is also valid. For each A -parameter ψ , we have the associated non-tempered Langlands parameter

$$\phi_\psi : \mathcal{L}_F \ni w \longmapsto \psi(w, \begin{pmatrix} |w|_F^{1/2} & 0 \\ 0 & |w|_F^{-1/2} \end{pmatrix}) \in {}^L G.$$

Here the “absolute value” $|\cdot|_F$ on $\mathcal{L}_F$ is the composite $|\cdot|_F : \mathcal{L}_F \rightarrow W_F^{\text{ab}} \xrightarrow{\text{rec}} F^\times \xrightarrow{||_F} \mathbb{R}_+^\times$. (rec denotes the reciprocity map in the local classfield theory.) In Arthur’s conjecture, it was imposed that the L -packet $\Pi_{\phi_\psi}(G)$ associated to ϕ_ψ should be contained in $\Pi_\psi(G)$. We also have the S -group $\mathcal{S}_\psi(G)$ as in the global case. We postulate the following:

Assumption 2.1. *There exists a bijection $\Pi_\psi(G) \ni \pi \longmapsto (\bar{s} \mapsto \langle \bar{s}, \pi \rangle_\psi) \in \Pi(\mathcal{S}_\psi(G))$. Here $\Pi(\mathcal{S}_\psi(G))$ is the set of isomorphism classes of irreducible representations of $\mathcal{S}_\psi(G)$.*

Now for $n = 4$, the possibilities of $\{(d_i, m_i)\}_i$ for elliptic A -parameters with non-trivial $SL(2, \mathbb{C})$ -component are given as follows.

(1) Stable cases. $\{(4, 1)\}, \{(2, 2)\}$.

(2) Endoscopic cases.

- (a) $\{(3, 1), (1, 1)\}$;
- (b) $\{(2, 1), (1, 2)\}$;
- (c) $\{(2, 1), (2, 1)\}$;
- (d) $\{(2, 1), (1, 1), (1, 1)\}$.

In the cases (1), (2.a), it follows from Assumption 2.1 that $\Pi_{\phi_\psi}(G) = \Pi_\psi(G)$, and we know from [Kon98] that all the contribution of the corresponding global A -packets belong to the residual spectrum. On the other hand, $\Pi_\psi(G) \setminus \Pi_{\phi_\psi}(G)$ is expected to be non-empty in the rest cases. We shall use the local θ -correspondence to construct the missing members.

3 Local θ -correspondence

Local Howe duality First let us recall the local θ -correspondence. We consider an m -dimensional (non-degenerate) hermitian space $(V, (\cdot, \cdot))$ and n -dimensional skew-hermitian space $(W, (\cdot, \cdot))$ over E . We write $G(V)$ and $G(W)$ for the unitary groups of V and W , respectively. If we define the symplectic space $(\mathbf{W}, \langle \cdot, \cdot \rangle)$ by

$$\mathbf{W} := V \otimes_E W, \quad \langle v \otimes w, v' \otimes w' \rangle := \frac{1}{2} \text{Tr}_{E/F}[(v, v')\sigma(\langle w, w' \rangle)],$$

Then $(G(V), G(W))$ form a so-called *dual reductive pair* in the symplectic group $Sp(\mathbf{W})$ of this symplectic space:

$$\iota_{V,W} : G(V) \times G(W) \ni (g, g') \longmapsto g \otimes g' \in Sp(\mathbf{W}).$$

Fixing a non-trivial character ψ_F of F , we have the metaplectic group of $\mathbf{W}$ which is a central extension

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow Mp_{\psi_F}(\mathbf{W}) \longrightarrow Sp(\mathbf{W}) \longrightarrow 1.$$

This admits a unique Weil representation ω_{ψ_F} on which $\mathbb{C}^1$ acts by the multiplication [RR93]. For each pair $\underline{\xi} = (\xi, \xi')$ of characters of $E^\times$ satisfying $\xi|_{F^\times} = \omega_{E/F}^n, \xi'|_{F^\times} = \omega_{E/F}^n$, we have the corresponding lifting $\tilde{\iota}_{V,W,\underline{\xi}} : G(V) \times G(W) \rightarrow Mp_{\psi_F}(\mathbf{W})$ of $\iota_{V,W}$:

$$\begin{array}{ccc} G(V) \times G(W) & \xrightarrow{\tilde{\iota}_{V,W,\underline{\xi}}} & Mp_{\psi_F}(\mathbf{W}) \\ \parallel & & \downarrow \\ G(V) \times G(W) & \xrightarrow{\iota_{V,W}} & Sp(\mathbf{W}) \end{array}$$

The composite $\omega_{V,W,\underline{\xi}} := \omega_\psi \circ \tilde{\iota}_{V,W,\underline{\xi}}$ is the *Weil representation* of the dual reductive pair $(G(V), G(W))$ associated to $\underline{\xi}$. It is the product of the Weil representations $\omega_{W,\underline{\xi}}$ of $G(V)$ and $\omega_{V,\underline{\xi}}$ of $G(W)$.

We write $\mathcal{R}(G(V), \omega_{W,\xi})$ for the set of isomorphism classes of irreducible admissible representations of $G(V)$ which appear as quotients of $\omega_{W,\xi}$. For $\pi_V \in \mathcal{R}(G(V), \omega_{W,\xi})$, the maximal π_V -isotypic quotient of $\omega_{V,W,\xi}$ is of the form $\pi_V \otimes \Theta_\xi(\pi_V, W)$ for some smooth representation $\Theta_\xi(\pi_V, W)$ of $G(W)$. Similarly we have $\mathcal{R}(G(W), \omega_{V,\xi'})$ and $\Theta_\xi(\pi_W, V)$ for each $\pi_W \in \mathcal{R}(G(W), \omega_{V,\xi'})$. The local Howe duality conjecture, which was proved by R. Howe himself if F is archimedean [How89] and by Waldspurger if F is a non-archimedean local field of odd residual characteristic [Wal90], asserts the following:

- (i) $\Theta_\xi(\pi_V, W)$ (resp. $\Theta_\xi(\pi_W, V)$) is an admissible representation of finite length of $G(W)$ (resp. $G(V)$), so that it admits an irreducible quotient.
- (ii) Moreover its irreducible quotient $\theta_\xi(\pi_V, W)$ (resp. $\theta_\xi(\pi_W, V)$) is unique.
- (iii) $\pi_V \mapsto \theta_\xi(\pi_V, W)$, $\pi_W \mapsto \theta_\xi(\pi_W, V)$ are bijections between $\mathcal{R}(G(V), \omega_{W,\xi})$ and $\mathcal{R}(G(W), \omega_{V,\xi'})$ converse to each other.

Adams' conjecture A link between the local θ -correspondence and A -packets is given by the following conjecture of J. Adams [Ada89]. Suppose $n \geq m$. Then we have an L -embedding $i_{V,W,\xi} : {}^L G(V) \rightarrow {}^L G(W)$ given by

$$i_{V,W,\xi}(g \rtimes w) := \begin{cases} \xi' \xi^{-1}(w) \begin{pmatrix} g & \\ & 1_{n-m} \end{pmatrix} \rtimes w & \text{if } w \in W_E, \\ \begin{pmatrix} & g \\ J_{n-m}^{n-m-1} & \end{pmatrix} \rtimes w_\sigma & \text{if } w = w_\sigma, \end{cases}$$

where w_σ is a fixed element in $W_F \setminus W_E$ and

$$J_n := \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & (-1)^{n-1} \end{pmatrix}$$

Let $T : SL(2, \mathbb{C}) \rightarrow \text{Cent}(i_{V,W,\xi}, \widehat{G}(W))$ be the homomorphism which corresponds to a regular unipotent element in $\text{Cent}(i_{V,W,\xi}, \widehat{G}(W)) \simeq GL(n-m, \mathbb{C})$ (the *tail representation* of $SL(2, \mathbb{C})$). Using this, we define the θ -lifting of A -parameters by

$$\theta_{V,W,\xi} : \Psi(G(V)) \ni \psi \longmapsto (i_{V,W,\xi} \circ \psi^\vee) \cdot T \in \Psi(G(W)).$$

Conjecture 3.1 ([Ada89] Conj.A). *The local θ -correspondence should be subordinated to the map of A -packets: $\Pi_\psi(G(V)) \mapsto \Pi_{\theta_{V,W,\xi}(\psi)}(G(W))$.*

Here we have said subordinated because $\mathcal{R}(G(V), \omega_{W,\xi})$ is not compatible with A -packets, that is, $\Pi_\psi(G(V)) \cap \mathcal{R}(G(V), \omega_{W,\xi})$ is often strictly smaller than $\Pi_\psi(G(V))$. But when these two are assured to coincide, we can expect more:

Conjecture 3.2 ([Ada89] Conj.B). *For V, W in the stable range, that is, the Witt index of W is larger than m , we have*

$$\Pi_{\theta_{V,W,\xi}(\psi)}(G(W)) = \bigcup_{V; \dim_E V = m} \theta_{\xi}(\Pi_{\psi}(G(V)), W).$$

Now we note that our situation is precisely that of Conj. 3.2 with $m = 2$ and $W = V \oplus -V$. Moreover, we find that the A -parameters in the cases (2.b), (2.c), (2.d) in § 2 are exactly those of the form

$$\theta_{V,W,\xi}(\psi), \quad \psi \in \Psi(G(V)).$$

ε -dichotomy We explain the construction of the A -packets when F is non-archimedean. We need one more ingredient.

Proposition 3.3 (ε -dichotomy). *Suppose $\dim_E V = 2$ and write W_1 for the hyperbolic skew-hermitian space $(E^2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$. Take an L -packet Π of $G_2(F) = G(W)$ and $\tau \in \Pi$ [Rog90, Ch.11].*

(i) $\tau \in \mathcal{R}(G(W), \omega_{V,\xi'})$ if and only if

$$\varepsilon(1/2, \Pi \times \xi \xi'^{-1}, \psi_F) \omega_{\Pi}(-1) \lambda(E/F, \psi_F)^{-2} = \omega_{E/F}(-\det V).$$

Here the ε -factor on the right hand side is the standard ε -factor for G_2 twisted by $\xi \xi'^{-1}$ defined by the Langlands-Shahidi theory [Sha90]. ω_{Π} is the central character of the elements of Π and $\lambda(E/F, \psi_F)$ is Langlands' λ -factor [Lan70].

(ii) If this is the case, we have $\theta_{\xi}(\tau, V) = (\xi^{-1} \xi')_{G(V)} \tau_V^{\vee}$. Here $(\xi^{-1} \xi')_{G(V)}$ denotes the character of $G(V)$ given by the composite

$$G(V) \xrightarrow{\det} U_{E/F}(1, F) \ni z/\sigma(z) \mapsto \xi^{-1} \xi'(z) \in \mathbb{C}^{\times}.$$

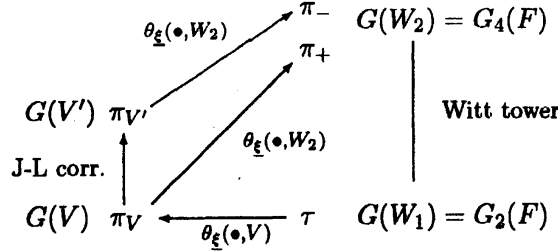
τ_V stands for the Jacquet-Langlands correspondent¹ of τ .

This is a special case of the ε -dichotomy of the local θ -correspondence for unitary groups over p -adic fields, which was proved for general unitary groups (at least for supercuspidal representations) in [HKS96]. But since we need to combine this with our description of the residual spectrum [Kon98], we have to use the Langlands-Shahidi ε -factors instead of Piatetski-Shapiro-Rallis's doubling ε -factors adopted by them. By this reason, we deduced this proposition from the analogous result for the unitary similitude groups [Har93] combined with the following description of the base change for G_2 .

Lemma 3.4. *Let $\tilde{\pi} = \omega \otimes \pi'$ be an irreducible admissible representation of the unitary similitude group $GU_{E/F}(2) \simeq (E^{\times} \times GL(2, F))/\Delta F^{\times}$, and write $\Pi(\tilde{\pi})$ for the associated L -packet of $G_2(F)$ consisting of the irreducible components of $\tilde{\pi}|_{G_2(F)}$. Then the standard base change of $\Pi(\tilde{\pi})$ to $GL(2, E)$ [Rog90, 11.4] is given by $\omega(\det) \pi'_E$, where π'_E is the base change lift of π' to $GL(2, E)$ [Lan80].*

¹In fact, the Jacquet-Langlands correspondence for unitary groups in two variables is defined only for L -packets and not for each member of the packets [LL79]. We know that $\tau \mapsto \tau_V$ certainly defines a bijection between Π and its Jacquet-Langlands correspondent. But we do not specify the bijection explicitly here. See Rem. 3.6 also.

Now we construct the A -packets. Our construction is summarized in the following picture.



Each A -parameter of our concern is of the form

$$\psi|_{\mathcal{L}_E \times SL(2, \mathbb{C})} = \psi_1|_{\mathcal{L}_E \times SL(2, \mathbb{C})} \oplus (\xi' \xi^{-1} \otimes \rho_2),$$

where ψ_1 is some A -parameter for G_2 . Take $\tau \in \Pi_{\psi_1}(G_2)$ and let $(V, (,))$ be the 2-dimensional hermitian space such that the condition of Prop. 3.3 (i) holds. If we write $\pi_V := \theta_{\xi}(\tau, V) \simeq (\xi \xi'^{-1})_{G(V)} \tau_V^{\vee}$, then the result of [Kud86] tells us $\pi_+ := \theta_{\xi}(\pi_V, W_2)$, ($\tau \in \Pi_{\psi_1}(G_2)$) form the local residual L -packet $\Pi_{\phi_{\psi}}(G_4)$. We now suppose that there exists a Jacquet-Langlands correspondent $\pi_{V'} \simeq (\xi \xi'^{-1})_{G(V')} \tau_{V'}^{\vee}$ of π_V on the unitary group $G(V')$ of the other (isometry class of) 2-dimensional hermitian space. Then Prop. 3.3 (i) tells us that $\pi_{V'} \notin \mathcal{R}(G(V'), \omega_{W_1, \xi})$. Yet its local θ -lifting $\pi_- := \theta_{\xi}(\pi_{V'}, W_2)$ to the larger group $G_4(F)$ still exists. This is the so-called *early lift* or the *first occurrence*. Following Conj. 3.2, we define

$$\Pi_{\psi}(G_4) := \{\pi_{\pm} \mid \tau \in \Pi_{\psi}(G_2)\}.$$

This gives sufficiently many members of the packet as predicted by Assumption 2.1.

Example 3.5. (i) Suppose $\Pi_{\psi_1}(G_2)$ is an L -packet consisting of supercuspidal elements. For $\tau \in \Pi_{\psi_1}(G_2)$, π_+ is the Langlands quotient $J_{P_1}^{G_4}(\xi' \xi^{-1} | \cdot |_E^{1/2} \otimes \tau)$, where P_1 is a parabolic subgroup with the Levi factor $R_{E/F} \mathbf{G}_m \times G_2$. On the other hand the early lift π_- of the supercuspidal τ is again supercuspidal. Thus $\Pi_{\psi}(G_4)$ consists of non-tempered members and supercuspidal elements.

(ii) On the contrary, we take $\xi = \xi'$ and consider $\Pi_{\psi_1}(G_2)$ consists of either the Steinberg representation δ_{G_2} or the trivial representation 1_{G_2} .

- δ_{G_2} lifts to $\pi_V = 1_{G(V)}$, where V is anisotropic. $\pi_{V'} = \delta_{G_2}$. $\pi_+ = J_{P_1}^{G_4}(| \cdot |_E^{1/2} \otimes \delta_{G_2})$ and π_- is an irreducible tempered but not square integrable representation.
- 1_{G_2} lifts to $\pi_V = 1_{G(V)}$ but V is hyperbolic this time. $\pi_{V'}$ is again $1_{G(V')}$ but this should be viewed as the Jacquet-Langlands correspondent of the A -packet $\{1_{G(V)}\}$. We have $\pi_+ = J_{P_2}^{G_4}(I_{\mathbf{B}}^{GL(2)E}(1 \otimes 1) | \det |_E^{1/2})$, where P_2 is the so-called Siegel parabolic subgroup with the Levi factor $GL(2, E)$. Obviously $\pi_- = J_{P_1}^{G_4}(| \cdot |_E^{1/2} \otimes \delta_{G_2})$. This last representation is shared by the two packets considered here.

Real case We end this section by some comments on the case $E/F = \mathbb{C}/\mathbb{R}$. Similar results are obtained by applying the argument of Adams-Barbasch [AB95]. In fact, the local θ -correspondence between unitary groups of the same size is described quite explicitly and in full generality in [Pau98]. Their argument also works in the present case. Let me explain some example.

We write $G_{p,q} = U(p, q)$. For a regular integral infinitesimal character $\lambda = (\lambda_1, \lambda_2)$ for $G_{1,1}$, consider the extended L -packet:

$$\Pi_\lambda = \{\delta_{1,1}^+, \delta_{1,1}^-, \delta_{2,0}, \delta_{0,2}\}$$

consisting of the discrete series representation of various $G_{p,q}$ with the infinitesimal character λ . The subscript p, q indicates that $\delta_{p,q}^\bullet$ lives on $G_{p,q}$. We can write $\xi'\xi^{-1}(z) = (z/\bar{z})^n$, $\forall z \in \mathbb{C}$ for some $n \in \mathbb{Z}$. An analogue of Prop. 3.3 in the real case asserts that the local θ -correspondence under the Weil representation $\omega_{V,W,\xi}$ gives a bijection

$$\theta_\xi : \Pi_\lambda \xrightarrow{\sim} \Pi_{n-\lambda},$$

where $n - \lambda = (n - \lambda_2, n - \lambda_1)$.

If λ is sufficiently regular, by which we mean $|\lambda_i - n| > 1$, then it is proved by J.-S. Li [Li90] that $\theta_\xi(\theta_\xi(\delta_{1,1}^\pm), W_2)$ is a non-tempered cohomological representation $A_q(\lambda')$, where the Levi factor of the θ -stable parabolic subalgebra $\mathfrak{q}$ is $\mathfrak{u}(1, 1) \oplus \mathfrak{u}(1)^2$. As for the other elements $\delta_{p,q} \in \Pi_{n-\lambda}$, $\theta_\xi(\delta_{p,q}, W_2)$ is a discrete series representation $A_q(\lambda')$. This time $\mathfrak{q}$ has the Levi factor $\mathfrak{u}(2) \oplus \mathfrak{u}(1)^2$. The resulting A -packet $\theta_\xi(\Pi_{n-\lambda})$ is exactly the cohomological A -packet defined by Adams-Johnson [AJ87].

For the complete list of the packets both in the archimedean and non-archimedean case, see our paper [KK].

One can easily check that the S -groups in the cases (2.b), (2.c), (2.d) satisfy $\mathcal{S}_\psi(G_4) \simeq \mathcal{S}_{\psi_1}(G_2) \times \mathbb{Z}/2\mathbb{Z}$. Now we define the bijection in Assumption 2.1 by

- $\langle \bar{s}, \pi_\pm \rangle_\psi := \langle \bar{s}, \tau \rangle_{\psi_1}$ on $\bar{s} \in \mathcal{S}_{\psi_1}(G_2)$;
- $\langle \cdot, \pi_\pm \rangle_\psi$ on $\mathbb{Z}/2\mathbb{Z}$ equals the sign character if π_- and trivial character otherwise.

For the other cases, only the first one in this definition is enough to give a complete bijection. This finishes our local task.

Remark 3.6. In the above, we do not mention the definition of the pairing $\langle \cdot, \cdot \rangle_{\psi_1}$. There are several choices for this, and we can choose one by fixing a non-trivial character ψ_F of F [LL79]. Also we did not specify the correspondence $\pi_V \mapsto \pi_{V'}$, which is again a subtle problem. In fact, we need to make a choice of (absolute) transfer factor as in [LL79] which again involves a choice of ψ_F (appearing in $\lambda(E/F, \psi_F)$ in the transfer factor). Using this specific transfer, we label the members of endoscopic L -packets of anisotropic unitary group. The correspondence $\pi_V \mapsto \pi_{V'}$ can be described in terms of these data, but we do not go into details here.

4 Multiplicity formula

We now go back to the global situation where E/F is a quadratic extension of number fields. We note that there always exists a homomorphism $\mathcal{S}_\psi(G_4) \ni \bar{s} \mapsto \bar{s}(v) \in \mathcal{S}_{\psi_v}(G_{4,v})$. We can now state the main result of this talk. Although we treat only the number field case, we believe the result holds also over function fields of one variable over a finite field of odd characteristic.

Theorem 4.1. *Let ψ be an A -parameter of CAP type for $G_4 = U_{E/F}(4)$. As was explained in § 1, we form the global A -packet $\Pi_\psi(G_4) := \bigotimes_v \Pi_{\psi_v}(G_{4,v})$. Then the multiplicity $m(\pi)$ of $\pi = \bigotimes_v \pi_v \in \Pi_\psi(G_4)$ in $L^2(G(F) \backslash G(\mathbb{A}))$ is given by*

$$m(\pi) = \frac{1}{|\mathcal{S}_\psi(G_4)|} \sum_{\bar{s} \in \mathcal{S}_\psi(G_4)} \epsilon_\psi(\bar{s}) \prod_v \langle \bar{s}(v), \pi_v \rangle_{\psi_v},$$

where the sign character ϵ_ψ is defined by

$$\epsilon_\psi = \begin{cases} \text{sgn}_{\mathcal{S}_\psi(G_4)} & \text{if } \psi_1 \text{ is a stable } L\text{-parameter} \\ & \text{and } \varepsilon(1/2, \psi_1 \otimes \xi \xi'^{-1}) = -1, \\ 1 & \text{otherwise.} \end{cases}$$

Here $\varepsilon(s, \psi_1 \otimes \xi \xi'^{-1})$ is the Artin root number attached to ψ_1 , which equals the standard ε -function for $\Pi_{\psi_1}(G_2) \times \xi \xi'^{-1}$.

The proof divides into two parts. Our local construction together with the global θ -correspondence shows that the multiplicity is no less than the right hand side. Note that we also relies on the multiplicity formula of Labesse-Langlands for unitary groups in two variables [LL79], [Rog90]. Then we prove a characterization of the image of such θ -lifts by poles of certain L -functions, which gives the converse inequality. This also shows that all the CAP forms for $U_{E/F}(4)$ are obtained in the above as the contribution of the A -packets we constructed. In particular the A -packets contains the sufficiently many members at least for global purposes, so that our Assumption 2.1 is justified.

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