

Approximate point spectra of m -complex symmetric operators and others

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Abstract

Let C be a conjugation on a complex Hilbert space $\mathcal{H}$. If $\{x_n\}$ is a sequence of unit vectors, then so is $\{Cx_n\}$. Under the assumption such that $(T - \lambda)x_n \rightarrow 0$ ($n \rightarrow \infty$), we show spectral properties concerning with a sequence $\{Cx_n\}$ of unit vectors.

1 Introduction and conjugation

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. First we introduce a conjugation C on $\mathcal{H}$.

Definition 1.1 Let $\mathcal{H}$ be a complex Hilbert space. For a mapping $C : \mathcal{H} \rightarrow \mathcal{H}$ is said to be *antilinear* if

$$C(ax + by) = \bar{a}Cx + \bar{b}Cy \quad (\forall a, b \in \mathbb{C}, \forall x, y \in \mathcal{H}).$$

An antilinear operator C is said to be a *conjugation* if

$$C^2 = I \quad \text{and} \quad \langle Cx, Cy \rangle = \langle y, x \rangle \quad (\forall x, y \in \mathcal{H}).$$

If C is a conjugation, then $\|Cx\| = \|x\|$ for all $x \in \mathcal{H}$, i.e., C is isometric. In this paper, when a sequence $\{x_n\}$ of unit vectors satisfies $(T - \lambda)x_n \rightarrow 0$ ($n \rightarrow \infty$), we show spectral properties concerning with a sequence $\{Cx_n\}$ of unit vectors.

2 m -Complex symmetric operator

Let $B(\mathcal{H})$ be the set of all bounded linear operators on a complex Hilbert space $\mathcal{H}$.

Definition 2.1 An operator $T \in B(\mathcal{H})$ is said to be *m -complex symmetric* if

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$$\delta_m(T; C) = \sum_{j=0}^m (-1)^j \binom{m}{j} T^{*j} \cdot CT^{m-j}C = 0.$$

It holds that $\delta_m(T; C) \cdot (CTC) - T^* \cdot \delta_m(T; C) = \delta_{m+1}(T; C)$.

Hence, if T is m -complex symmetric, then T is n -complex symmetric for all $n \geq m$.

Theorem 2.2 *Let T be an m -complex symmetric operator and $\{x_n\}$ be a sequence of unit vectors. For $\lambda \in \mathbb{C}$, if $(T - \lambda)x_n \rightarrow 0$ ($n \rightarrow \infty$), then $\langle (T - \lambda)^m Cx_n, Cx_n \rangle \rightarrow 0$ ($n \rightarrow \infty$). Hence, if $(T - \lambda)x = 0$, then $\langle (T - \lambda)^m Cx, Cx \rangle = 0$.*

Proof. Since $(T - \lambda)x_n \rightarrow 0$ and $C(T - \lambda)^m C = - \sum_{j=1}^m (-1)^j \binom{m}{j} (T^{*j} - \bar{\lambda}^j) CT^{m-j}C$, it holds

$$\langle (T - \lambda)^m Cx_n, Cx_n \rangle = - \sum_{j=1}^m (-1)^j \binom{m}{j} \langle (T^j - \lambda^j)x_n, CT^{m-j}Cx_n \rangle.$$

Hence we have Theorem 2.2. $\square$

Corollary 2.3 *Under the assumption of Theorem 2.2, we have:*

$$(1) \langle (T^* - \bar{\lambda})^m x_n, x_n \rangle \rightarrow 0,$$

$$(2) \langle (T^k - \lambda^k)Cx_n, Cx_n \rangle \rightarrow 0 \text{ for all } k \in \mathbb{N}.$$

Example 2.4 Let $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $Cx = \begin{pmatrix} \bar{x}_2 \\ x_1 \end{pmatrix}$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ on $\mathbb{C}^2$. Then for a vector $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, it holds $Tx = 0$. But since $Cx = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have

$$\langle TCx, Cx \rangle = 1 \neq 0.$$

Theorem 2.5 *Let T be an m -complex symmetric operator and $\{x_n\}$ be a sequence of unit vectors. For $\lambda \in \mathbb{R}$, if $(T - \lambda)x_n \rightarrow 0$, then $(T^* - \lambda)^m Cx_n \rightarrow 0$. Hence, if $(T - \lambda)x = 0$, then $(T^* - \lambda)^m Cx = 0$.*

Proof. Since $\lambda \in \mathbb{R}$, $(T - \lambda)x_n \rightarrow 0$ and

$$C(T^* - \lambda)^m C = - \sum_{j=1}^m (-1)^j \binom{m}{j} CT^{*m-j}C(T^j - \lambda^j),$$

we have

$$(T^* - \lambda)^m Cx_n = \sum_{j=1}^m (-1)^j \binom{m}{j} CT^{*m-j}C(T^j - \lambda^j)x_n.$$

Therefore we have Theorem 2.5. $\square$

3 $[m, C]$ -Symmetric operator

Definition 3.1 An operator $T \in B(\mathcal{H})$ is said to be $[m, C]$ -symmetric if

$$\alpha_m(T; C) = \sum_{j=0}^m (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^j = 0.$$

Then it holds $(CTC) \cdot \alpha_m(T; C) - \alpha_m(T; C) \cdot T = \alpha_{m+1}(T; C)$.

Hence, if T is $[m, C]$ -symmetric, then T is $[n, C]$ -complex symmetric for all $n \geq m$.

Also if T is $[m, C]$ -symmetric, then so is T^* .

Theorem 3.2 Let T be $[m, C]$ -symmetric and $\{x_n\}$ be a sequence of unit vectors. For $\lambda \in \mathbb{C}$, if $(T - \lambda)x_n \rightarrow 0$, then $(T - \bar{\lambda})^m Cx_n \rightarrow 0$. Hence, if, for $\lambda \in \mathbb{C}$, $(T - \lambda)x = 0$, then $(T - \bar{\lambda})^m Cx = 0$.

Proof. Since T^* is $[m, C]$ -symmetric, $\alpha_m(T^*, C) = 0$ and

$$\alpha_m(T^*, C)^* = \sum_{j=0}^m (-1)^j \binom{m}{j} T^{m-j} \cdot CT^j C = 0.$$

Hence

$$\begin{aligned} 0 &= \left(\sum_{j=0}^m (-1)^j \binom{m}{j} T^{m-j} \cdot CT^j C \right) Cx_n \\ &= (T - \bar{\lambda})^m Cx_n + \sum_{j=1}^m (-1)^j \binom{m}{j} T^{m-j} \cdot (CT^j C - \bar{\lambda}^j) Cx_n. \quad \square \end{aligned}$$

If T is $[m, C]$ -symmetric, then so is T^k for any $k \in \mathbb{N}$ (see [4]). Hence we have following corollary.

Corollary 3.3 Under the assumption of Theorem 3.2, it holds

$$\|(T^k - \bar{\lambda}^k)^m Cx_n\| \rightarrow 0$$

for all $k \in \mathbb{N}$.

Example 3.4 Let $T = \begin{pmatrix} 2i & 1 \\ 1 & -2i \end{pmatrix}$ and $Cx = \begin{pmatrix} \overline{x_2} \\ \overline{x_1} \end{pmatrix}$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ on $\mathbb{C}^2$. Then $CTC =$

T and T is $[1, C]$ -symmetric. For an eigenvalue $\sqrt{3}i$ and an eigen-vector $x = \begin{pmatrix} 1 \\ (\sqrt{3} - 2)i \end{pmatrix}$, it holds

$$(T - \sqrt{3}i)Cx = \begin{pmatrix} 4\sqrt{3} - 6 \\ -2\sqrt{3}i \end{pmatrix} \neq 0 \text{ and } (T + \sqrt{3}i)Cx = 0.$$

4 Skew m -complex operator

Definition 4.1 An operator $T \in B(\mathcal{H})$ is said to be *skew m -complex symmetric* if

$$\gamma_m(T; C) = \sum_{j=0}^m \binom{m}{j} T^{*j} \cdot CT^{m-j}C = 0.$$

Since it holds that

$$T^* \cdot \gamma_m(T; C) + \gamma_m(T; C) \cdot CTC = \gamma_{m+1}(T; C),$$

if T is skew m -complex symmetric, then T is skew n -complex symmetric for all $n \geq m$.

Theorem 4.2 Let T be a skew m -complex symmetric operator and $\{x_n\}$ be a sequence of unit vectors. For $\lambda \in \mathbb{C}$, if $(T - \lambda)x_n \rightarrow 0$ ($n \rightarrow \infty$), then $\langle (T + \lambda)^m Cx_n, Cx_n \rangle \rightarrow 0$ ($n \rightarrow \infty$). Hence, if $(T - \lambda)x = 0$, then $\langle (T + \lambda)^m Cx, Cx \rangle = 0$.

Proof. Since $(T - \lambda)x_n \rightarrow 0$ and $C(T + \lambda)^m C = \sum_{j=1}^m \binom{m}{j} \bar{\lambda}^j \cdot CT^{m-j}C$,

$$\langle (T + \lambda)^m Cx_n, Cx_n \rangle = - \sum_{j=1}^m \binom{m}{j} \langle (T^j - \lambda^j)x_n, CT^{m-j}Cx_n \rangle \quad \square$$

Example 4.3 If T is m -complex symmetric, then so is T^n for every $n \in \mathbb{N}$. But there exists a skew 1-complex symmetric operator T such that T^2 is not skew 1-complex symmetric. For example, let

$$T = \begin{pmatrix} 1+i & 0 \\ 0 & -1-i \end{pmatrix} \quad \text{and} \quad Cx = \begin{pmatrix} \overline{x_2} \\ x_1 \end{pmatrix} \quad \text{for } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ on } \mathbb{C}^2.$$

Then it is easy to see $CTC = \begin{pmatrix} -1+i & 0 \\ 0 & 1-i \end{pmatrix} = -T^*$ and hence T is skew 1-complex symmetric. But since $T^2 = \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix}$, we have $CT^2C = T^{2*}$ and hence T^2 is complex symmetric and not skew 1-complex symmetric.

Theorem 4.4 Let T be a skew m -complex symmetric operator and $\{x_n\}$ be a sequence of unit vectors. For $\lambda \in \mathbb{C}$, if $(T - \lambda)x_n \rightarrow 0$ ($n \rightarrow \infty$), then $(T^* + \bar{\lambda})^m Cx_n \rightarrow 0$ ($n \rightarrow \infty$). Hence, if $(T - \lambda)x = 0$, then $\langle (T^* + \bar{\lambda})^m Cx, Cx \rangle = 0$.

Proof. Since $(T - \lambda)x_n \rightarrow 0$, $(CT^jC - \bar{\lambda}^j)Cx_n \rightarrow 0$ and

$$C(\gamma_m(T; C))^*C = \sum_{j=0}^m \binom{m}{j} T^{*m-j} \cdot CT^{m-j}C,$$

it holds

$$0 = (T^* + \bar{\lambda})^m Cx_n + \sum_{j=1}^m \binom{m}{j} T^{*m-j} \cdot (CT^j C - \bar{\lambda}^j) Cx_n.$$

Hence, we have Theorem 4.4. $\square$

Corollary 4.5 *Let T be skew m -complex symmetric. Then:*

- (1) *If $\lambda \in \sigma_a(T)$, then $-\bar{\lambda} \in \sigma_a(T^*)$.*
- (2) *If $\lambda \in \sigma_p(T)$, then $-\bar{\lambda} \in \sigma_p(T^*)$.*

By Theorem 4.4 since $0 \in \sigma_a((T^* + \bar{\lambda})^m)$, by the spectral mapping theorem of the approximate point spectrum, $0 \in \sigma_a(T^* + \bar{\lambda})$ and hence $-\bar{\lambda} \in \sigma_a(T^*)$.

5 Skew $[m, C]$ -symmetric operator

Definition 5.1 An operator $T \in B(\mathcal{H})$ is said to be *skew $[m, C]$ -symmetric* if

$$\zeta_m(T; C) := \sum_{j=0}^m \binom{m}{j} CT^{m-j} C \cdot T^j = 0.$$

It holds $CTC \cdot \zeta_m(T; C) + \zeta_m(T; C) \cdot T = \zeta_{m+1}(T; C)$.

Therefore if T is skew $[m, C]$ -symmetric, then T is skew $[n, C]$ -symmetric for all $n \geq m$. If T is skew $[m, C]$ -symmetric, then it holds

$$0 = C(\zeta_m(T; C))^* C = \sum_{j=0}^m \binom{m}{j} CT^{*j} C \cdot T^{*m-j} = \zeta_m(T^*; C)$$

and hence so is T^* .

Theorem 5.2 *Let T be a skew $[m, C]$ -symmetric operator and $\{x_n\}$ be a sequence of unit vectors. For $\lambda \in \mathbb{C}$, if $(T - \lambda)x_n \rightarrow 0$, then $(T^* + \bar{\lambda})^m Cx_n \rightarrow 0$. Hence, if $(T - \lambda)x = 0$, then $(T^* + \bar{\lambda})^m Cx = 0$.*

Proof. Since $(T - \lambda)x_n \rightarrow 0$ and $C(\zeta_m(T^*; C))^* C = \sum_{j=0}^m \binom{m}{j} T^{m-j} \cdot CT^j C = 0$,

$$0 = (T^* + \bar{\lambda})^m Cx_n + \sum_{j=1}^m \binom{m}{j} T^{m-j} \cdot (CT^j C - \bar{\lambda}^j) Cx_n.$$

Hence, we have Theorem 5.2. $\square$

Corollary 5.3 *Let T be skew $[m, C]$ -symmetric. Then:*

- (1) *If $\lambda \in \sigma_a(T)$, then $-\bar{\lambda} \in \sigma_a(T^*)$.*

(2) If $\lambda \in \sigma_p(T)$, then $-\bar{\lambda} \in \sigma_p(T^*)$.

By Theorem 5.2 since $0 \in \sigma_a((T^* + \bar{\lambda})^m)$, by the spectral mapping theorem of the approximate point spectrum, $0 \in \sigma_a(T^* + \bar{\lambda})$ and hence $-\bar{\lambda} \in \sigma_a(T^*)$.

Example 5.4 Let

$$T = \begin{pmatrix} 1 & 2i \\ 2i & -1 \end{pmatrix} \quad \text{and} \quad Cx = \begin{pmatrix} \overline{x_2} \\ \overline{x_1} \end{pmatrix} \quad \text{for } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ on } \mathbb{C}^2.$$

Then it holds $CTC = -T$ and hence T is skew $[1, C]$ -symmetric. For the eigenvalue $\sqrt{3}i$ of T and the corresponding eigenvector $x = \begin{pmatrix} 1 \\ \frac{\sqrt{3}+i}{2} \end{pmatrix}$, we have

$$(T + \sqrt{3}i)Cx = \begin{pmatrix} 2\sqrt{3}i \\ -\sqrt{3} + 3i \end{pmatrix} \neq 0 \quad \text{and} \quad (T - \sqrt{3}i)Cx = 0.$$

Theorem 5.5 Let T be a skew $[m, C]$ -symmetric operator and $\{x_n\}$ be a sequence of unit vectors. For $\lambda \in \mathbb{C}$, if $(T - \lambda)x_n \rightarrow 0$, then $\langle (T^* + \lambda)^m Cx_n, Cx_n \rangle \rightarrow 0$. Hence, if $(T - \lambda)x = 0$, then $\langle (T^* + \lambda)^m Cx, Cx \rangle = 0$.

Proof. Since $CT^{*m}C = -\sum_{j=1}^m \binom{m}{j} T^{*j} \cdot CT^{*m-j}C$,

$$C(T^* + \lambda)^m C = -\sum_{j=1}^m \binom{m}{j} (T^{*j} - \bar{\lambda}^j) \cdot CT^{*m-j}C.$$

Hence we have Theorem 5.5. $\square$

Example 5.6 If T is $[m, C]$ -symmetric, then so is T^n for every $n \in \mathbb{N}$. But there exists a skew $[1, C]$ -symmetric operator T such that T^2 is not skew $[1, C]$ -symmetric. For example, let

$$T = \begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix} \quad \text{and} \quad Cx = \begin{pmatrix} \overline{x_2} \\ \overline{x_1} \end{pmatrix} \quad \text{for } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ on } \mathbb{C}^2.$$

Then it is easy to see $CTC = \begin{pmatrix} 1 & 2i \\ 2i & -1 \end{pmatrix} = -T$ and hence T is skew $[1, C]$ -symmetric.

But since $T^2 = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$, we have $CT^2C = T^2$. Hence T^2 is $[1, C]$ -symmetric and not skew $[1, C]$ -symmetric.

6 Square hyponormal operator

We begin with the definition of square hyponormal operators.

Definition 6.1 An operator $T \in B(\mathcal{H})$ is said to be *square hyponormal* if T^2 is hyponormal.

Following results are famous.

- (1) If $\ker(T - z) \perp \ker(T - w)$ for any distinct nonzero eigenvalues z and w , then T has SVEP.
- (2) Let p be polynomial. If $p(T)$ has SVEP, then T has SVEP.

Hence, if T is square hyponormal, then T has SVEP.

In general, T is 2-hyponormal if $\begin{pmatrix} I & T^* \\ T & T^*T \end{pmatrix} \geq 0$

We have many papers about 2-hyponormal operators. So T is said to be *square hyponormal* if T^2 is hyponormal. About 2-hyponormal operators, please see “R. Curto and Woo Young Lee, Towards a model theory for 2-hyponormal operators, Integr. Equat. Oper. Theory, 44(2002), 290-315”.

Basic properties are the following:

Theorem 6.2 Let T be square hyponormal. Then the following statements hold.

- (1) If T is invertible, then so is T^{-1} .
- (2) If $n = 2k \in \mathbb{N}$ is even, then T^n is $\frac{1}{k}$ -hyponormal.
- (3) If $S \in B(\mathcal{H})$ and $S \simeq T$, then S is square hyponormal.
- (4) If $T - t$ are square hyponormal for all $t > 0$, then T is hyponormal.
- (5) If M is an invariant subspace for T , then $T|_M$ is square hyponormal.

By Aluthge and Wang’ result, T is hyponormal, then T^2 is semi-hyponormal. But we have many examples non hyponormal operator T which T^2 is hyponormal.

Curto and Han studied algebraically hyponormal operators.

For T , we set the following property:

$$(*) \quad \sigma(T) \cap (-\sigma(T)) \subset \{0\}$$

Lemma 6.3 Let T satisfy (*). If z is an isolated point of $\sigma(T)$, then z^2 is an isolated point of $\sigma(T^2)$.

Proof. If $z = 0$, then it is clear. If $z \neq 0$, then proof follows from $T^2 - z^2 = (T + z)(T - z)$ and (*). $\square$

Theorem 6.4 Let T be square hyponormal and satisfy (*), then $\sigma(T) = \{\bar{z} : z \in \sigma_a(T)\}$.

Theorem 6.5 Let T be square hyponormal and satisfy (*), M be an invariant subspace for T such that $\sigma(T|_M) = \{z\}$. Then:

- (1) If $z = 0$, then $(T|_M)^2 = 0$.
- (2) If $z \neq 0$, then $T|_M = z$.

Theorem 6.5 *Let T be square hyponormal and satisfy $(*)$. Then:*

- (1) *Let $Tx = zx$ and $Ty = wy$. If $z \neq w$, then $\langle x, y \rangle = 0$.*
- (2) *Similar result holds for approximate eigenvalues.*

Theorem 6.6 *Let T be square hyponormal and satisfy $(*)$. Let $Tx = zx$ ($z \neq 0$). Then $\ker(T - z) = \ker(T^2 - z^2) \subset \ker(T^{*2} - \bar{z}^2) = \ker(T^* - \bar{z})$.*

Remark About proofs and other results, please see [1] - [5].

References

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