

**Holographic studies on semi-classical saddles
in three-dimensional gravity**

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Abstract

In this thesis, we discuss the holographic dictionary for the semi-classical saddle point geometries in the three-dimensional gravity with a positive or negative cosmological constant. There has been a long-standing discussion on the integration contour of the gravitational path integral, which can be rephrased as the set of saddle point geometries contributing to the semi-classical approximation. We attack this problem by applying the holographic principle, especially the $\text{AdS}_3/\text{CFT}_2$ and the dS_3/CFT_2 correspondences. We aim to determine the integration contour of the gravitational path integral so as to match with the CFT calculations. We consider a concrete model of the $\text{AdS}_3/\text{CFT}_2$ such that the CFT side can be described by Liouville field theory, and the dS_3/CFT_2 is constructed by an analytic continuation from the $\text{AdS}_3/\text{CFT}_2$. An advantage to consider Liouville field theory is that the exact forms of its correlation functions can be derived and have good analytic properties, which makes it possible to rigorously perform the analytic continuation.

After reviewing some necessary knowledge about the AdS/CFT , the dS/CFT and Liouville field theory, we first focus on the dS_3/CFT_2 . From the CFT calculations, we find that the Stokes phenomenon, which is a non-trivial transition of the set of relevant saddle points, is observed along the analytic continuation. After taking into account the Stokes phenomenon, we can find that the leading contribution in the semi-classical approximation is indeed consistent with the physical expectation for the expanding universe because it is identical to the Gibbons-Hawking entropy. After that, we move on to the gravity side. By analyzing the gravitational path integral in the so-called mini-superspace model, we confirm that a consistent integration contour with the CFT calculations can be taken.

We also discuss the Euclidean $\text{AdS}_3/\text{CFT}_2$ in the similar way. The CFT results imply that infinitely many complex saddle points contribute to the semi-classical approximation of the CFT correlation functions. We again discuss the mini-superspace model and find that a natural integration contour that sums up real hyperbolic metrics can reproduce the CFT result, which picks up the infinitely many complex saddle point geometries. Another motivation to study $\text{AdS}_3/\text{CFT}_2$ comes from another type of Stokes phenomenon observed in the CFT correlation functions. When the conformal dimensions of inserted operators are increased, the correlation function shows the Stokes phenomenon, where after the transition the relevant saddle point may correspond to a kind of wormhole geometry with higher genus slices. We also find the corresponding Stokes phenomenon in the mini-superspace model by assuming the same integration contour. Although the complete understanding about wormholes has not yet been given, we report some additional relevant calculations and discuss physical interpretations.

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1 Introduction

One of the remaining problems in theoretical physics is to understand the quantum gravity, i.e., a quantum theory including gravitational effects. One of several approaches to studying quantum gravity is the holographic principle [1, 2], which states a duality between the quantum gravity and a quantum field theory living in a codimension-1 subspace of the spacetime. In particular, the holography has been well studied for gravity theories in asymptotically anti-de Sitter (AdS) spacetimes, which is dual to a conformal field theory (CFT). This special class of holography is called the AdS/CFT correspondence [3]. Thanks to the nice property of strong-weak coupling duality, the AdS/CFT can be used in both directions; studying gravity via CFTs or analyzing strongly coupled field theories via gravity. This thesis aims to understand the quantum gravity from CFT calculations.

Many interesting aspects of a generic quantum theory lie in non-perturbative quantum effects, which play important roles in studying confinement, superconductors, monopoles, instantons, and so on. Non-perturbative effects are not tractable by perturbative expansions, but some phenomena can be analyzed in the semi-classical regime $\hbar \ll 1$. In the path integral formalism, the saddle point approximation

$$\int_{\mathcal{C}} \mathcal{D}\phi e^{\frac{i}{\hbar} I[\phi]} \sim \sum_{n: \text{saddles}} e^{\frac{i}{\hbar} I[\phi_n]} \quad (1.1)$$

can be used to capture some non-perturbative contributions to the path integral in $\hbar \ll 1$. Here the saddle points ϕ_n are solutions to the saddle point equation $\delta I / \delta \phi = 0$. In this procedure, an important question is the following: Which set of saddle points should be taken into account? In fact, this question can be answered from the data of the integration contour $\mathcal{C}$ in (1.1) by applying the Picard-Lefschetz theory. Therefore, when the path integral is completely defined including the integration contour, in principle we can perform the semi-classical approximation of every physical quantity.

For our purpose to understand the quantum gravity, it would be natural to study non-perturbative effects in gravitational systems. In the semi-classical regime, the gravitational path integral

$$\int_{\mathcal{C}} \mathcal{D}g_{\mu\nu} e^{\frac{i}{\hbar} I_{\text{grav}}[g_{\mu\nu}]} \quad (1.2)$$

is thought of as the integral over a set of spacetime metrics, which include spacetimes with non-trivial topologies. The non-trivial saddle point geometries of the gravitational path integral include important gravitational effects such as black holes [4] and the initial state of universe [5]. These are interesting regions in our universe because the quantum effects of gravity will be relevant there. Therefore, applying the method explained above to the gravitational path integral may give us some information about those extreme regions of universe. However, there is a huge

distinction from the ordinary setup of path integral: we do not know the full definition of the gravitational path integral (1.2), especially the integration contour $\mathcal{C}$, due to the lack of knowledge on the quantum gravity. Indeed, old works [6–8] in this motivation could not give the exact answer about which saddle points are relevant, although the detailed investigations on the saddle point structure of the gravitational path integral were given. Thus we have to rely on physical intuitions about which saddles should be taken. Besides, another proposal of a condition for the relevant saddle point geometries is recently given by [9] based on a mathematical argument [10]. As commented in [9], this has not been proven to be either necessary or sufficient condition for the saddle to contribute. Nevertheless the consistency checks for some known examples of saddle points has been performed, so it would be worth discussing the condition whenever we are interested in the gravitational saddle points.

The main idea of this thesis is to attack this problem by using the holographic duality. We consider three-dimensional gravity theories with both positive and negative cosmological constants. The negative case is related with a CFT through the AdS/CFT correspondence. Even for de Sitter (dS) spacetime with positive cosmological constant, analogous holographic dualities have been proposed. One of them is known as the dS/CFT correspondence [11–13], where the dual theory is expected to be a non-unitary CFT living at the future (and/or past) boundary. To apply the dS/CFT to solving our problem, we have to construct a concrete model of the dual CFT. We will focus on the model proposed by [14, 15], where the dual CFT is constructed by an analytic continuation from the AdS/CFT. Our first task is calculating the dual CFT quantities in the semi-classical regime. In evaluating the saddle point approximations of the partition function or correlation functions of CFT, we can get an interesting observation [16, 17]: The set of contributing saddle points are drastically changed when the cosmological constant is analytically continued from negative to positive. Such a transition in the saddle point approximation is generally known as the Stokes phenomenon. One of the statements by [16, 17] is that we need to seriously take care of the Stokes phenomenon in order to obtain the consistent result with the gravity side.

The next thing we have to do is to pull back the CFT results to the gravity side. We utilize the three-dimensional version of mini-superspace method, see e.g. [6] for four-dimensional case. As mentioned above, we do not know the integration contour $\mathcal{C}$ in (1.2). In this thesis, we determine the contour $\mathcal{C}$ by comparing the mini-superspace analysis with the CFT results. Furthermore, we give geometrical interpretations to the obtained saddle points. In particular, for AdS case, we find that the infinite number of complex saddle points needs to be taken into account to match with CFT results, and each saddle is interpreted as a complex geometry constructed by continuing the three-dimensional disk into the imaginary direction. Here, there is a technical subtlety in analyzing the saddle point structure. In the mini-superspace method, the gravity theory exactly lies on the point where the Stokes phenomenon occurs, which means that the saddle point approximation cannot be applied to the theory. One way to avoid this problem is to slightly shift a parameter in order for the theory to leave the critical point. The literatures so far do not seem to do this seriously, so that our treatment will also be supposed to refine the previous discussions. We have to note that the theory might depend on how to make a shift on the parameter.

The organization of this thesis is as follows. We first give reviews on the AdS/CFT in §2 and on the dS/CFT in §3. In §4, we summarize the basics on the two-dimensional Liouville field theory, which will be important in evaluating the correlation functions of the dual CFT. The remaining sections are the main part of this thesis. We start with focusing on the dS/CFT in §5. As described above, we consider a model of the dS/CFT constructed as the analytic continuation from a known model of the AdS/CFT, where the correlation functions of the CFT side are

described by Liouville field theory. We aim to survey the saddle points that contribute to the gravitational path integral from the CFT calculations, and in other words the integration contour $\mathcal{C}$. We first classify the complex geometries that solves the complexified Einstein solutions. It is useful to discuss it in the first-order formalism, where the action is given by two Chern-Simons actions. CFT calculations are given for correlation functions with two, three and four operator insertions, and extending to further insertions are also discussed. Finally we give the direct calculations in the gravity side, by applying the so-called mini-superspace method. The data of the integration contour, which has not been given a priori, will be determined by comparing with the CFT results. In §6, we move on to AdS. Although the strategy is almost same as the dS case, the result is quite different. In order to be consistent with semi-classical calculations of CFT side, we argue that infinite complex geometries should be taken into account as the relevant saddle points. The distinction from dS result can be understood as the Stokes phenomenon, the transition of the contributing saddle points. The gravity theory of our interest has another type of Stokes phenomenon, which can be geometrically understood as the transition between disk-type geometry and wormhole-type geometry. The similar transition is also discussed in the recently proposed “average CFT” [18]. Among this thesis, §3.3, §3.4, §5 and §6 are based on the author’s own works.

2 AdS/CFT

In 1997, Maldacena [3] proposed a conjecture that the semi-classical limit of the quantum gravity on AdS_{d+1} spacetime is equivalent to a limit of a CFT_d associated with the d -dimensional asymptotic boundary of AdS_{d+1} , by considering field theories living on some configurations of branes in the superstring (or M) theory. This duality has been one of the biggest subjects of high energy physics because it can be expected to give a definition of the quantum gravity non-perturbatively, not only the classical limit. Therefore this duality is thought of as a realization of the holographic principle [1, 2]. The purpose of studying the AdS/CFT correspondence is to understand the “holographic dictionary,” which translates the words between AdS side and CFT side. In this section, we mainly show some knowledge about the AdS/CFT that will be relevant to the later discussions. Therefore the large part of this section is devoted to a special dimensional case $\text{AdS}_3/\text{CFT}_2$.

2.1 GKPW dictionary

First let us consider general dimensions $\text{AdS}_{d+1}/\text{CFT}_d$. Shortly after the Maldacena’s discovery, a precise formulation of this duality was proposed as [19, 20]

$$\mathcal{Z}[\phi_0] = \left\langle \exp \int d^d x \phi_0(\vec{x}) \mathcal{O}(\vec{x}) \right\rangle, \quad (2.1)$$

where the left-hand side denotes the partition function in the gravity side defined by the path integral

$$\mathcal{Z}[\phi_0] = \int_{\phi|_{\text{bdy}}=\phi_0} \mathcal{D}\phi e^{-I[\phi]} \quad (2.2)$$

over all the fields ϕ , which may include the metric field $g_{\mu\nu}$, with boundary conditions $\phi|_{\text{bdy}} = \phi_0$ being imposed at the asymptotic boundary of AdS_{d+1} . The right-hand side of (2.1) is the generating functional of correlation functions in CFT side, where ϕ_0 plays a role of a source that generates an operator $\mathcal{O}$. In this sense, (2.1) states that there is a correspondence between a field

ϕ in AdS side and an operator $\mathcal{O}$ in CFT side. The relation (2.1) is a fundamental dictionary of the AdS/CFT correspondence, called GKPW dictionary.

Let us show a simple example of GKPW dictionary. We consider a massive scalar field

$$I[\phi] = \int d^d x \sqrt{g} \left[\frac{1}{2} (\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 \right] \quad (2.3)$$

defined on a fixed background of Poincaré EAdS_{d+1}

$$ds^2 = \ell_{\text{AdS}}^2 \frac{dz^2 + dx_i dx^i}{z^2}. \quad (2.4)$$

The equation of motion $(-\nabla^2 + m^2)\phi = 0$ with respect to ϕ is explicitly expressed as

$$-\partial_z^2 \phi + \frac{d-1}{z} \partial_z \phi - \partial_i \partial^i \phi + \frac{m^2 \ell_{\text{AdS}}^2}{z^2} \phi = 0. \quad (2.5)$$

In the asymptotic region $z \ll 1$, this equation has two independent solutions z^Δ and $z^{d-\Delta}$ with

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 \ell_{\text{AdS}}^2}. \quad (2.6)$$

We write the dominant term in $z \ll 1$ as $\phi(z, \vec{x}) \sim z^{d-\Delta} \phi_0(\vec{x})$. By using $\phi_0(\vec{x})$, we can express the solution to (2.5) as [20]

$$\phi(z, \vec{x}) = \int d^d x' K_\Delta(z; \vec{x}, \vec{x}') \phi_0(\vec{x}'), \quad (2.7)$$

where $K_\Delta(z; \vec{x}, \vec{x}')$ is the bulk-to-boundary propagator

$$K_\Delta(z; \vec{x}, \vec{x}') = \pi^{-d/2} \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)} \frac{z^\Delta}{(z^2 + |\vec{x} - \vec{x}'|^2)^\Delta}. \quad (2.8)$$

Thanks to the expansion in small z

$$K_\Delta(z; \vec{x}, \vec{x}') = \delta^{(d)}(\vec{x} - \vec{x}') z^{d-\Delta} + \pi^{-d/2} \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)} \frac{z^\Delta}{|\vec{x} - \vec{x}'|^{2\Delta}} + \dots, \quad (2.9)$$

this solution (2.7) indeed realizes the asymptotic behavior of $\phi(z, \vec{x})$. Substituting (2.7) into the action (2.3), we obtain the on-shell action

$$I[\phi_0] = -(\Delta - d/2) \pi^{-d/2} \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)} \int d^d x d^d x' \frac{\phi_0(\vec{x}) \phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2\Delta}}. \quad (2.10)$$

Therefore we have evaluated the saddle point approximation of the left-hand side of (2.1). Assuming the GKPW dictionary, the two-point function of the dual CFT is obtained by differentiating $e^{-I[\phi_0]}$ with respect to ϕ_0 twice, then

$$\langle \mathcal{O}(\vec{x}) \mathcal{O}(\vec{x}') \rangle \sim \frac{1}{|\vec{x} - \vec{x}'|^{2\Delta}}. \quad (2.11)$$

This means that the dual operator $\mathcal{O}$ has the scaling dimension Δ . Therefore we can conclude that a bulk massive scalar field ϕ with mass m corresponds to an CFT scalar operator $\mathcal{O}$ with scaling dimension Δ given by (2.6).

Here we make some comments:

- The other asymptotic dependence z^Δ of ϕ , coming from the second term in (2.9), also has a physical interpretation. From the solution (2.7), the coefficient $A(\vec{x})$ of $\phi(z, \vec{x}) \sim z^\Delta A(\vec{x})$ takes the form

$$A(\vec{x}) = \pi^{-d/2} \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)} \int d^d x' \frac{\phi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2\Delta}}. \quad (2.12)$$

From this expression, $A(\vec{x})$ can be interpreted as the expectation value of $\mathcal{O}(\vec{x})$ under a source $\phi_0(\vec{x}')$. Thus two coefficients of $z^{d-\Delta}$ and z^Δ are interpreted as a source and a physical response, respectively.

- In the above discussion, we have implicitly taken the positive root (2.6) as Δ . This is the only possible choice to satisfy the unitarity bound $\Delta > (d-2)/2$, i.e. $m^2 \ell_{\text{AdS}}^2 > -d^2/4 + 1$ to accomplish the normalizable mode. However, we can also consider the case $-d^2/4 < m^2 \ell_{\text{AdS}}^2 < -d^2/4 + 1$, where both $\Delta_\pm = d/2 \pm \sqrt{d^2/4 + m^2 \ell_{\text{AdS}}^2}$ satisfy the unitarity bound $\Delta > (d-2)/2$, so that both $\Delta_\pm$ can be candidates for Δ . These two choices lead to two different boundary field theories, which correspond to two different quantizations of a bulk scalar field theory. Those two theories are connected to each other under a Legendre transformation [21].
- There is another formulation of dictionary between fields and operators, called extrapolate dictionary or BDHM dictionary [22], which has been shown to be equivalent to GKPW dictionary [23].

2.2 $\text{AdS}_3/\text{CFT}_2$

Here we focus on $\text{AdS}_3/\text{CFT}_2$. Since this is a special case $d = 2$ of the previous subsection, of course, everything we described there also holds in this case. In addition, $d = 2$ case has many specific features coming from the topological feature of AdS_3 and the enhanced symmetry of CFT_2 .

An advantage to study lower-dimensional gravity is that the theory should be topological, i.e., there are no gravitational wave modes in the bulk. Precisely speaking, in three-dimensional Einstein gravity, the Einstein equations completely fix all the elements of Riemann tensor $R_{\mu\nu}{}^\alpha{}_\beta$, leaving no remaining degrees of freedom corresponding to the Weyl tensor. The topological feature of three-dimensional Einstein gravity can also be seen from the fact that the Einstein-Hilbert action can be expressed by using the Chern-Simons actions [24, 25]. The detailed description of this formalism will be explained in §2.2.1. Despite the lack of gravitational fluctuations in the bulk, the three-dimensional Einstein gravity has a rich variety of solutions in the existence of a negative cosmological constant: There exist black hole solutions [26] (see [27] for a review) called BTZ black holes. Such a geometry is locally equivalent to AdS_3 , and globally a quotient of AdS_3 . In this way, the topological feature of the gravity theory highly restricts the solutions; AdS_3 and its quotients. Hence in $d = 2$ case it would be easier to study the correspondence on the gravitational part, i.e. the spectrum of the dual CFT_2 states corresponding to the geometric solutions. Therefore many people have been studying the CFT_2 that is holographically dual to the pure gravity theory with a negative cosmological constant. Such CFT_2 is often called “holographic CFT_2 .” Even though the holographic CFT_2 has not yet been completely understood, it is expected to have some properties explained in the remainder of this subsection.

Around ten years before the AdS/CFT was discovered, Brown and Henneaux [28] had shown that the asymptotic symmetry of three-dimensional asymptotically AdS spacetime is generated

by the Virasoro algebra with the central charge

$$c = \frac{3\ell_{\text{AdS}}}{2G}. \quad (2.13)$$

This is the first data of the holographic CFT_2 . From this expression, we can find that the semi-classical limit $G \rightarrow 0$ corresponds to the large c limit $c \rightarrow \infty$.

In the semi-classical limit, the gravitational path integral is approximated by the contributions from the saddle points

$$\mathcal{Z} = \int \mathcal{D}g_{\mu\nu} e^{-I_{\text{EH}}[g_{\mu\nu}]} \sim \sum_{g_{\mu\nu}: \text{saddle points}} e^{-I_{\text{EH}}[g_{\mu\nu}]}. \quad (2.14)$$

Therefore, in the semi-classical limit, the objects that correspond to the CFT states are the saddle points of $\mathcal{Z}$. Thanks to the state-operator correspondence of CFT, a semi-classical dictionary on the gravity sector would be

$$\text{A saddle point geometry in AdS} \longleftrightarrow \text{A configuration of heavy operators in CFT}.$$

Here a “heavy” operator means the conformal weight h (and $\bar{h}$) of the operator is very large in the semi-classical limit $c \rightarrow \infty$, typically $h = \mathcal{O}(c)$, which is required so that it creates a back reaction. A more concrete correspondence can be obtained by identifying the ADM Hamiltonian in the AdS side and the CFT Hamiltonian $L_0 + \bar{L}_0$. For example, a non-spinning asymptotically AdS solution, such as the BTZ black hole, with the energy (=ADM mass) E corresponds to an operator with the scaling dimension Δ in the following relation:

$$\ell_{\text{AdS}} E = \Delta. \quad (2.15)$$

The Virasoro descendants correspond to the graviton excitations.

One approach to investigate the spectrum of the holographic CFT_2 is to consider the torus partition function, which takes the universal form

$$Z(\tau, \bar{\tau}) = \sum_{a,b} \mathcal{M}_{ab} \chi_a(\tau) \bar{\chi}_b(\bar{\tau}), \quad (2.16)$$

where $\mathcal{M}_{ab}$ is a matrix that has the information about the spectrum of the CFT and $\chi_a(\tau), \bar{\chi}_b(\bar{\tau})$ are the left and right Virasoro characters. As explained above, the semi-classical saddles in the gravity side naively include the vacuum AdS_3 and the BTZ black holes. The authors of [29] evaluates the gravitational partition function by summing over the geometries that have the torus with complex structure τ as their boundary, which amounts to summing over the $SL(2, \mathbb{Z})$ images of the vacuum AdS_3 . However, one can find that the resulting partition function cannot be that of any unitary CFT_2 ; the density of states is not positive definite. In order to obtain the appropriate partition function, some additional geometries would need to be included as saddle points. In [29] and the following papers [30–32], several possibilities for such additional contributions are considered. One possibility is to include the geometries that have conical defects, which can also be constructed as a quotient of AdS_3 . This would give a motivation to consider such singular geometries in the context of the AdS/CFT correspondence.

2.2.1 Chern-Simons gravity

Here, we review the Chern-Simons formulation of three-dimensional gravity [24, 25]. This formalism will be useful in various situations in the study of three-dimensional gravity. Later we show two uses; deriving boundary gravitons and extending to higher-spin gravity.

First let us explain the Chern-Simons theory itself. The Chern-Simons functional $I_{\text{CS}}[A]$ with the gauge group G , associated with the Lie algebra $\mathfrak{g}$, is given by

$$I_{\text{CS}}[A] = \frac{1}{4\pi} \int \text{tr} \left[A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right], \quad (2.17)$$

where A is a connection 1-form, which takes $\mathfrak{g}$ -valued, and the trace is taken in the fundamental representation of $\mathfrak{g}$. The Chern-Simons theory is a topological field theory associated with the action $kI_{\text{CS}}[A]$ with a constant k called the Chern-Simons level. Under the gauge transformation, $I_{\text{CS}}[A]$ is invariant up to a $2\pi\mathbb{Z}$ valued shift. Therefore if k is an integer, the path integral measure $\int \mathcal{D}A e^{ikI_{\text{CS}}[A]}$ is gauge invariant. This is the most famous class of Chern-Simons theories, which are equivalent to the two-dimensional Wess-Zumino-Witten (WZW) model associated with an affine Lie algebra $\widehat{G}_k$ [33].

In applications to gravity, we are sometimes interested in cases where k is complex-valued. Before diving into the gravity theory, here we discuss how to formulate the Chern-Simons theory with a complex level. First, let us consider the Chern-Simons theory with a complex group. The multivaluedness of $I_{\text{CS}}[A]$ only appears in its real part. Therefore we can define the gauge-invariant action in more general way as [34]

$$I = -s \text{Im } I_{\text{CS}}[A] + \ell \text{Re } I_{\text{CS}}[A], \quad s \in \mathbb{C}, \ell \in \mathbb{Z}, \quad (2.18)$$

and this can be rewritten as

$$I = \frac{t}{2} I_{\text{CS}}[A] + \frac{\tilde{t}}{2} I_{\text{CS}}[A^*], \quad (2.19)$$

with $t = \ell + is$, $\tilde{t} = \ell - is$, where A^* denotes the complex conjugate of A . Thus the action has to take the form of (2.19) in order for the path integral to be well-defined. The discussion so far is only based on the single-valuedness of e^{iI} . However, the path integral is not necessarily well-defined even if this condition holds. Indeed, the path integral defined by the standard integration region of A is not well-defined for an imaginary s . Therefore we need to perform an analytic continuation from a real s to define a convergent path integral along a deformed integration region [35]. On the other hand, if the action of our interest does not take the form (2.19), the factor e^{iI} is not invariant under the full gauge transformation. In fact, the non-zero shift of $I_{\text{CS}}[A]$ comes from a large gauge transformation (i.e. a gauge transformation that is not connected to an identity). Therefore such a theory is less symmetric; only invariant under the local gauge transformations [36].

Let us consider the three-dimensional Einstein gravity with a non-zero cosmological constant¹

$$I_{\text{EH}} = -\frac{1}{16\pi G} \int d^3x \sqrt{g} (R - 2\Lambda) \quad (2.20)$$

Even though we are interested in both Euclidean and Lorentzian gravities, here we always adopt the Euclidean notation with the weight $e^{-I_{\text{EH}}}$ in path integral. When we rewrite it in the first order formalism in terms of the vielbein e^a and the spin connection ω^a , the Einstein-Hilbert action can be expressed by two Chern-Simons actions as [25]

$$I_{\text{EH}}[e, \omega] = k \left(I_{\text{CS}}[A] - I_{\text{CS}}[\tilde{A}] \right). \quad (2.21)$$

¹We can also apply this formalism to $\Lambda = 0$, but here we only consider $\Lambda \neq 0$ for simplicity.

Here k is related to the cosmological constant as $k = 1/(4G\sqrt{-\Lambda})$, that is,

$$\begin{aligned} k &= \frac{\ell_{\text{AdS}}}{4G} & \text{for} & \quad \Lambda = -\frac{1}{\ell_{\text{AdS}}^2} < 0, \\ k &= -i\frac{\ell_{\text{dS}}}{4G} & \text{for} & \quad \Lambda = \frac{1}{\ell_{\text{dS}}^2} > 0 \end{aligned} \quad (2.22)$$

The relations of $A, \tilde{A}$ with e^a, ω^a are different among the four cases; AdS, EAdS, dS, and EdS².

- For EAdS or dS, $A, \tilde{A}$ are expressed as³

$$A = (\omega^a + i\sqrt{|\Lambda|}e^a)J_a, \quad \tilde{A} = (\omega^a - i\sqrt{|\Lambda|}e^a)J_a, \quad (2.23)$$

where J_a ($a = 0, 1, 2$) are a basis of $\mathfrak{su}(2)$ for EAdS and of $\mathfrak{sl}(2, \mathbb{R})$ for dS (see appendix A for the notations of the Lie algebras). Due to the presence of i in the coefficients of the second terms, the Lie algebra associated with the gauge group is $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ for EAdS and $\mathfrak{sl}(2, \mathbb{R})_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ for dS, where the subscript $\mathbb{C}$ means the complexification of the Lie algebra. Therefore, these two theories have a same complex gauge group generated by $\mathfrak{sl}(2, \mathbb{C})$, which is locally consistent with the isometry group $SO(1, 3)$, but have different real forms. In each case, A and $\tilde{A}$ are related to each other by some conjugation: For EAdS, $\tilde{A} = -A^\dagger$, and for dS, $\tilde{A} = A^*$. In both cases, the actions take the form (2.19) with $\ell = 0$ and $s = -2ik$, so they are invariant under the large gauge transformations. According to (2.22), s is real for dS and purely imaginary for EAdS, so precisely we need to perform an analytic continuation for EAdS. The metric tensor in these cases is given by

$$g_{\mu\nu} = -\frac{1}{2|\Lambda|} \text{tr} \left[(A_\mu - \tilde{A}_\mu)(A_\nu - \tilde{A}_\nu) \right]. \quad (2.24)$$

- For AdS or EdS, $A, \tilde{A}$ are expressed as

$$A = (\omega^a + \sqrt{|\Lambda|}e^a)J_a, \quad \tilde{A} = (\omega^a - \sqrt{|\Lambda|}e^a)J_a. \quad (2.25)$$

Here J_a ($a = 0, 1, 2$) are a basis of $\mathfrak{sl}(2, \mathbb{R})$ for AdS and of $\mathfrak{su}(2)$ for EdS (see Appendix A for the notations of generators). Therefore the gauge group is $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ for AdS and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ for EdS, which are locally consistent with the isometry groups $SO(2, 2)$ of AdS₃ and $SO(4)$ of dS₃. Hence A and $\tilde{A}$ are independent gauge fields. In contrast to the previous cases, these theories are not in the form (2.19), so only the local gauge transformations remain as the symmetry⁴. The metric tensor in these cases is given by

$$g_{\mu\nu} = \frac{1}{2|\Lambda|} \text{tr} \left[(A_\mu - \tilde{A}_\mu)(A_\nu - \tilde{A}_\nu) \right]. \quad (2.26)$$

These relations for $A, \tilde{A}$ are summarized in table 1. So far we have not taken care of the global structure of gauge groups since the classical action only depends on the local structure, i.e.

²Henceforth we sometimes call the Lorentzian Einstein gravity with a negative cosmological constant as simply “AdS gravity,” and same for EAdS, dS, and EdS.

³There are some literatures (e.g. [37]) to express EAdS Chern-Simons gravity as the Wick rotation from the Chern-Simons gravity for Lorentzian AdS, i.e. $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern-Simons theory. One can see that this approach is equivalent to ours by redefining the generators J_a as (A.12).

⁴If we fix k to be an integer in AdS case, of course, the theory remains the full gauge symmetry. Although an integral k is not natural in our context from the expression $k = \ell_{\text{AdS}}/4G$, such a theory has been considered in [38] to construct the dual gravity theory to the extremal CFTs.

	AdS ($\Lambda < 0$)	dS ($\Lambda > 0$)
Lorentzian	$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$	$\mathfrak{sl}(2, \mathbb{R})_{\mathbb{C}}$
Euclidean	$\mathfrak{su}(2)_{\mathbb{C}}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

Table 1: Gauge algebras for Chern-Simons gravity

the Lie algebra. Nevertheless, the global structure is relevant to non-local objects defined in Chern-Simons theory. As an example of such objects, one often consider the holonomy matrix defined as the path-ordered integral

$$H[A] = \mathcal{P} \exp \oint A \quad (2.27)$$

along a closed curve. The holonomy matrix plays an important role to characterize black holes in terms of Chern-Simons gravity, since it captures the non-trivial topology of spacetimes. Indeed, in AdS Chern-Simons gravity with $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, for the BTZ black holes $H[A]$ along a spatial cycle takes a non-trivial matrix [39]. Since the global AdS₃ spacetime has only shrinkable loops, the holonomy matrix should take the trivial element of the gauge group. However, global AdS₃ actually leads to

$$H[A_{\text{global AdS}_3}] = H[\tilde{A}_{\text{global AdS}_3}] = -\mathbf{1}. \quad (2.28)$$

Therefore we have to identify as $\mathbf{1} \sim -\mathbf{1}$, which amounts to considering a quotient gauge group $SL(2, \mathbb{R})/\mathbb{Z}_2 = SO(2, 1)$ [40].

We have to note that the argument of Chern-Simons gravity is nothing but the equivalence of the action. The gravitational path integral should include the summation over different topologies. Even though the Chern-Simons theory itself has the well-defined path integral measure, it may not give the non-perturbative definition of the gravitational path integral because the measure is defined on a fixed background.

2.2.2 Boundary gravitons

One advantage to consider Chern-Simons description of gravity is that its correspondence with WZW model can be applied to investigating the dual CFT₂, although WZW model itself is not the exact dual theory. This is analogous to the edge modes in the context of the quantum Hall effect whose effective theory is described by Chern-Simons theory. In AdS₃ gravity, the “edge mode” theory emerging at the asymptotic boundary of a spacetime is analogously called boundary gravitons. This type of study was first discussed in [41], and later refined in [42]. Here we briefly review their discussions about the canonical quantization.

As explained in the previous subsection, Lorentzian Einstein gravity with a negative cosmological constant is described by $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ Chern-Simons theory with the action (2.21). Let us consider the global AdS₃

$$ds^2 = -(r^2 + 1) dt^2 + \frac{dr^2}{r^2 + 1} + r^2 d\theta^2. \quad (2.29)$$

In this subsection, we set $\ell_{\text{AdS}} = 1$ for simplicity. In terms of gauge fields (2.25),

$$A = \begin{pmatrix} \frac{1}{2\sqrt{r^2+1}} dr & -\frac{\sqrt{r^2+1}-r}{2} dx^+ \\ \frac{1}{\sqrt{r^2+1}+r} dx^+ & -\frac{1}{2\sqrt{r^2+1}} dr \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} -\frac{1}{2\sqrt{r^2+1}} dr & -\frac{\sqrt{r^2+1}+r}{2} dx^- \\ \frac{1}{\sqrt{r^2+1}-r} dx^- & \frac{1}{2\sqrt{r^2+1}} dr \end{pmatrix}, \quad (2.30)$$

where $x^\pm = \pm t + \theta$. Since the global AdS_3 has cylinder topology, the equations of motion $F = \tilde{F} = 0$ lead to the expressions of $A, \tilde{A}$ in the pure gauge:

$$A = g^{-1}dg, \quad \tilde{A} = \tilde{g}^{-1}d\tilde{g}. \quad (2.31)$$

with $SL(2, \mathbb{R})$ -valued functions $g, \tilde{g}$.

We would like to perform the canonical quantization. It is useful to separate

$$A = A_t dt + A_i dx^i, \quad \tilde{A} = \tilde{A}_t dt + \tilde{A}_i dx^i, \quad (2.32)$$

where i runs the spatial coordinates, then each Chern-Simons action after discarding the boundary contributions takes the form

$$I_{\text{CS}}^{(\text{bulk})}[A] = \frac{k}{4\pi} \int d^3x \text{Tr}[A_\theta \partial_t A_r - A_r \partial_t A_\theta + 2A_t F_{r\theta}]. \quad (2.33)$$

A_t plays the role of the Lagrange multiplier imposing $F_{ij} = 0$. In order to impose the asymptotically AdS_3 condition, we have to add an unconventional boundary term to the action⁵. From (2.30), we can see the asymptotically AdS_3 condition implies $A_- = 0, \tilde{A}_+ = 0$. Those conditions are accomplished if we consider the following action:

$$I = I_{\text{CS}}^{(\text{bulk})}[A] - I_{\text{CS}}^{(\text{bulk})}[\tilde{A}] + I_{\text{bdy}}, \quad (2.34)$$

where $I_{\text{CS}}^{(\text{bulk})}$ is given by (2.33) and

$$I_{\text{bdy}} = \frac{k}{4\pi} \oint d^2x \text{Tr} [A_\theta^2 + \tilde{A}_\theta^2]. \quad (2.35)$$

Now we are ready to perform the reduction to two dimensions. From (2.31), the remaining components $A_i, \tilde{A}_i$ after integrating out $A_t, \tilde{A}_t$ can also be expressed as

$$A_i = g^{-1} \partial_i g, \quad \tilde{A}_i = \tilde{g}^{-1} \partial_i \tilde{g}. \quad (2.36)$$

Substituting them into (2.34), the action reduces to two chiral WZW actions

$$I = I_{\text{CWZW-}}[g] + I_{\text{CWZW+}}[\tilde{g}]. \quad (2.37)$$

However, this is not the ordinary non-chiral WZW model due to the asymptotic AdS_3 condition imposed on $g, \tilde{g}$. To reduce the theory further, it is useful to parametrize them by using the Gauss decomposition

$$g = \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & \Psi \\ 0 & 1 \end{pmatrix}, \quad \tilde{g} = \begin{pmatrix} 1 & -\tilde{F} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\lambda}^{-1} & 0 \\ 0 & \tilde{\lambda} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tilde{\Psi} & 1 \end{pmatrix}. \quad (2.38)$$

In terms of λ, F, Ψ , the asymptotically AdS_3 condition can be written as

$$\lambda = \sqrt{\frac{r}{F'}}, \quad \Psi = -\frac{F''}{2rF'}, \quad \text{as } r \rightarrow \infty, \quad (2.39)$$

where the prime $'$ denotes the derivative with respect to θ . The same form holds for $\tilde{\lambda}, \tilde{F}, \tilde{\Psi}$. Furthermore, we parametrize the boundary restriction of F as $F|_{r \rightarrow \infty} = \tan(\phi/2)$. From the

⁵See [43] for an alternative covariant construction of the boundary action.

single-valuedness of g , ϕ should satisfy the monotonicity $\phi' > 0$ and $\phi(\theta + 2\pi, t) = \phi(\theta, t)$, which means $\phi \in \text{Diff}(S^1)$. Finally we obtain

$$I = I_+[\phi] + I_-[\tilde{\phi}] \quad (2.40)$$

with

$$I_{\pm}[\phi] = -\frac{k}{4\pi} \int d^2x \left(\frac{\phi'' \partial_{\pm} \phi'}{\phi^2} - \phi' \partial_{\pm} \phi \right). \quad (2.41)$$

This theory is called Alekseev-Shatashvili theory [44].

The discussions so far were given by [41]. What is refined in [42] is the following: The expressions (2.31) and (2.36) are actually not in one-to-one correspondence. Replacing $g(t, r, \theta)$ in (2.31) with $h(t)g(t, r, \theta)$ leads to the same spatial components (2.36). Therefore we have to identify those connected by such a quasi-local $PSL(2, \mathbb{R})$ transformation $h(t)$. In terms of ϕ , this identification can be expressed as

$$\tan\left(\frac{\phi(\theta, t)}{2}\right) \sim \frac{a(t) \tan\left(\frac{\phi(\theta, t)}{2}\right) + b(t)}{c(t) \tan\left(\frac{\phi(\theta, t)}{2}\right) + d(t)}, \quad (2.42)$$

where $a(t), b(t), c(t), d(t)$ satisfy $a(t)d(t) - b(t)c(t) = 1$ for each t . Indeed, (2.41) is invariant under such a $PSL(2, \mathbb{R})$ transformation. Thus precisely $\phi \in \text{Diff}(S^1)/PSL(2, \mathbb{R})$.

It is known that the path integral formulation with the action (2.41) can be interpreted as the geometric quantization of a coadjoint orbit of the Virasoro group [44]. Coadjoint orbit method is a mathematical method to construct representations of non-compact Lie groups, where an orbit in the coadjoint space $\mathfrak{g}^*$ can be regarded as a representation space of Lie group G . For the Virasoro group, coadjoint orbits have been classified [45]. The theory (2.41) with $\phi \in \text{Diff}(S^1)/PSL(2, \mathbb{R})$ corresponds to the identity representation, and its Hilbert space is given by just the identity Verma module $\mathcal{V}_1$. Thus we can conclude that the boundary graviton theory on the global AdS_3 is two copies of the quantization of $\text{Diff}(S^1)/PSL(2, \mathbb{R})$, and the Hilbert space is

$$\mathcal{H} = \mathcal{V}_1 \times \mathcal{V}_1. \quad (2.43)$$

The boundary graviton theory we have obtained in this section is not a full CFT in the usual sense due to the lack of modular invariance. Our derivation fixed the background geometry as the global AdS_3 . A different background leads to a different boundary graviton theory, i.e., the quantization of a different coadjoint orbit. Therefore we have to note that the derived theory does not have any information about the spectrum of the holographic CFT_2 .

Although the original claim of [41] is that the dual CFT is described by Liouville field theory, this is obviously not correct as described above. However, there is a deep relationship between Liouville field theory and the Alekseev-Shatashvili theory (2.41) [46, 47]. In fact, the Liouville theory placed on a cylinder with specific FZZT [48, 49] or ZZ [50] types of boundary conditions being imposed is equivalent to the Alekseev-Shatashvili theory associated with a representation of the Virasoro group. This property can be applied to some calculations for 1d Schwarzian theory.

2.2.3 Minimal model holography

Another advantage to adopt the Chern-Simons formulation is that it is easier to extend to higher-spin gravity, which is a gravity theory including the higher rank symmetric massless tensor fields $\varphi_{\mu\nu\rho\cdots}$ as well as the metric field $g_{\mu\nu}$. A theory with higher-spin fields admits a larger gauge symmetry generated by

$$\varphi_{\mu\nu\rho\cdots} \mapsto \varphi_{\mu\nu\rho\cdots} + \nabla_{(\mu} \xi_{\nu\rho\cdots)}, \quad (2.44)$$

with a traceless tensor $\xi_{\nu\rho\cdots}(x)$. Therefore higher-spin gravity theories are more restricted than the ordinary Einstein gravity theory. An example of a consistent higher-spin gravity is known as the Vasiliev theory [51–53] (see also review articles [54, 55]), where the infinite tower of higher-spin fields with spins $s = 2, 3, \dots$ as well as a set of scalar fields are included for the consistency of theory.

In the context of holography, higher-spin gravity theories play an important role of serving concrete models of holography. An example is the relation between the four-dimensional Vasiliev theory on AdS_4 and the three-dimensional $O(N)$ vector model with large N limit [56]. However, since the four-dimensional Vasiliev theory has to necessarily include infinite tower of higher-spin fields, this example is far away from the familiar Einstein gravity. On the other hand, the situation would be rather relaxed in three dimensions as explained shortly. In three dimensions, the Vasiliev theory with the infinite tower of higher-spin fields is described by the Chern-Simons gravity with the higher-spin algebra $\mathfrak{hs}(\lambda)$, which is a kind of generalization of $\mathfrak{sl}(N)$ to a non-integer N . Indeed, for an integer $\lambda = N$, the higher-spin algebra $\mathfrak{hs}(\lambda)$ is truncated into $\mathfrak{sl}(N)$. Gaberdiel and Gopakumar proposed [57] (see [58] for a review) that the three-dimensional Vasiliev theory including additional two complex scalar fields with mass $\ell_{\text{AdS}}^2 m^2 = -1 + \lambda^2$ is holographically dual to the two-dimensional W_N minimal model, which is defined as a coset

$$\frac{SU(N)_k \times SU(N)_1}{SU(N)_{k+1}}, \quad (2.45)$$

with taking the 't Hooft limit, which is the limit $N, k \rightarrow \infty$ with keeping $\lambda = N/(N+k)$ fixed. This is actually the first version of the minimal model holography. A different version is proposed in [59] (see also [60]) by considering a duality among the coset models of the form (2.45) with the same central charge, which is called triality relation. By using the triality, one can see the dual CFT is equivalent to another coset model (2.45) with keeping N fixed. By using the expression of the central charge of (2.45)

$$c = (N-1) \left(1 - \frac{N(N+1)}{(N+k)(N+k+1)} \right), \quad (2.46)$$

we can obtain two candidates for k that gives fixed N and c :

$$k = -N - \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{4N(N^2-1)}{c-N+1}}. \quad (2.47)$$

Here we focus on the plus branch. In the large c limit, the level k behaves as

$$k = -N - \frac{N(N^2-1)}{c} + \mathcal{O}(c^{-2}). \quad (2.48)$$

Therefore the second version of minimal model holography states that the dual CFT_2 is the coset model (2.45) with fixed N and $k \rightarrow -N$ limit. An advantage to consider this limit is that the $SU(N)_k$ part of (2.45) will be dominant in the large c limit. Therefore in this regime the

coset can be approximated as a single $SU(N)_{-N-0}$ WZW model. This feature will be important in an analysis of a concrete model of dS_3/CFT_2 in §3.3.

From the latter version, it is clear that the dual CFT is non-unitary because k can be non-integer. An advantage to consider the latter version is that N can be set to a finite number. In this case, the dual gravity theory is $\mathfrak{sl}(N, \mathbb{R})$ Vasiliev theory, which includes a finite tower of higher spin fields with spins $s = 2, 3, \dots, N$. In particular, we mainly focus on $N = 2$ to study the dual CFT to the Einstein gravity, i.e. without higher-spin fields. This is an advantage to consider three dimensions rather than four dimensions, where the example is only known for the theory with full tower of higher-spin fields.

We make some comments on the minimal model holography.

- In the dual W_N minimal model (2.45) especially in the second version, we consider non-integer level k . In such a case the model is not the W_N minimal model in the usual sense as a rational CFT. Precisely speaking, our model should be regarded as the “generalized minimal model” [61, 62] (or its generalization to $N > 2$), whose spectrum consists of all the degenerate states.
- The Virasoro minimal model is a class of CFT_2 that does not have any other symmetry than Virasoro symmetry with $c < 1$. This is reminiscent of Liouville field theory, which also has only Virasoro symmetry though $c > 1$. The three-point function of Liouville field theory was proposed in [63, 64], and later another derivation by using the bootstrap approach is given by [65], where the existence of the degenerate operators are assumed and the three-point coefficients are constrained by the crossing symmetry with inserting degenerate operators. Due to the facts that the data used in the bootstrap procedure in [65] is same as the minimal model and the resulting correlation functions have a good analytic behavior, the correlation functions of the Virasoro minimal model can be thought of as an analytic continuation of Liouville correlation functions. In the same way, the correlation functions of the W_N coset model with general N are identical to those of the Toda field theory, as directly checked by [66] by using the free field representation of the coset theory.
- By investigating the representations of the coset (2.45), we can analyze the semi-classical saddle points of the gravity side. We can classify the solutions of bulk equation of motions [40] by using $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ Chern-Simons gravity and find the set of solutions that corresponds to the CFT spectrum. As a result, one can see that the “conical surplus” geometry with a negative deficit angle should exist. This is an interpretation in the gravity side on how the non-unitarity mentioned below (2.47) arises.

2.3 Holographic entanglement entropy

A powerful point of the AdS/CFT is that it may give non-perturbative correspondences. A typical example from this point of view is the holographic entanglement entropy formula, or Ryu-Takayanagi formula [67]. This is the formula that relates the entanglement entropy defined for CFT to a geometric quantity in the gravity side.

For a generic quantum system, associated with a Hilbert space $\mathcal{H}$, entanglement entropy is defined as follows: We take a subsystem A and assume that the Hilbert space $\mathcal{H}$ can be expressed as the tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ of the Hilbert spaces associated with A and its complement $\bar{A}$. Given a pure state $|\psi\rangle \in \mathcal{H}$, we define the reduced density state

$$\rho_A = \text{tr}_{\bar{A}} |\psi\rangle \langle \psi| . \quad (2.49)$$

Entanglement entropy for a state $|\psi\rangle$ and a subsystem A is then defined as the von Neumann entropy for ρ_A :

$$S(\rho_A) = -\text{tr}[\rho_A \log \rho_A]. \quad (2.50)$$

This quantity is known as the almost unique quantity measuring the amount of the entanglement of a pure state $|\psi\rangle$ between A and $\bar{A}$ [68]. See also [69] for a comprehensive review on quantum entanglement and entanglement measures.

In quantum field theories, a useful way to calculate entanglement entropy is so-called replica method [70]. There, deforming the expression (2.50) as

$$S(\rho_A) = -\left. \frac{\partial}{\partial n} \log \text{tr}[(\rho_A)^n] \right|_{n=1}, \quad (2.51)$$

the factor $\text{tr}[(\rho_A)^n]$ can be evaluated by the path integral on the n -replicated manifold. In particular, in CFT₂ case, we can apply strong properties of two-dimensional conformal symmetry to the calculation of $\text{tr}[(\rho_A)^n]$. The final result for CFT₂ is [70]

$$S(\rho_A) = \frac{c}{3} \log \frac{L}{\epsilon} + \mathcal{O}(\epsilon^0) \quad (2.52)$$

where L is the length of the subsystem A and ϵ is the lattice spacing. The first logarithmic term is known to be universal in CFT₂, so the formula even holds in the holographic systems.

The argument of [67] is that the entanglement entropy for the vacuum state and a subsystem A is identical to the following formula

$$S_A = \min_{\partial\Gamma=\partial A} \left\{ \frac{\text{Area}(\Gamma)}{4G} \right\}, \quad (2.53)$$

where the right-hand side accounts for the minimal area surface among the bulk codimension-2 surfaces whose boundaries are fixed to ∂A , in the unit of $4G$. Here let us check that this formula indeed reproduces (2.52) in AdS₃/CFT₂. We consider the subsystem A as an interval $-L/2 < x < L/2$ on the boundary of Poincaré AdS₃

$$ds^2 = \ell_{\text{AdS}}^2 \frac{dz^2 - dt^2 + dx^2}{z^2}. \quad (2.54)$$

The area functional for a curve $x(z)$ is given by

$$A[x] = \int_{\epsilon}^{z_*} dz \frac{\sqrt{1 + x'(z)^2}}{z}, \quad (2.55)$$

where $z = \epsilon$ is the UV cutoff and $z = z_*$ is the turning point satisfying $dz/dx|_{z_*} = 0$. The solution to the variation problem for this functional with boundary condition $x(\epsilon) = \pm L/2$ is

$$x^2 + z^2 = \left(\frac{L}{2} \right)^2. \quad (2.56)$$

The formula (2.53) then leads to

$$S_A = \frac{\ell_{\text{AdS}}}{2G} \log \frac{L}{\epsilon} + \mathcal{O}(\epsilon^0) = \frac{c}{3} \log \frac{L}{\epsilon} + \mathcal{O}(\epsilon^0). \quad (2.57)$$

In the final equality we have used the Brown-Henneaux formula (2.13).

Let us finally summarize the developments on the formula for holographic entanglement entropy. The original Ryu-Takayanagi formula [67] is applicable for the setup with time reversal symmetry, and so also the Euclidean AdS/CFT that can be continued from such the Lorentzian AdS/CFT. This formula has been derived by [71]. Shortly after the proposal of Ryu-Takayanagi formula, the generalization to Lorentzian geometry with nontrivial time evolution is proposed by [72], where the minimization in (2.53) is replaced with extremization as

$$S_A = \min_{\partial\Gamma=\partial A}^{\text{ext}} \left\{ \frac{\text{Area}(\Gamma)}{4G} \right\}, \quad (2.58)$$

and its derivation has been given by [73]. These formulae only have classical contributions from a fixed geometry. It is also important to extend them to the generalized entanglement entropy, including the contributions from matter fields on the fixed background. This extension has been considered by [74] by using the analogous method to Wald entropy [75, 76], and finally leads to the “quantum extremal surface” formula [77]. In particular, the latter formula plays an essential role in recent developments on the information loss of black holes [78–82].

3 dS/CFT

One interesting and important direction to understand the holographic principle is to try going beyond AdS. The most natural target would be dS spacetime. The metric of the global dS_{d+1} is given by

$$\text{d}s^2 = \ell_{\text{dS}}^2 (-\text{d}\tau^2 + \cosh^2 \tau \text{d}\Omega_d^2), \quad (3.1)$$

where $\text{d}\Omega_d^2$ is the metric of the unit sphere S^d . Both dS and AdS spacetimes are maximally symmetric space with constant curvatures, and dS spacetime can be realized as an analytic continuation from EAdS. One can easily see this, for example, by focusing on the Poincaré coordinate of EAdS_{d+1}

$$\text{d}s^2 = \ell_{\text{AdS}}^2 \frac{\text{d}z^2 + \text{d}x_i \text{d}x^i}{z^2}. \quad (3.2)$$

By taking the continuations

$$z \rightarrow -i\eta, \quad \ell_{\text{AdS}} \rightarrow -i\ell_{\text{dS}}, \quad (3.3)$$

it leads to

$$\text{d}s^2 = \ell_{\text{dS}}^2 \frac{-\text{d}\eta^2 + \text{d}x_i \text{d}x^i}{\eta^2}, \quad (3.4)$$

which is called the planar coordinates (or inflationary coordinates) $\mathcal{F}$ of dS_{d+1} with $\eta \in (-\infty, 0)$. The future boundary $\mathcal{I}^{\text{out}}$ is located at $\eta = 0$ and the horizon corresponds to $\eta = -\infty$. This coordinate can cover only a half of the global dS, see figure 1. The other half $\mathcal{P}$ is covered by the metric with the same form with (3.4) but $\eta \in (0, \infty)$ by taking $\eta \rightarrow -\eta$. We will mainly consider the future half. The dS spacetime has an analogous asymptotic structure to AdS. A big difference is that the asymptotic boundary of the global dS is space-like and located at the future and past infinity, and there is no time-like boundary. This fact makes studies of dS holography hard compared with the AdS/CFT correspondence, in which the notion of times of the two dual theories is identical.

Therefore there are various proposals of dS holography, coming from the difference of places where the holographic screen is located.

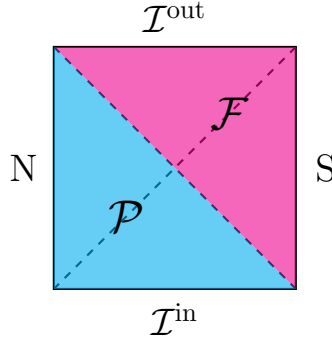


Figure 1: The Penrose diagram for dS_{d+1} . The global coordinates (3.1) cover the entire space-time. The planar coordinates (3.4) with $\eta \in (-\infty, 0)$ cover the future half $\mathcal{F}$, and the past half $\mathcal{P}$ is covered by the same metric with $(0, \infty)$.

The first proposal of dS holography was given by [11] (see also [12, 13]), where the holographic screen is located at the future asymptotic boundary. We call this type of dS holography as the dS/CFT correspondence. Since the boundary of dS corresponds to the ordinary asymptotic boundary of AdS under the analytic continuation (3.3), the asymptotic symmetry of dS is similar to that of AdS but with a different complex structure. In three dimensions, the asymptotic symmetry of dS_3 is generated by the Virasoro algebra with a purely imaginary central charge [11, 13]

$$c = -i \frac{3\ell_{\text{dS}}}{2G}. \quad (3.5)$$

In general dimensions dS_{d+1} ,

$$c \sim (-i)^{d-1} \frac{\ell_{\text{dS}}^{d-1}}{G_N^{(d+1)}}, \quad (3.6)$$

which does not take positive real values when $d \not\equiv 1 \pmod{4}$. Therefore the dual CFT is obviously non-unitary. Such a CFT is highly unphysical theory in the sense that no physical system described by such a theory is known. Nevertheless, the dS/CFT can be actually applied to the calculations of gravitational quantities [13].

There are also other proposals of dS holography, in the motivation of finding the holographic duality with physical systems, which include the dS/dS correspondence [83–85], the surface/state duality [86], the static patch holography [87–89], and so on. See [90–92] for other recent developments.

3.1 Field-operator dictionary in dS/CFT

In this subsection, let us discuss the dictionary between a bulk field and a CFT operator in the dS/CFT, in analogy with GKPW relation of the AdS/CFT discussed in §2.1.

Field theories on de Sitter spacetime are known to have one parameter family of the candidate vacuum states [93, 94]. Among them, two special vacua are physically interesting; the Euclidean vacuum $|E\rangle$ [95, 96], which is constructed as the analytic continuation from the $d+1$ -sphere S^{d+1} , and the out (in) vacuum $|out\rangle$ ($|in\rangle$), which has no particle on $\mathcal{I}^{\text{out}}$ ($\mathcal{I}^{\text{in}}$). The existence of multiple vacua is analogous to special cases of the AdS/CFT where both $\Delta_{\pm}$ modes are normalizable as commented in §2.1. In dS case, both two asymptotic modes are indeed

normalizable for any mass as seen later. Therefore the field-operator dictionary of the dS/CFT also has a variety of one parameter family. In discussing the dictionaries, two different patches (3.1) and (3.4) of dS_{d+1} also play important roles.

Here we only summarize the outline of discussions. See [97, 98] for the detailed calculations.

3.1.1 Planar coordinates

Let us start with repeating the similar discussion to §2.1 in the planar coordinates. In the planar coordinates (3.4), the equation of motion for a scalar field ϕ is

$$\partial_\eta^2 \phi - \frac{d-1}{\eta} \partial_\eta \phi - \partial_i \partial^i \phi + \frac{m^2 \ell_{\text{dS}}^2}{\eta^2} \phi = 0. \quad (3.7)$$

In the asymptotic regime $\eta \sim 0$, this equation has two independent solutions $|\eta|^{\Delta_\pm}$ with

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - m^2 \ell_{\text{dS}}^2}. \quad (3.8)$$

These take real-valued when $m^2 \ell_{\text{dS}}^2 \leq d^2/4$, and imaginary-valued when $m^2 \ell_{\text{dS}}^2 > d^2/4$. In either case, both $\Delta_\pm$ are normalizable for any m . This is a distinct point from the AdS case, where either Δ or $d - \Delta$ could not be normalizable as long as m^2 satisfies a condition $m^2 \ell_{\text{AdS}}^2 > -d^2/4 + 1$. Henceforth we only focus on the region $m^2 \ell_{\text{dS}}^2 > d^2/4$ for simplicity. In this case, the two modes $\Delta_\pm$ are imaginary and complex conjugate to each other.

Let us perform the mode expansion of ϕ . When we focus on $\mathcal{F}$,

$$\phi^{\text{out}}(\eta, x) = \sum_{n=0}^{\infty} (a_n^{\text{out}, \Delta_+} \phi_n^{\text{out}, \Delta_+}(\eta, x) + a_n^{\text{out}, \Delta_-} \phi_n^{\text{out}, \Delta_-}(\eta, x)), \quad (3.9)$$

where $\phi_n^{\text{out}, \Delta_\pm}$ are the eigenmodes that asymptotically behave as

$$\phi_n^{\text{out}, \Delta_\pm} \sim |\eta|^{\Delta_\pm + n} \quad (3.10)$$

in $|\eta| \sim 0$. Since $\Delta_\pm$ are complex conjugate to each other, $\phi^{\text{out}, \Delta_-} = (\phi^{\text{out}, \Delta_+})^*$, and then the coefficient operators satisfy

$$a_n^{\text{out}, \Delta_-} = (a_n^{\text{out}, \Delta_+})^\dagger, \quad [a_n^{\text{out}, \Delta_+}, a_m^{\text{out}, \Delta_-}] = \delta_{nm}. \quad (3.11)$$

Therefore, a straightforward definition of vacuum is

$$a_n^{\text{out}, \Delta_+} |\text{out}\rangle = 0, \quad (3.12)$$

and this $|\text{out}\rangle$ is nothing but the out state. In this case, the field ϕ^{out} corresponds to a CFT operator with scaling dimension Δ_- . If we define the vacuum by $a_n^{\text{out}, \Delta_-} |\text{out}\rangle = 0$, then the corresponding operator will have scaling dimension Δ_+ .

The almost same discussion can be applied to the field defined in $\mathcal{P}$ patch, then

$$\phi^{\text{in}}(\eta, x) = \sum_{n=0}^{\infty} (a_n^{\text{in}, \Delta_+} \phi_n^{\text{in}, \Delta_+}(\eta, x) + a_n^{\text{in}, \Delta_-} \phi_n^{\text{in}, \Delta_-}(\eta, x)), \quad (3.13)$$

and the eigenfunctions and coefficients are defined in the same way as above. If the in vacuum $|\text{in}\rangle$ is defined as

$$a_n^{\text{in}, \Delta_+} |\text{in}\rangle = 0, \quad (3.14)$$

then the corresponding CFT operator has the scaling dimension Δ_- .

When the in or out vacuum is assumed, the field-operator dictionary is one-to-one correspondence. However, constructing the Euclidean vacuum needs further discussions. In order to construct the Euclidean vacuum, it will be helpful to discuss in the global coordinates, in which the metric is related to the sphere metric by an analytic continuation.

3.1.2 Global coordinates

The equation of motion on the global coordinates is

$$\partial_\tau^2 \phi + d \tanh \tau \partial_\tau \phi - \frac{1}{\cosh^2 \tau} \nabla_{S^d}^2 \phi + m^2 \ell_{\text{dS}}^2 \phi = 0. \quad (3.15)$$

Decomposing as $\phi = y_L(\tau) Y_L^j(\Omega)$ with the spherical harmonics Y_L^j satisfying⁶

$$\nabla_{S^d}^2 Y_L^j = -L(L+d-1) Y_L^j, \quad Y_L^j(\Omega_A) = (-1)^L Y_L^j(\Omega), \quad (3.16)$$

where Ω_A denotes the antipodal point to Ω , the equation of motion reduces to

$$y_L'' + d \tanh \tau y_L' + \left(m^2 \ell_{\text{dS}}^2 + \frac{L(L+d-1)}{\cosh^2 \tau} \right) y_L = 0. \quad (3.17)$$

One can find two solutions to this equation

$$y_L^{\text{in}}(\tau) = \frac{2^{L+d/2+1}}{\sqrt{\mu}} \cosh^L \tau e^{(L+d/2-i\mu)\tau} {}_2F_1 \left(L + \frac{d}{2}, L + \frac{d}{2} - i\mu, 1 - i\mu, -e^{2\tau} \right), \quad (3.18)$$

$$y_L^{\text{out}}(\tau) = \frac{2^{L+d/2+1}}{\sqrt{\mu}} \cosh^L \tau e^{-(L+d/2+i\mu)\tau} {}_2F_1 \left(L + \frac{d}{2}, L + \frac{d}{2} + i\mu, 1 + i\mu, -e^{-2\tau} \right), \quad (3.19)$$

where $\mu = \sqrt{m^2 - d^2/4}$. Note that these two solutions are related by

$$y_L^{\text{out}}(\tau) = y_L^{\text{in}}(-\tau)^*, \quad (3.20)$$

which leads to the antipodal relation between “in” and “out” modes

$$\phi_{L,j}^{\text{in}}(x_A) = \begin{cases} \phi_{L,j}^{\text{out}}(x)^* & \text{even dimension (odd } d) \\ \phi_{L,j}^{\text{in}}(x)^* & \text{odd dimension (even } d) \end{cases} \quad (3.21)$$

where $x = (\tau, \Omega)$ and $x_A = (-\tau, \Omega_A)$. Furthermore, the resulting solutions $\phi^{\text{in}}, \phi^{\text{out}}$ are related by a transformation

$$\phi_{L,j}^{\text{in}} = A \phi_{L,j}^{\text{out}} + B \phi_{L,j}^{\text{out}*}, \quad (3.22)$$

with some complex coefficients A, B . Especially, for odd dimensional dS spacetime (i.e. even d), the coefficients take $A = 1, B = 0$, then $\phi_{L,j}^{\text{in}} = \phi_{L,j}^{\text{out}}$.

In terms of the global coordinates, the two vacuum states $|\text{in}\rangle$ and $|\text{out}\rangle$ introduced in the previous subsection are defined by using the annihilation operators associated with ϕ^{in} and ϕ^{out} , respectively. In odd dimensions, due to the equivalence of the two modes, we have $|\text{in}\rangle = |\text{out}\rangle$.

Next, let us consider the Euclidean vacuum. We consider another solution:

$$\phi_{L,j}^{\text{E}} = y_L^{\text{E}}(\tau) Y_L^j(\Omega), \quad (3.23)$$

⁶Note that the spherical harmonics used here is different from the usual definition, see eq. (3.6) in [97].

where

$$y_L^E = C_L \cosh^L \tau e^{(L+d/2+i\mu)\tau} {}_2F_1 \left(L + \frac{d}{2}, L + \frac{d}{2} + i\mu, 2L + d, 1 + e^{2\tau} \right) \quad (3.24)$$

with a constant C_L depending on L . This solution is regular at the pole $\tau = -i\pi/2$ of the lower Euclidean hemisphere. We define the vacuum state $|E\rangle$ defined by using the annihilation operators associated with $\phi_{L,j}^E$ as the Euclidean vacuum.

We have seen that various vacuum states can be constructed by using different solutions to the equation of motion. The existence of various vacua comes from the fact that both two modes with $\Delta_{\pm}$ are normalized, which can lead to other solutions as various linear combinations.

By using the explicit forms of $\phi_{L,j}^{\text{in}}$ and $\phi_{L,j}^E$, we can check the transformation from the Euclidean modes to the “in” modes

$$\phi_{L,j}^{\text{in}} = \frac{1}{1 - e^{-2\pi\mu}} \left(\phi_{L,j}^E - (-1)^{d/2} e^{-\pi\mu} \phi_{L,j}^{E*} \right). \quad (3.25)$$

We can also construct more general modes by transforming as

$$\phi_{L,j}^{\alpha} = \frac{1}{\sqrt{1 - e^{\alpha+\alpha^*}}} \left(\phi_{L,j}^E + e^{\alpha} \phi_{L,j}^{E*} \right), \quad (3.26)$$

which leads to so-called α -vacua $|\alpha\rangle$. Among α -vacua, the Euclidean vacuum is physically interesting in the sense that the Green’s function

$$G(x, x') = \sum_{L,j} \phi_{L,j}^E(x) \phi_{L,j}^{E*}(x') \quad (3.27)$$

has the anti-podal singularity at $x' = x_A$ unless $\alpha = 0$, which is unphysical in the Minkowski limit where the cosmological horizon will disappear.

Finally let us discuss the holographic dictionaries in the global coordinates. A solution ϕ generally corresponds to four CFT operators $\mathcal{O}_{\Delta_{\pm}}^{\text{in}}, \mathcal{O}_{\Delta_{\pm}}^{\text{out}}$. Among them, $\mathcal{O}_{\Delta_+}^{\text{in}}$ and $\mathcal{O}_{\Delta_-}^{\text{in}}$ couple to positive and negative frequency modes of ϕ on $\mathcal{I}^{\text{in}}$, respectively. Similarly, $\mathcal{O}_{\Delta_+}^{\text{out}}$ and $\mathcal{O}_{\Delta_-}^{\text{out}}$ couple to negative and positive frequency modes of ϕ on $\mathcal{I}^{\text{out}}$, respectively. The existence of four operators is a difference from the dS/CFT for the planar coordinates. However, since “in” and “out” modes are related as (3.22), only two of those four operators are independent. The Green functions for each vacuum state are evaluated in [97, 98]. From the result, we can predict that the CFT interpretation of the parameter α is a marginal deformation by a composite operator $\mathcal{O}_{\Delta_+} \mathcal{O}_{\Delta_-}$, which has conformal dimension $(1, 1)$. The corresponding CFT states in the case of Euclidean vacuum have been recently constructed in [99].

3.2 dS₃/CFT₂

As in the AdS₃/CFT₂, the $d = 2$ case is an important target to study the gravity sector of the dS/CFT. Here, assuming that the dual CFT₂ to dS₃ exists, we explain its properties.

As explained above, the central charge takes the purely imaginary value

$$c = -i \frac{3\ell_{\text{dS}}}{2G} \equiv -i c^{(g)}. \quad (3.28)$$

Throughout this paper we use the notation $c^{(g)}$ for the imaginary part. The semi-classical limit $G \rightarrow 0$ then corresponds to the large $c^{(g)}$ limit.

In the similar way to $\text{AdS}_3/\text{CFT}_2$, the saddle points of the gravitational path integral would correspond to those of the CFT side. The dictionary between the bulk energy and the scaling dimension of the heavy operator is derived as the analytic continuation from (2.15), then

$$\ell_{\text{dS}} E = \Delta^{(g)}, \quad (3.29)$$

where $\Delta^{(g)}$ is the imaginary part of the scaling dimension:

$$\Delta = -i\Delta^{(g)}. \quad (3.30)$$

We will also use the similar notation $h = -ih^{(g)}$ for conformal weights.

The asymptotic boundary is now at the future infinity, so we can regard the gravity side of the dictionary as the wave functional of universe

$$\Psi[h_{\mu\nu}] = \int_{g_{\mu\nu}|_{\text{bdy}}=h_{\mu\nu}} \mathcal{D}g_{\mu\nu} e^{iI_{\text{EH}}[g_{\mu\nu}]}. \quad (3.31)$$

To define this quantity, we have to further impose an initial boundary condition. In the semi-classical limit, this can be approximated again by saddle points

$$\Psi \sim \sum_{\text{saddles}} e^{S_{\text{GH}}/2 + iI}. \quad (3.32)$$

Here the real part in an exponent $S_{\text{GH}}/2$ comes from the initial condition and are understood as the Gibbons-Hawking entropy. The most famous example of the saddle point would be the no-boundary saddle point by Hartle and Hawking [5], in which

$$S_{\text{GH}} = \frac{\pi \ell_{\text{dS}}}{2G} \quad (3.33)$$

is identical to the dS entropy [95], i.e., the entropy from the cosmological horizon.

In the three-dimensional gravity with a positive cosmological constant, there does not exist any black hole geometry. Indeed, the three-dimensional version of the Schwarzschild-de Sitter geometry

$$\frac{ds^2}{\ell_{\text{dS}}^2} = -(1 - 8GE - r^2) dt^2 + \frac{dr^2}{1 - 8GE - r^2} + r^2 d\phi^2. \quad (3.34)$$

has only one horizon at $r_+ = \sqrt{1 - 8GE}$, which can be regarded as the cosmological horizon. When $E = 0$, this reduces to the static patch of the pure dS_3 . In another useful coordinate $r = r_+ \sin \theta$ ($0 < \theta < \pi/2$), the metric after the Wick rotation $t \rightarrow -i\tau$ is expressed as

$$\frac{ds^2}{\ell_{\text{dS}}^2} = r_+^2 \cos^2 \theta d\tau^2 + d\theta^2 + r_+^2 \sin^2 \theta d\phi^2. \quad (3.35)$$

Since $\theta = 0$ and $\theta = \pi/2$ correspond to the center $r = 0$ and the horizon $r = r_+$, respectively, one can see that the inverse temperature is $\beta = 2\pi/r_+$ and the geometry has a conical defect with a deficit angle $2\pi(1 - r_+)$ at the center $r = 0$. Therefore the geometry (3.34) is not a black hole, but has a conical defect. In the CFT side, this saddle point corresponds to two heavy operator insertion with $h = -i\ell_{\text{dS}} E/2$.

3.3 Concrete examples

One difficulty to study the dS/CFT is that there are only a few known concrete examples. In the case of AdS/CFT, the constructions of most concrete models rely on the top-down construction from the superstring theory. However, it is known that there is no dS compactification from supergravity [100]. Furthermore, it is known to be hard to construct a supersymmetric theory on the dS background. Therefore, it is not reasonable to rely on the top-down construction from the superstring theory, and also supersymmetry itself.

As mentioned in §2.2.3, there are some bottom-up constructions of examples of the AdS/CFT that do not have supersymmetry, but include higher-spin fields. The first example of the dS/CFT was proposed for 4d/3d [101], where the Vasiliev theory on dS₄ is conjectured to be dual to the large N $Sp(N)$ vector model, by considering an analytic continuation $N \rightarrow -N$ from the $O(N)$ vector model [56].

Concrete examples for 3d/2d has been proposed in [14, 15], where the dual CFT₂ is constructed by an analytic continuation of the minimal model holography explained in §2.2.3. In particular, we focus on the second version of minimal model holography: The dual CFT₂ to the Vasiliev theory on AdS₃ is the coset model (2.45) with the level k behaving as (2.48). Here let us consider the $N = 2$ case, where the gravity side does not include higher-spin fields and the dual CFT₂ becomes the (generalized) Virasoro minimal model.

$$\frac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}}. \quad (3.36)$$

In this case, the behavior of k in the large c is

$$k = -2 - \frac{6}{c} + \mathcal{O}(c^{-2}). \quad (3.37)$$

Let us perform the analytic continuation $\ell_{\text{AdS}} \rightarrow -i\ell_{\text{dS}}$. Since the central charge is

$$c = -ic^{(g)} = -i\frac{3\ell_{\text{dS}}}{2G}, \quad (3.38)$$

the level k behaves as

$$k \rightarrow -2 - \frac{6i}{c^{(g)}}. \quad (3.39)$$

Therefore we can predict that the dual CFT₂ to the Vasiliev theory on dS₃ is the coset model (3.36), and the semi-classical limit $G \rightarrow 0$ corresponds to the limit of k to -2 from the imaginary direction. Recalling that the original minimal model holography has already considered the generalized minimal model with non-integer k , it would be straightforward to extend k to complex values (see e.g. [62] for discussions with complex k). In this limit, as commented in §2.2.3, the first $SU(2)_k$ WZW part is dominant in (3.36) because the central charge of the theory is dominated by that of $SU(2)_k$ part. Therefore we can propose that the dual CFT to the Einstein gravity with a positive cosmological constant can be described as $SU(2)_k$ WZW model in the limit (3.39) [14, 15]. The remaining parts of (3.36) are interpreted as additional matter sectors whose contributions are subleading in the semi-classical limit.

By using this proposal, we can compute some gravitational quantities in an efficient way. As a simple example, let us consider the square of the wave functional of universe $|\Psi|^2$, which is approximated by the gravitational partition function $\mathcal{Z}[S^3]$ on the sphere. By using the

dS/CFT correspondence, this corresponds to the square of the WZW partition function without any operator insertions

$$\mathcal{Z}[S^3] \sim |Z_{\text{WZW}}[S^2]|^2. \quad (3.40)$$

See the introduction of [15] for a more detailed discussion on this relation. Since the WZW partition function without any operator insertions has the expression by means of an element of the modular S-matrix

$$Z_{\text{WZW}}[S^2] = \mathcal{S}_0^0 = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right), \quad (3.41)$$

The gravitational partition function $\mathcal{Z}[S^3]$ in the limit (3.39) can be evaluated as

$$\mathcal{Z}[S^3] \sim \exp\left(\frac{\pi c^{(g)}}{3}\right) = \exp\left(\frac{\pi \ell_{\text{dS}}}{2G}\right). \quad (3.42)$$

The leading contribution in the argument of the exponential is identical to the de Sitter entropy (3.33). This identification would be a support for our conjecture.

We can further consider operator insertions in WZW model corresponding to the three-dimensional version of Schwarzschild-de Sitter geometry. See [15] for such cases. Furthermore, the reduction of this model to two dimensions is discussed in [102].

As commented in §2.2.3, correlation functions in the CFT side of the minimal model holography can also be described by those of Liouville field theory. Since the correlation functions of Liouville field theory have good analytic properties as explained later, this fact is often useful to discuss the analytic continuation to dS₃/CFT₂. Indeed, we will mainly analyze the Liouville field theory to investigate the structure of dS₃/CFT₂ in the main part of this thesis.

3.4 Pseudo entropy in dS/CFT

It would be natural to expect that quantum entanglement play some important roles to understand de Sitter quantum gravity. In this subsection, we discuss how to extend the holographic entanglement entropy formula (2.53) to the dS/CFT, following [103, 104].

We would like to define the entanglement entropy in the dual CFT, which is defined on $\mathbb{S}^d$ at the future $\tau = \tau_\infty \gg 1$ of the global dS_{d+1} (3.1) with a large radius $\cosh \tau_\infty \simeq \frac{1}{2}e^{\tau_\infty}$. We choose an arbitrary direction in $\mathbb{S}^d$ as an imaginary time t_E and take a subsystem A on a time slice $t_E = 0$. Note that the imaginary time t_E is different from the time τ in the global dS. A state on the time slice is prepared by path integral as usual. However, since the CFT is non-unitary, a ket state $|\Psi\rangle$ and a bra state $\langle\Psi|$, which are prepared by the path integrals of the past and future halves of $\mathbb{S}^d$ with the same insertions of operators, are different in general, i.e., $\langle\Psi| \neq |\Psi\rangle^\dagger$. This fact can be seen by thinking of $|\Psi\rangle$ and $\langle\Psi|$ as the semi-infinite imaginary time evolution from the “vacuum”, corresponding to without operator insertions, with non-Hermitian Hamiltonian. Therefore we should call the operator

$$\mathcal{T} = |\Psi\rangle \langle\Psi| \quad (3.43)$$

as a transition matrix instead of density matrix following [105]. Since $\langle\Psi| \neq |\Psi\rangle^\dagger$, the operator $\mathcal{T}$ is not Hermitian. For this reason, henceforth we call the quantity

$$S_A = -\text{tr}[\mathcal{T}_A \log \mathcal{T}_A], \quad \mathcal{T}_A = \text{tr}_{\bar{A}} \mathcal{T} \quad (3.44)$$

as pseudo entropy, which can take imaginary values. Thus in the dS/CFT, we expect that the holographic relation analogous to holographic entanglement entropy in the AdS/CFT holds for pseudo entropy rather than entanglement entropy.

Here we restrict ourselves to dS₃/CFT₂ for simplicity. See [104] for calculations in higher dimensions. We assume the dS background is the Hartle-Hawking geometry

$$ds^2 = \ell_{\text{dS}}^2 (-d\tau^2 + \cosh^2 \tau (dt_E^2 + \cos^2 t_E d\theta^2)), \quad (\tau > 0) \quad (3.45)$$

glued with the Euclidean part corresponding to the initial state

$$ds^2 = \ell_{\text{dS}}^2 (d\tau_E^2 + \cos^2 \tau_E (dt_E^2 + \cos^2 t_E d\theta^2)), \quad (0 < \tau_E < \pi), \quad (3.46)$$

along $\tau = \tau_E = 0$. On the boundary $\tau = \tau_\infty$, we take a subsystem A on a time slice $t_E = 0$ as an arc of a region $-\phi_0/2 < \phi < \phi_0/2$. In this case, the pseudo entropy of the CFT dual to dS₃ is expected to dual to the geodesic length. However, there seems to be a problem that we cannot connect the two edges ∂A by a single geodesic because a geodesic emanating from an edge of A should be timelike so that it cannot turn back to the other edge of A in the global dS. Indeed, if we naively apply the geodesic formula, the length takes an imaginary value

$$\begin{aligned} D &= \ell_{\text{dS}} \cos^{-1}(\cosh^2 \tau_\infty \cos \phi_0 - \sinh^2 \tau_\infty) \\ &\simeq \ell_{\text{dS}} \cos^{-1} \left(-\frac{1}{2} e^{2\tau_\infty} \cos^2 \frac{\phi_0}{2} \right) \\ &= \pi \ell_{\text{dS}} - 2i \ell_{\text{dS}} \log \left(e^{\tau_\infty} \cos \frac{\phi_0}{2} \right). \end{aligned} \quad (3.47)$$

Hence this is not a geodesic in the usual sense. Nevertheless we can give an interpretation to this “geodesic length”. We consider a possibility that a timelike geodesic emanating from an edge of A goes through the Euclidean part of geometry and turns back to the other edge of A , as depicted in figure 2. Let us assume this configuration and impose the “extremality” condition that both the real and imaginary parts of the length is extremized.

We fix the points that the two timelike curves from $\phi_0/2$ and $-\phi_0/2$ on the future boundary will attach at $\tau = 0$ as $\tilde{\phi}_1$ and $\tilde{\phi}_2$. First we consider the real part. The geodesic length of the Euclidean part is obtained by varying the length functional

$$\ell_{\text{dS}} \int dt_E \sqrt{1 + \cos^2 t_E \left(\frac{d\phi(t_E)}{dt_E} \right)^2}, \quad (3.48)$$

which leads to a condition

$$\tilde{\phi}_1 - \tilde{\phi}_2 = \pi. \quad (3.49)$$

The resulting length is then

$$\pi \ell_{\text{dS}}. \quad (3.50)$$

Therefore the geodesic takes the largest length in the hemisphere, depicted as a green line in figure 2. Next we consider the imaginary part. The sum of geodesic length with fixing $\tilde{\phi}_1$ and

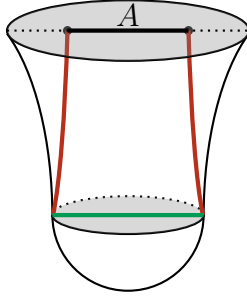


Figure 2: The geodesic configuration whose length reproduces the pseudo entropy in the dual CFT. The red curve means the timelike geodesic, which contributes to the imaginary part of the pseudo entropy, and the green curve is the spacelike geodesic giving the real part, which goes through the boundary of the hemisphere $\mathbb{B}^3$.

$\tilde{\phi}_2$ is⁷

$$\begin{aligned} & \ell_{\text{dS}} \cos^{-1} \left[\cosh \tau_{\infty} \cos \left(\frac{\phi_0}{2} - \tilde{\phi}_1 \right) \right] + \ell_{\text{dS}} \cos^{-1} \left[\cosh \tau_{\infty} \cos \left(\frac{\phi_0}{2} + \tilde{\phi}_2 \right) \right] \\ & \simeq -i \ell_{\text{dS}} \log \left[e^{2\tau_{\infty}} \cos \left(\frac{\phi_0}{2} - \tilde{\phi}_1 \right) \cos \left(\frac{\phi_0}{2} + \tilde{\phi}_2 \right) \right] \\ & = -i \ell_{\text{dS}} \log \left[-\frac{1}{2} e^{2\tau_{\infty}} (\cos \phi_0 + \cos 2\tilde{\phi}_1) \right], \end{aligned}$$

where in the second equality we have used the condition (3.49). From the condition that the derivative with $\tilde{\phi}_1$ vanishes, we have $\sin 2\tilde{\phi}_1 = 0$, leading to

$$\tilde{\phi}_1 = \frac{\pi}{2}, \quad \tilde{\phi}_2 = -\frac{\pi}{2}. \quad (3.51)$$

Then the contribution from the timelike geodesic reads

$$-2i \ell_{\text{dS}} \log \left(e^{\tau_{\infty}} \sin \frac{\phi_0}{2} \right). \quad (3.52)$$

Thus the sum of (3.50) and (3.52) is identical to the naive calculation (3.47).

From the calculations above, the holographic pseudo entropy is

$$\begin{aligned} S_A &= \frac{1}{4G_N} \left[-2i \ell_{\text{dS}} \log \left(e^{\tau_{\infty}} \sin \frac{\phi_0}{2} \right) + \pi \ell_{\text{dS}} \right] \\ &= -i \frac{c^{(g)}}{3} \log \left(\frac{2 \sin \frac{\phi_0}{2}}{\epsilon} \right) + \frac{\pi c^{(g)}}{6}, \quad (\epsilon \equiv 2e^{-\tau_{\infty}}). \end{aligned} \quad (3.53)$$

Thus the pseudo entropy always takes a complex value. In particular, the real part is a half of de Sitter entropy (3.33). This fact actually holds in higher dimensions [104]. Even though the physical interpretation of this quantity in the gravity side has not yet been clear, our result indicates that the pseudo entropy has information about the initial state of the universe in some way.

Furthermore, in [103, 104], the relation with “timelike entanglement entropy,” which is defined for a timelike subregion instead of the ordinary spacelike subsystem, is also discussed. The pseudo entropy in the dS/CFT discussed in this subsection can be connected to the timelike entanglement entropy in the AdS/CFT by an analytic continuation.

⁷We take the principal branch of the arccos function beyond $[-1, 1]$ as $\cos^{-1} z = -i \log(z + i\sqrt{1-z^2})$.

4 Liouville Field Theory

In this section, we review Liouville field theory. Liouville field theory is one of the most attracting models of two-dimensional CFT due to the following features

- It does not have additional symmetry other than the symmetry generated by Virasoro algebra.
- It is a non-rational CFT.
- The exact data of CFT including the spectrum of primary states and OPE coefficients are known.

From these facts, Liouville field theory can be expected to be a CFT that is similar to the holographic CFT₂ but easy to calculate, so it is often helpful to analyze this theory to study holography.

For our purpose, we focus on the complex solutions to the Liouville equation of motion in the appropriate parameter region. The contents in this section mainly follows [36]. See also [106, 107] for review articles on Liouville field theory.

4.1 Basics of Liouville Field Theory

In this section, we explain the definitions and basic properties of some notions in Liouville field theory on a sphere. The action of Liouville field theory is given by

$$I_L[\phi] = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \left[\hat{g}^{ab} \partial_a \phi \partial_b \phi + Q \hat{\mathcal{R}} \phi + 4\pi\mu e^{2b\phi} \right], \quad (4.1)$$

where $Q = b + b^{-1}$ is called background charge. Tentatively we take parameters b, μ to be real values, though we will discuss the extension to complex values later. We set the reference metric as the flat metric $\hat{g}_{ab} = \delta_{ab}$. We are sometimes interested in “physical metric” $g_{ab} = e^{2b\phi} \hat{g}_{ab}$ as well. In the complex coordinates $z = \sigma^1 + i\sigma^2$, $\bar{z} = \sigma^1 - i\sigma^2$, a conformal transformation is generated by a holomorphic function $z' = f(z)$. Under the transformation, the field ϕ is not exactly scalar, but gains the effect of the conformal anomaly as

$$\phi'(z', \bar{z}') = \phi(z, \bar{z}) - \frac{Q}{2} \log \left| \frac{\partial f}{\partial z} \right|^2. \quad (4.2)$$

Therefore, ϕ itself is not a primary operator in the quantum Liouville theory. Nevertheless, an exponential operator

$$V_\alpha(z, \bar{z}) = e^{2\alpha\phi(z, \bar{z})}, \quad (4.3)$$

is still primary with conformal weight

$$h = \bar{h} = \alpha(Q - \alpha). \quad (4.4)$$

The parameter α is called Liouville momentum and often parameterized as

$$\alpha = \frac{Q}{2} + iP, \quad P \in \mathbb{R}, \quad (4.5)$$

In terms of P , the conformal weight takes the form

$$h = \bar{h} = \frac{Q^2}{4} + P^2. \quad (4.6)$$

The stress tensor has an additional term from the background charge

$$T(w) = -(\partial\phi)^2 + Q\partial^2\phi. \quad (4.7)$$

compared to the free scalar theory. As a result, the central charge is

$$c = 1 + 6Q^2. \quad (4.8)$$

From this expression, one can see that the large central charge limit $c \rightarrow \infty$ is equivalent to $b \rightarrow 0$.

In studying Liouville field theory in the flat reference metric, it is more convenient to express the action in another way [64]. First, defining Liouville theory on S^2 corresponds to imposing the following boundary condition on the Liouville field on $\mathbb{C}$:

$$\phi(z, \bar{z}) = -2Q \log |z| + \mathcal{O}(1) \quad (4.9)$$

at large $|z|$. Considering the large disk D with radius R , the Liouville action respecting this boundary condition can be expressed as

$$I_L[\phi] = \frac{1}{4\pi} \int_D d^2\sigma \left(\partial_a \phi \partial_a \phi + 4\pi\mu e^{2b\phi} \right) + \frac{Q}{\pi} \oint_{\partial D} \phi d\theta + 2Q^2 \log R, \quad (4.10)$$

where θ is a coordinate of ∂D . In studying the semi-classical regime $b \ll 1$, it is convenient to introduce $\varphi = 2b\phi$ so that the Liouville action approximates

$$b^2 I_L[\varphi] = \frac{1}{16\pi} \int_D d^2\sigma \left(\partial_a \varphi \partial_a \varphi + 16\lambda e^\varphi \right) + \frac{1}{2\pi} \oint_{\partial D} \varphi d\theta + 2 \log R + \mathcal{O}(b^2), \quad (4.11)$$

where we have introduced $\lambda = \pi\mu b^2$, which will have an important physical parameter in the later discussions. In these notations, the equation of motion is

$$\partial\bar{\partial}\varphi = 2\lambda e^\varphi, \quad (4.12)$$

and the boundary condition at $|z| \rightarrow \infty$ is

$$\varphi(z, \bar{z}) = -2 \log |z| + \mathcal{O}(1). \quad (4.13)$$

In the following discussions, we will consider the limit $b \rightarrow 0$ with keeping λ fixed so that the equation of motion (4.12) has non-trivial solutions in $b \rightarrow 0$. Therefore we regard λ as a fixed parameter rather than depending on b^2 .

4.2 Heavy operator insertions

As discussed in §2.2, heavy operators play important roles in studying the gravity sector of the holographic dictionary. In terms of Liouville notations, a heavy operator is defined as a primary operator $V_\alpha(z, \bar{z})$ whose momentum α scales in b as

$$\alpha = \frac{\eta}{b} + \mathcal{O}(1) \quad (4.14)$$

with a constant η . For such an operator, the conformal weight scales as $h = \mathcal{O}(c)$.

When we consider the correlation functions inserting n heavy operators with $\alpha_i = \eta_i/b$ ($i = 1, \dots, n$)

$$\langle V_{\alpha_1}(z_1, \bar{z}_1) \cdots V_{\alpha_n}(z_n, \bar{z}_n) \rangle = \int \mathcal{D}\varphi e^{-I_L[\varphi]} e^{\frac{\eta_1}{b^2} \varphi(z_1, \bar{z}_1)} \cdots e^{\frac{\eta_n}{b^2} \varphi(z_n, \bar{z}_n)}, \quad (4.15)$$

the saddle point equation is changed from the Liouville equation of motion (4.12) as

$$\partial\bar{\partial}\varphi = 2\lambda e^\varphi - 4\pi \sum_i \eta_i \delta^2(z - z_i). \quad (4.16)$$

Near each insertion point z_i , let us assume that the delta function term $\delta^2(z - z_i)$ is dominant in the right-hand side, so the solution φ approximates

$$\varphi(z, \bar{z}) = -4\eta_i \log |z - z_i| + \mathcal{O}(1). \quad (4.17)$$

In the similar discussion to deriving (4.10), this can be regarded as a boundary condition imposed on φ at $z \rightarrow z_i$. Therefore a saddle point of (4.15) should be a solution of the Liouville equation (4.12) with imposing the boundary conditions (4.13) and (4.17). This can be rephrased as stabilizing the modified action

$$\begin{aligned} b^2 I_L[\varphi] = & \frac{1}{16\pi} \int_{D - \cup_i d_i} d^2\sigma (\partial_a \varphi \partial_a \varphi + 16\lambda e^\varphi) + \frac{1}{2\pi} \oint_{\partial D} \varphi d\theta + 2 \log R \\ & - \sum_i \left(\frac{\eta_i}{2\pi} \oint_{d_i} \varphi d\theta_i + 2\eta_i^2 \log \epsilon_i \right), \end{aligned} \quad (4.18)$$

where d_i is a small disk with radius ϵ_i around z_i .

Let us elaborate on the effect of the heavy operator insertions a bit more. In terms of the physical metric $g_{ab} = e^\varphi \delta_{ab}$, at near z_i we have

$$ds^2 \sim \frac{1}{r^{4\eta_i}} (dr^2 + r^2 d\theta^2). \quad (4.19)$$

By changing the coordinates as

$$r' = \frac{1}{1 - 2\eta_i} r^{1-2\eta_i}, \quad \theta' = (1 - 2\eta_i)\theta, \quad (4.20)$$

we obtain the flat metric

$$ds^2 \sim dr'^2 + r'^2 d\theta'^2 \quad (4.21)$$

with an unusual periodicity $\theta' \sim \theta' + 2\pi(1 - 2\eta_i)$. This means that the geometry has a conical defect with deficit angle $\delta_i = 4\pi\eta_i$ around each z_i if $0 < \eta_i < 1/2$. The inequality $0 < \eta_i < 1/2$ is called the Seiberg bound. When the Seiberg bound is satisfied, we can check the assumption that the first term in the right-hand side of (4.16) is subleading.

This is not the end of the story. We can obtain an additional constraint from the condition that the solution to (4.16) gives the genus 0 compact geometry without boundary. Note that the original Liouville equation of motion (4.12) (for positive λ) can be interpreted as imposing a negative constant curvature since the Ricci scalar for the physical metric $g_{ab} = e^\varphi \delta_{ab}$ is given by $\mathcal{R} = -2e^{-\varphi} \partial\bar{\partial}\varphi$. From the Gauss-Bonnet theorem $\frac{1}{4\pi} \int_{S^2} d^2x \sqrt{g} \mathcal{R} = 2$, we have

$$-\frac{1}{2\pi} \int d^2\sigma \partial\bar{\partial}\varphi = 2. \quad (4.22)$$

Combining this with (4.16),

$$\lambda \int d^2\sigma e^\varphi = 2\pi \left(\sum_i \eta_i - 1 \right). \quad (4.23)$$

Therefore, as long as $\lambda > 0$, a real saddle point can exist only if $\sum_i \eta_i > 1$. Intuitively, this condition can be interpreted as follows: Since the curvature apart from the singularities is negative due to the Liouville equation of motion, we have to insert enough heavy operators to make the integrated curvature positive. For general Riemann surfaces, the condition is extended to $\sum_i \eta_i + g - 1 > 0$. Note that this condition is for the compact Riemann surfaces without boundary, otherwise the constraint is relaxed.

In summary, we have obtained the condition

$$\begin{aligned} 0 < \eta_i < \frac{1}{2}, \quad i = 1, \dots, n \\ \sum_i \eta_i > 1 \end{aligned} \tag{4.24}$$

for a real saddle point to exist, which gives a negative curvature geometry. However, we will be also interested in positive curvature geometries. Let us discuss the condition for such a compact geometry with a positive curvature to exist in the three defects case $\delta_i = 4\pi\eta_i$ ($i = 1, 2, 3$), with forgetting the Liouville theory. If such a geometry exists, it can be mapped to a subregion of S^2 [108] (see also §4.2 of [36]). The image of the map shapes a triangle on S^2 surrounded by three geodesics. Each internal angle of the spherical triangle is $\pi(1 - 2\eta_i)$. Since the sum of the three angles of a spherical triangle should be larger than π , we obtain a condition

$$\sum_i \eta_i < 1. \tag{4.25}$$

Furthermore, the existence of the spherical triangle requires additional conditions

$$\eta_i + \eta_j - \eta_k > 0, \quad i \neq j \neq k. \tag{4.26}$$

The generalization to geometries with more conical defects will be discussed in §5.2.3 by taking triangulation. Now it is natural to ask a question: Can we realize the geometry with conditions (4.25) and (4.26) as a solution of Liouville field theory? This question is a big motivation of [36], where analytic continuation of Liouville theory is generally discussed. As described above, since η_i do not satisfy the condition (4.24), it cannot be realized as a *real* saddle point. However, it can be accomplished as a *complex* saddle point, i.e., a solution φ to the saddle point equation in the complex region $\mathbb{C}$. As we will see later, the Stokes phenomenon occurs when we leave the original region (4.24), and then the complex saddles contribute to the partition function or the correlation functions.

Finally let us make a comment on complexifications of parameters. When we consider complex η_i 's, we can repeat the same discussion and obtain the same inequalities on η_i obtained in this subsection after just replacing η_i with $\text{Re } \eta_i$. Here we have fixed b to be a real parameter, but this will also be complexified later.

4.3 Complex solutions

As described in the previous subsection, the saddle points for the correlation function (4.15) are the solutions to

$$\partial\bar{\partial}\varphi = 2\lambda e^\varphi \tag{4.27}$$

with boundary conditions

$$\varphi(z, \bar{z}) \sim -2\log|z|, \quad \text{as } z \rightarrow \infty, \tag{4.28}$$

$$\varphi(z, \bar{z}) \sim -4\eta_i \log|z - z_i|, \quad \text{as } z \rightarrow z_i. \tag{4.29}$$

Here we summarize the complex solutions with two- and three-insertions. See [36] for derivations of the results.

4.3.1 Two-point solution

As is well known in CFT, the two-point function is non-zero only when $h_1 = h_2$, so we restrict ourselves to $\eta_1 = \eta_2 \equiv \eta$. In this case, the solution is

$$e^\varphi = -\frac{1}{\lambda} \frac{1}{\left(\kappa |z - z_1|^{2\eta} |z - z_2|^{2-2\eta} + \frac{1}{\kappa(1-2\eta)^2 |z_{12}|^2} |z - z_1|^{2-2\eta} |z - z_2|^{2\eta} \right)^2}, \quad (4.30)$$

where κ can take an arbitrary complex number except for the one such that the solution is singular. Therefore the solution φ should be complex. There is no other ambiguities in (4.30), but φ has another ambiguity of choosing the branch as

$$\begin{aligned} \varphi^{(N)}(z, \bar{z}) = & i\pi + 2\pi i N - \log \lambda \\ & - 2 \log \left(\kappa |z - z_1|^{2\eta} |z - z_2|^{2-2\eta} + \frac{1}{\kappa(1-2\eta)^2 |z_{12}|^2} |z - z_1|^{2-2\eta} |z - z_2|^{2\eta} \right), \end{aligned} \quad (4.31)$$

with $N \in \mathbb{Z}$. Here we assume the Seiberg bound $0 < \text{Re } \eta < 1/2$. One can check that the solution approximates

$$\varphi^{(N)}(z, \bar{z}) \sim -4\eta \log |z - z_i| + C_i, \quad \text{as } z \rightarrow z_i, \quad (4.32)$$

$$\varphi^{(N)}(z, \bar{z}) \sim -4 \log |z| + C_\infty, \quad \text{as } z \rightarrow \infty, \quad (4.33)$$

where the constants are

$$C_1 = 2\pi i \left(N + \frac{1}{2} \right) - \log \lambda - 2 \log \kappa + (4\eta - 4) \log |z_{12}| \quad (4.34)$$

$$C_2 = 2\pi i \left(N + \frac{1}{2} \right) - \log \lambda + 2 \log \kappa + 4 \log(1 - 2\eta) \quad (4.35)$$

$$C_\infty = 2\pi i \left(N + \frac{1}{2} \right) - \log \lambda - 2 \log \kappa - 2 \log \left(1 + \frac{1}{\kappa^2(1-2\eta)^2 |z_{12}|^2} \right) \quad (4.36)$$

Therefore (4.31) is indeed a saddle point of the two-point function. As a special case, $\eta = 0$ leads to

$$\varphi^{(N)} = i\pi + 2\pi i N - \log \lambda - 2 \log(1 + |z|^2), \quad (4.37)$$

which actually gives the minus of the S^2 metric

$$ds^2 = 4e^{\varphi^{(N)}} dz d\bar{z} = -\frac{1}{\lambda} \frac{4dzd\bar{z}}{(1 + |z|^2)^2}. \quad (4.38)$$

Next, we would like to evaluate the on-shell action by inserting (4.31) to the action $I_L[\varphi]$ of the form (4.18). In doing so, it is useful to first evaluate the on-shell value of the derivative $b^2 dI_L/d\eta$ [64]. The bulk integral in (4.18) depends on η only through $\varphi^{(N)}$, so the derivative $dI_L/d\eta$ has no contribution from the bulk term. Therefore only the surface terms are relevant,

but the implicit dependence through $\varphi^{(N)}$ is again irrelevant due to the variation invariance, then

$$\begin{aligned} b^2 \frac{dI_L[\varphi^{(N)}]}{d\eta} &= -C_1 - C_2 \\ &= -2\pi i(2N+1) + 2\log\lambda + (4-8\eta)\log|z_{12}| - 4\log(1-2\eta). \end{aligned} \quad (4.39)$$

Integrating by η , we obtain the on-shell action

$$\begin{aligned} b^2 I_L[\varphi^{(N)}] &= 2\pi i \left(N + \frac{1}{2} \right) (1-2\eta) + (2\eta-1)\lambda + 4(\eta-\eta^2)\log|z_{12}| \\ &\quad + 2[(1-2\eta)\log(1-2\eta) - (1-2\eta)], \end{aligned} \quad (4.40)$$

where the integral constant has been determined by evaluating explicitly the on-shell action for $\eta=0$ case (4.37).

4.3.2 Three-point solution

Let us move on to the case with three insertions. In this case, the solution takes a more complicated form than the two-point case. As we have seen in the two-point case, the important data to evaluate the on-shell action is just $\mathcal{O}(1)$ contributions when φ is expanded around the insertion points. Therefore we only show the approximations of the solution

$$\varphi^{(N)}(z, \bar{z}) \sim -4\eta \log|z - z_i| + C_i, \quad \text{as } z \rightarrow z_i, \quad (4.41)$$

$$\varphi^{(N)}(z, \bar{z}) \sim -4\log|z| + C_\infty, \quad \text{as } z \rightarrow \infty, \quad (4.42)$$

where the constants are

$$\begin{aligned} C_1 &= 2\pi i N - \log\lambda - (1-2\eta_1) \log \frac{|z_{12}|^2 |z_{13}|^2}{|z_{23}|^2} \\ &\quad - \log \frac{\gamma(\eta_1 + \eta_2 - \eta_3) \gamma(\eta_1 + \eta_3 - \eta_2) \gamma(\eta_1 + \eta_2 + \eta_3 - 1)}{\gamma(2\eta_1)^2 \gamma(\eta_2 + \eta_3 - \eta_1)}, \end{aligned} \quad (4.43)$$

$$\begin{aligned} C_2 &= 2\pi i N - \log\lambda - (1-2\eta_2) \log \frac{|z_{12}|^2 |z_{23}|^2}{|z_{13}|^2} \\ &\quad - \log \frac{\gamma(\eta_1 + \eta_2 - \eta_3) \gamma(\eta_2 + \eta_3 - \eta_1) \gamma(\eta_1 + \eta_2 + \eta_3 - 1)}{\gamma(2\eta_2)^2 \gamma(\eta_1 + \eta_3 - \eta_2)}, \end{aligned} \quad (4.44)$$

$$\begin{aligned} C_3 &= 2\pi i N - \log\lambda - (1-2\eta_3) \log \frac{|z_{23}|^2 |z_{13}|^2}{|z_{12}|^2} \\ &\quad - \log \frac{\gamma(\eta_3 + \eta_2 - \eta_1) \gamma(\eta_1 + \eta_3 - \eta_2) \gamma(\eta_1 + \eta_2 + \eta_3 - 1)}{\gamma(2\eta_3)^2 \gamma(\eta_1 + \eta_2 - \eta_3)}, \end{aligned} \quad (4.45)$$

As in the similar way to the two-point case, it is convenient to first compute

$$b^2 \frac{\partial I_L[\varphi^{(N)}]}{\partial \eta_i} = -C_i, \quad i = 1, 2, 3. \quad (4.46)$$

The explicit expression of the on-shell action actually depends on the region of η_i . For example, if η_i satisfy (4.24), then

$$\begin{aligned}
b^2 I_L[\varphi^{(N)}] = & 2\pi i N \left(1 - \sum_i \eta_i \right) + \left(\sum_i \eta_i - 1 \right) \log \lambda \\
& + (\delta_1 + \delta_2 - \delta_3) \log |z_{12}|^2 + (\delta_2 + \delta_3 - \delta_1) \log |z_{23}|^2 + (\delta_3 + \delta_1 - \delta_2) \log |z_{13}|^2 \\
& + F(\eta_1 + \eta_2 - \eta_3) + F(\eta_2 + \eta_3 - \eta_1) + F(\eta_3 + \eta_1 - \eta_2) + F(\eta_1 + \eta_2 + \eta_3 - 1) \\
& - F(2\eta_1) - F(2\eta_2) - F(3\eta_3) - F(0),
\end{aligned} \tag{4.47}$$

where $\delta_i = \eta_i(1 - \eta_i)$ is the leading coefficient of the conformal weights $h_i \sim \delta_i/b^2$ and $F(x)$ is a function defined in (B.5).

4.4 Correlation functions

A strong property of Liouville field theory is that the OPE coefficient has been perfectly determined [63, 64] and it shows good analytic behavior. Here we summarize the exact correlation functions of the Liouville field theory. From the conformal symmetry, two- and three-point functions of a generic CFT are restricted as the following forms:

$$\langle V_\alpha(z_1, \bar{z}_1) V_\alpha(z_2, \bar{z}_2) \rangle = \frac{B(\alpha)}{|z_1 - z_2|^{4h}}, \tag{4.48}$$

$$\langle V_{\alpha_1}(z_1, \bar{z}_1) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_3}(z_3, \bar{z}_3) \rangle = \frac{C(\alpha_1, \alpha_2, \alpha_3)}{|z_1 - z_2|^{2(h_1+h_2-h_3)} |z_2 - z_3|^{2(h_2+h_3-h_1)} |z_3 - z_1|^{2(h_3+h_1-h_2)}}, \tag{4.49}$$

where α_i takes the form (4.5) corresponding to a state in the spectrum of Liouville theory. Note that the operator with $-P$ is equivalent to that with P . In other words, V_α and $V_{Q-\alpha}$ are equivalent operators in the sense that those give a same state under the state-operator correspondence. Therefore, those two operators are related in the following form:

$$V_\alpha(z, \bar{z}) = R(\alpha) V_{Q-\alpha}(z, \bar{z}), \tag{4.50}$$

where $R(\alpha)$ is called the reflection coefficient.

The three-point coefficient of Liouville theory has obtained in [63, 64] as

$$\begin{aligned}
C(\alpha_1, \alpha_2, \alpha_3) = & \left[\pi \mu \gamma(b^2) b^{2-2b^2} \right]^{(Q-\sum_i \alpha_i)/b} \\
& \times \frac{\Upsilon'_b(0) \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1 + \alpha_2 + \alpha_3 - Q) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon_b(\alpha_2 + \alpha_3 - \alpha_1) \Upsilon_b(\alpha_3 + \alpha_1 - \alpha_2)}.
\end{aligned} \tag{4.51}$$

See appendix B for the definition and properties of a special function Υ_b . From this formula, we can obtain $B(\alpha)$ and $R(\alpha)$ by considering $\lim_{\alpha_3 \rightarrow 0} \langle V_\alpha V_\alpha V_{\alpha_3} \rangle = \langle V_\alpha V_\alpha \rangle$ and $C(\alpha_1, \alpha_2, \alpha_3) = R(\alpha) C(Q - \alpha_1, \alpha_2, \alpha_3)$, respectively. We then obtain

$$B(\alpha) = R(\alpha) = \left[\pi \mu \gamma(b^2) \right]^{(Q-2\alpha)/b} \frac{1}{b^2} \gamma \left(\frac{2\alpha}{b} - 1 - \frac{1}{b^2} \right) \gamma(2b\alpha - b^2). \tag{4.52}$$

Therefore, in this convention, the coefficient of the two-point function with identical two operators is not normalized to be 1, in contrast to most cases of CFT_2 . Instead, the conventional normalization in Liouville field theory is fixed as

$$\langle V_{Q-\alpha}(z_1, \bar{z}_1) V_\alpha(z_2, \bar{z}_2) \rangle = \frac{1}{|z_1 - z_2|^{4h}}, \quad (4.53)$$

which makes $B(\alpha)$ and $R(\alpha)$ identical. This normalization has some advantages. One is that the coefficients $B(\alpha), C(\alpha_1, \alpha_2, \alpha_3)$ are meromorphic functions, which makes it possible to analytically continue the parameters b, α beyond the standard regime. On the other hand, the standard convention $B(\alpha)$ in most CFT_2 makes the three-point coefficient $C(\alpha_1, \alpha_2, \alpha_3)$ multi-valued [62]. Furthermore, the correlation functions in this convention are consistent with the unit normalization of the path integral formalism (4.15).

From the general arguments of two-dimensional CFT, every correlation function with higher than four insertions can be deduced once we know the OPE coefficients. For example, a four-point function

$$G(z, \bar{z}) = \langle V_1(0) V_2(z, \bar{z}) V_3(1) V_4(\infty) \rangle, \quad (4.54)$$

with three of the insertion points being fixed by using $SL(2, \mathbb{C})$ global conformal symmetry, can be decomposed as

$$G(z, \bar{z}) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dP}{2\pi} C\left(\alpha_1, \alpha_2, \frac{Q}{2} - iP\right) C\left(\alpha_3, \alpha_4, \frac{Q}{2} + iP\right) \mathcal{F}_{34}^{12}(h_P|z) \mathcal{F}_{34}^{12}(h_P|\bar{z}). \quad (4.55)$$

Here we have defined the conformal blocks $\mathcal{F}_{34}^{12}(h_P|z)$ associated with the conformal weight h_P of the intermediate state. In fact, this decomposition (4.55) holds as long as the four insertions satisfy

$$\text{Re } \alpha_1 + \text{Re } \alpha_2 > \text{Re}(Q/2), \quad \text{Re } \alpha_3 + \text{Re } \alpha_4 > \text{Re}(Q/2). \quad (4.56)$$

When the inserted operators are in Liouville spectrum, i.e. $\alpha_i \in Q/2 + i\mathbb{R}$, they trivially satisfy the conditions. (4.56) are the conditions for the integral (4.55) to converge. This can be rephrased by considering the pole structure of the DOZZ formula (4.51). Let us analyze the pole structure of $C(\alpha_3, \alpha_4, Q/2 + iP)$. The zeros of $\Upsilon_b(x)$ are analyzed in [109] (see also [62]), where it was shown that the simple zeros of $\Upsilon_b(x)$ are at

$$x \in -b\mathbb{N} - \frac{1}{b}\mathbb{N} \quad \text{or} \quad x \in Q + b\mathbb{N} + \frac{1}{b}\mathbb{N} \quad (4.57)$$

($\mathbb{N}$ includes 0) and it has no poles. Therefore $\Upsilon_b(x)$ has two bunches of poles, one of which starts from 0 and the other starts from Q in the opposite directions. The poles of DOZZ formula (4.51) can only come from the Upsilon functions in the denominator. The condition that any bunch of poles of Upsilon functions in the denominator of (4.51) does not cross the real axis in the complex P -plane is given by (4.56). When we leave this region, some bunches of poles cross the real axis. In analytically continuing parameters, we have to deform the integration contour so as not to cross the poles crossing the real axis. In moving it back to the contour along the real axis, it picks up the contributions from the poles of OPE coefficients. We then have

$$\begin{aligned} G(z, \bar{z}) &= \frac{i}{2} \sum_{P: \text{poles crossing } \mathbb{R}} \text{Res}_P \left[C\left(\alpha_1, \alpha_2, \frac{Q}{2} - iP\right) C\left(\alpha_3, \alpha_4, \frac{Q}{2} + iP\right) \mathcal{F}_{34}^{12}(h_P|z) \mathcal{F}_{34}^{12}(h_P|\bar{z}) \right] \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \frac{dP}{2\pi} C\left(\alpha_1, \alpha_2, \frac{Q}{2} - iP\right) C\left(\alpha_3, \alpha_4, \frac{Q}{2} + iP\right) \mathcal{F}_{34}^{12}(h_P|z) \mathcal{F}_{34}^{12}(h_P|\bar{z}). \end{aligned} \quad (4.58)$$

The set of poles that contribute in (4.58) highly depends on the parameters α_i, b . We will encounter an example of this calculation in §5.2.3.

5 Holographic study of semi-classical saddles in dS gravity

One important motivation to study the quantum gravity is to understand the beginning of the universe. It can be rephrased as the boundary condition of the wave functional of universe

$$\Psi[h_{\mu\nu}] = \int_{g_{\mu\nu}|_{\text{initial}}=?}^{g_{\mu\nu}|_{\text{late}}=h_{\mu\nu}} \mathcal{D}g_{\mu\nu} e^{iI_{\text{EH}}[g_{\mu\nu}]} \quad (5.1)$$

at the beginning of the universe $T = 0$, in addition to the boundary condition at the late time. Hartle and Hawking [5, 110] suggested the “no-boundary” condition, which states that there are no boundary in the past. In other words, the initial state can be prepared as the Euclidean path integral over compact geometries without boundary.

To compare it with the real world, we are interested in the semi-classical limit. As in usual quantum mechanics, the semi-classical limit of path integral can be evaluated by its saddle points, which are solutions of the saddle point equations $\delta I_{\text{EH}}/\delta g_{\mu\nu} = 0$. In general, there are (sometimes infinitely) many solutions to these equations. In order to precisely determine which saddle points contribute in the semi-classical limit, we have to identify the integration contour defining the path integral (5.1). One straightforward idea in the gravitational path integral is to consider the Euclideanized path integral by summing over all Euclidean real metrics. This is the ordinary approach to define path integral in most quantum field theories. However, there is a serious problem in the case of gravity that the Euclidean Einstein-Hilbert action is unbounded below as we will see an example later, which makes the path integral along the real metrics divergent. Therefore, continuing this approach needs to consider the complex saddle points, which are the complexified solutions $g_{\mu\nu}$ to the saddle point equations. This would be natural in some sense because the no-boundary geometry by Hartle and Hawking can also be regarded as a complex saddle point.

The authors of [6–8] considered this problem by using the Picard-Lefschetz theory (see e.g. [35, 111] for detailed descriptions of the method), and recently reconsidered by [112, 113]. In this approach, the relevant saddle points can be determined by deforming the integration contour in the complex plane into a sum of the steepest descents (or Lefschetz thimbles). However, since we have no knowledge of the correct integration contour to define the gravitational path integral, we cannot completely argue the contributing saddle points in principle. Therefore we have been restricted to discuss the contours from the physical intuition. One important question is to find a criterion to determine the relevant saddle points and the defining contour.

In [16, 17], an approach to this question by using dS_3/CFT_2 is proposed in the case of the three-dimensional gravity with a positive cosmological constant. In particular, throughout this section, we consider a concrete model explained in §3.3, which is the analytic continuation from the minimal model holography for AdS_3 . As commented in the last sentence of §3.3, the correlation functions of the dual CFT, which is a generalized minimal model, can be described by those of Liouville field theory. The reason has already been explained in §2.2.3. We have two advantages to consider this concrete model. An advantage to consider Liouville field theory is that its correlation functions have good analytic properties as seen in §4. Thanks to such rich properties, the dual CFT_2 to dS_3 can be constructed as an analytic continuation $c \rightarrow -ic^{(g)}$ from the ordinary Liouville theory. By calculating the CFT quantities, the semi-classical saddles of the wave functional of universe are studied through the relation between the saddle points of the both sides explained in §3.2. One might think that the results do not make so much sense because we rely on a specific model of holography. However, at least in the semi-classical approximation, the results show the universal behavior in the holographic CFT_2 because the gravity sector would dominate the path integral at the leading order $\mathcal{O}(G^{-1})$ in the semi-classical

regime as one can see from the explicit form of the action. Thus we can expect to be able to extract the universal feature of holography by using a concrete model. Indeed, the universal large c behavior of holographic CFT₂ is known to be expressed by using DOZZ formula [18, 114].

In this section, we review the discussions of [16, 17] and a part of [115, 116]. We first classify the complex solutions of the complexified Einstein equations, which will be candidates for the saddle points in the semi-classical limit, by means of Chern-Simons gravity. After that, we calculate correlation functions of the CFT side and investigate the relevant saddle points contributing in the semi-classical limit. Then we will move on to the gravity side. We utilize the mini-superspace method to collect the saddle point geometries and determine the contour of gravitational path integral such that it picks up the relevant saddle points expected from the CFT side. Finally we explain the recent related works by using a similar method in order to make it clear the difference of them from ours.

5.1 Complex saddle points in dS₃ gravity

In this subsection, we aim to classify the complex solutions to Einstein equations with positive cosmological constant.

Let us start with focusing on the geometry of the type used for the no-boundary proposal by Hartle and Hawking [5]. Following [9], we consider a complexified metric of three-sphere with the ansatz,

$$ds^2 = \ell_{\text{dS}}^2 \left[\left(\frac{d\theta(u)}{du} \right)^2 du^2 + \cos^2 \theta(u) d\Sigma^2 \right]. \quad (5.2)$$

First we set $d\Sigma^2$ as the metric of $\mathbb{S}^2$ without conical defects. Note that we only complexify one direction, which is represented by a holomorphic function $\theta(u)$. The metric of $\mathbb{S}^3$ is obtained by setting $\theta = u$, while the metric of dS₃ is given by $\theta = iu$. For no-boundary proposal, we require that the universe starts from nothing and approaches to dS₃. We may set the range of the parameter u as $0 \leq u < \infty$. At $u = 0$, we require that the universe vanishes, which can be realized with $\theta = (n + 1/2)\pi$. For $u \rightarrow \infty$, the universe becomes de Sitter space by setting $\theta = iu$. We can glue the two geometries at $u = 1$. We may set

$$\theta = \left(n + \frac{1}{2} \right) \pi (1 - u) \quad (0 \leq u \leq 1), \quad \theta = i(u - 1) \quad (u > 1). \quad (5.3)$$

As discussed around (3.32), the wave functional can be put of the form

$$\Psi[h] \sim \sum_n \exp(S_{\text{GH}}^{(n)}/2 + i\mathcal{I}^{(n)}). \quad (5.4)$$

The sum is over the family of geometries corresponding to saddle points. The term $S_{\text{GH}}^{(n)}$ comes from the Euclidean geometry realized for $0 \leq u \leq 1$ and the phase factor $\exp(i\mathcal{I}^{(n)})$ from the Lorentzian region realized for $u > 1$. The contribution to $S_{\text{GH}}^{(n)}$ from the geometry labeled by n can be evaluated as

$$S_{\text{GH}}^{(n)} = \frac{(2n + 1)\pi\ell_{\text{dS}}}{2G}. \quad (5.5)$$

When $n = 0$, this reduces to the ordinary de Sitter entropy (3.33). Therefore the $n = 0$ saddle should be included in the physically reasonable gravity theory. This has been one of the guiding principles for investigating the appropriate integration contour of the gravitational path integral.

The label n of the saddles given by (5.3) can be understood as the elements of homotopy group $\pi_3(S^3)$ of the Euclidean sector. To make it clearer, it is helpful to consider the Chern-Simons formalism of three-dimensional gravity. As summarized in §2.2.1, the Chern-Simons action receives a shift of the form $2\pi\mathbb{Z}$ under a large gauge transformation, which leaves Lorentzian dS_3 gravity invariant though Euclidean dS_3 is not invariant. Thus a large gauge transformation changes only the Euclidean sector, leading to the transformation from one to another among (5.3). Let us explicitly show the saddles for Euclidean dS_3 . In this case, the Chern-Simons gauge fields $A, \tilde{A}$ are independent $\mathfrak{su}(2)$ -valued connections. The static dS_3 are given by

$$A = b^{-1}db + b^{-1}ab, \quad \tilde{A} = bdb^{-1} + b\tilde{a}b^{-1}, \quad b = e^{\theta J_0}, \quad (5.6)$$

with

$$a = J_2(d\phi + d\tau), \quad \tilde{a} = J_2(d\phi - d\tau). \quad (5.7)$$

Here $x^\pm = \tau \pm \phi$ and the $\mathfrak{su}(2)$ generators J_a are given by (A.1). Let us introduce the holonomy matrix along the temporal cycle

$$\mathcal{P}e^{\oint_\tau A} = b e^\Omega b^{-1}. \quad (5.8)$$

For the above solution, the eigenvalues of Ω are $(\pi i, -\pi i)$. The action of large gauge transformation changes the eigenvalues of Ω as $(2\pi i(n + 1/2), -2\pi i(n + 1/2))$ with integer n .⁸ A configuration of gauge fields with the holonomy matrix is given by

$$a = J_2(d\phi + (2n + 1)d\tau), \quad \tilde{a} = J_2(d\phi - (2n + 1)d\tau). \quad (5.9)$$

The metric from the configuration of gauge fields can be read off as

$$\ell_{\text{dS}}^{-2} ds^2 = (2n + 1)^2 \sin^2 \theta d\tau^2 + d\theta^2 + \cos^2 \theta d\phi^2. \quad (5.10)$$

By calculating the on-shell action on this solution, we can reproduce (5.5).

Next, let us extend the above discussions to include conical defects. Since the Lorentzian sector is still invariant under large gauge transformations, we only focus on Euclidean solutions. A geometry with a conical defect can be constructed by replacing $d\Sigma^2$ in (5.2) with the 2-sphere having the conical defect. One example is the ordinary Schwarzschild-de Sitter geometry (3.35), whose Chern-Simons expression is given by (5.6) with

$$a = r_+ J_2(d\phi + d\tau), \quad \tilde{a} = r_+ J_2(d\phi - d\tau). \quad (5.11)$$

Therefore the following discussions are almost same as the above case without defects. We again consider holonomy along the thermal cycle

$$\mathcal{P}e^{\oint_\tau A} = b^{-1} e^\Omega b. \quad (5.12)$$

The eigenvalues of Ω are $(\pi i, -\pi i)$ for the Schwarzschild-de Sitter solution. A large gauge transformation changes it as $(2\pi i(n + 1/2), -2\pi i(n + 1/2))$, then we obtain

$$\ell_{\text{dS}}^{-2} ds^2 = d\theta^2 + (2n + 1)^2 r_+^2 \sin^2 \theta d\tau^2 + r_+^2 \cos^2 \theta d\phi^2. \quad (5.13)$$

⁸We may require that the holonomy matrix (5.8) is trivial. The center of gauge group $SL(N) \times SL(N)$ is $\mathbb{Z}_2$ for even N , and the trivial holonomy means that the holonomy matrix is given by a center of the gauge group $\pm \mathbb{1}$. Thus, the trivial holonomy condition allows $n \in \mathbb{Z} + 1/2$, but such a case is not considered here as it is not generated by a large gauge transformation.

The classical action for the configuration is

$$-S (\equiv S_{\text{GH}}^{(n)}) = (2n+1) \frac{\pi \ell_{\text{dS}} r_+}{2G} = (2n+1) \frac{\pi \ell_{\text{dS}} \sqrt{1-8GE}}{2G}. \quad (5.14)$$

In the Chern-Simons formulation of gravity theory, there are no criteria to choose the proper set of saddle points labeled by n . In the next subsection, we try to determine the proper set from its dual CFT description.

5.2 CFT calculations

We would like to calculate the correlation functions and the partition function in the dual CFT₂. First let us explain the setup of the dual CFT₂. As explained in the beginning of this section, we consider the Liouville theory, the basics of which is summarized in §4. We perform the analytic continuation $c \rightarrow -ic^{(g)}$, with $c^{(g)} = 3\ell_{\text{dS}}/2G$ ⁹. In terms of the Liouville notations, the large $c^{(g)}$ approximation is rephrased as

$$\frac{1}{b^2} = -\frac{ic^{(g)}}{6} - \frac{13}{6} + \mathcal{O}\left(\frac{1}{c^{(g)}}\right). \quad (5.15)$$

The specific feature of this limit is the $\mathcal{O}(1)$ term $-13/6$. Compared with the real c case, where the $\mathcal{O}(1)$ is, of course, neglected in the leading order approximation, this term will actually be relevant in the later calculations because the leading term is purely imaginary.

We will consider the heavy operator insertions, which correspond to back-reacted geometries. In the three-dimensional gravity with a positive cosmological constant, there exist conical defect geometries (3.34) with deficit angle $2\pi(1-r_+) = 2\pi(1-\sqrt{1-8GE})$. Recalling that the deficit angle created by inserting a heavy operator $\alpha = \eta/b$ is $4\pi\eta$ (see §4.2), the operator corresponding to (3.34) is given by

$$\eta = \frac{1 - \sqrt{1-8GE}}{2}. \quad (5.16)$$

This is consistent with the operator dictionary (3.29) in the dS/CFT. In this section, we only focus on this type of operator insertions.

5.2.1 Two-point functions and partition function

As discussed in §4.4, the exact expression of Liouville two-point function is known with the coefficient (4.52), which we write again here:

$$\langle V_\alpha(z_1) V_\alpha(z_2) \rangle = |z_{12}|^{-4\alpha(Q-\alpha)} \frac{2\pi}{b^2} [\pi\mu\gamma(b^2)]^{(Q-2\alpha)/b} \gamma\left(\frac{2\alpha}{b} - 1 - \frac{1}{b^2}\right) \gamma(2b\alpha - b^2) \delta(0). \quad (5.17)$$

Here $\gamma(x)$ is

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \quad (5.18)$$

⁹Here, we think of the central charge c as an exact quantity including full quantum corrections since it is fixed by commutation relations among Virasoro generators. The relation to gravitational parameters may receive quantum corrections, but they should not violate the condition $c^{(g)} \in \mathbb{R}$ as the physical gravitational parameters take real values.

We then read off the set of semi-classical saddles in the regime (5.15) from the exact expression. Note that $1/b^2$ satisfies $-\pi < \arg \frac{1}{b^2} < -\pi/2$. For $b \sim 0$, the two-point function reduces to

$$\langle V_\alpha(z_1)V_\alpha(z_2) \rangle \sim \delta(0)|z_{12}|^{-4\eta(1-\eta)/b^2} \lambda^{(1-2\eta)/b^2} \left[\frac{\gamma(b^2)}{b^2} \right]^{(1-2\eta)/b^2} \gamma\left(\frac{2\eta-1}{b^2}\right). \quad (5.19)$$

At the region of $b \sim 0$, we can easily see¹⁰

$$\left[\frac{\gamma(b^2)}{b^2} \right]^{(1-2\eta)/b^2} \sim \exp \left[\frac{2-4\eta}{b^2} \left(\ln \frac{1}{b^2} + \pi i \right) \right], \quad (5.20)$$

where πi term comes from the difference of branches between $2 \ln \frac{1}{b^2}$ and $\ln \frac{1}{b^4}$. Furthermore, we can find (see [36])

$$\gamma\left(\frac{2\eta-1}{b^2}\right) \sim \left(e^{-\pi i(1-2\eta)/b^2} - e^{\pi i(1-2\eta)/b^2} \right) \exp \left[\frac{4\eta-2}{b^2} \left(\ln(1-2\eta) + \ln \frac{1}{b^2} + \pi i - 1 \right) \right] \quad (5.21)$$

by recalling that now $\text{Re } b^{-2} < 0$ from (5.15). Here we have used Stirling's approximation

$$\Gamma(x) = \exp(x \ln x - x + \mathcal{O}(\ln x)), \quad (5.22)$$

for $\text{Re } x > 0$ and

$$\Gamma(x) = \frac{1}{e^{\pi i x} - e^{-\pi i x}} \exp(x \ln(-x) - x + \mathcal{O}(\ln(-x))), \quad (5.23)$$

for $\text{Re } x < 0$, and an identity $\ln(-1/b^2) = \ln(1/b^2) + \pi i$. In summary, the two-point function behaves near $b \sim 0$ as

$$\begin{aligned} \langle V_\alpha(z_1)V_\alpha(z_2) \rangle &\sim \delta(0)|z_{12}|^{-4\eta(1-\eta)/b^2} \lambda^{(1-2\eta)/b^2} \\ &\times \left(e^{-\pi i(1-2\eta)/b^2} - e^{\pi i(1-2\eta)/b^2} \right) \exp \left\{ -\frac{2}{b^2} [(1-2\eta) \ln(1-2\eta) - (1-2\eta)] \right\}. \end{aligned} \quad (5.24)$$

This is the same as the sum of $\exp(-I_L[\varphi^{(N)}])$, where $I_L[\varphi^{(N)}]$ is given by (4.40), over $N = -1, 0$. For $b^{-2} \sim -ic^{(g)}/6$ with $c^{(g)} \gg 1$, the absolute value of two-point function behaves as

$$|\langle V_\alpha(z_1)V_\alpha(z_2) \rangle| \sim \left| e^{\frac{\pi c^{(g)}}{6} \sqrt{1-8G_N E}} - e^{-\frac{\pi c^{(g)}}{6} \sqrt{1-8G_N E}} \right|, \quad (5.25)$$

where we have used (5.16). Notice that the $|z_{12}|$ dependence is canceled due to the purely imaginary power of $-4\eta(1-\eta)/b^2$. With (5.4) and (5.14), we can read off the contributions corresponding to those to the Gibbons-Hawking entropy. Compared with (5.14), we can see that the possible saddles are with $N = -1, 0$. This reproduces the result obtained from the criteria of allowable complex geometry in [9, 10].

The partition function can be obtained by sending the Liouville momentum as $\alpha \rightarrow 0$. The exact formula for two-point functions then reduces to

$$Z_{\text{Liouville}} = \frac{2\pi}{b^2} \left[\lambda \frac{\gamma(b^2)}{b^2} \right]^{1/b^2} \gamma(-b^2) \gamma\left(-1 - \frac{1}{b^2}\right) \delta(0) \quad (5.26)$$

¹⁰Throughout this paper, we promise that $\log z$ for $z \in \mathbb{C}$ takes the principal value, satisfying $-\pi < \text{Im } \log z \leq \pi$. By using this, a^z with $a, z \in \mathbb{C}$ is defined as $a^z \equiv e^{z \log a}$. Note that some properties, e.g. $\log(ab) = \log a + \log b$, may be lost for complex case.

We can discuss the $b \rightarrow 0$ limit in the same way as above, then we obtain

$$Z_{\text{Liouville}} \sim \left(e^{-\pi i/b^2} - e^{\pi i/b^2} \right) \lambda^{1/b^2} \sim \left(e^{\frac{\pi c^{(g)}}{6}} - e^{-\frac{\pi c^{(g)}}{6}} \right) \lambda^{-i \frac{c^{(g)}}{6}} \quad (5.27)$$

in the regime (5.15). Comparing with (5.4), we obtain the de Sitter entropy

$$S_{\text{GH}} = \frac{\pi c^{(g)}}{3} = \frac{\ell_{\text{dS}}}{2G}. \quad (5.28)$$

as the leading contribution. Usually people do not care much about overall factors of the partition function of field theories because it is just the normalization factor that should be divided in the definition of correlation functions. However, the overall normalization of wave functional of universe has the physical interpretation as the de Sitter entropy, so it is worth being careful about the overall factor. Our convention is consistent with defining Liouville path integral measure as

$$\int \mathcal{D}\phi e^{-I_{\text{Liouville}}[\phi]} \quad (5.29)$$

with the overall constant 1. Indeed, the partition function in our setup has been evaluated in the path integral approach in [15], which results in

$$Z_{\text{CFT}} \simeq C e^{\frac{\pi c^{(g)}}{6}} \lambda^{-i \frac{c^{(g)}}{6}}, \quad (5.30)$$

where C is a constant independent of $c^{(g)}$. Although this lacks the subleading contribution, the leading behavior matches with the above calculation.

Finally let us explain why we set the parameter $\lambda = \pi b^2 \mu$ to be real. Let us compare the partition function (5.27) with (5.4). We have already seen that the part S_{GH} agrees among two dual descriptions. Therefore, we should identify as

$$\mathcal{I} = \frac{c^{(g)}}{6} \ln \lambda. \quad (5.31)$$

Since $\mathcal{I}$ is real, the above identification implies that λ should be real as well. In other words, if the Liouville theory at large central charge is dual to the geometry for Hartle-Hawking wave function, then λ should be real.

5.2.2 Three-point functions

We would like to evaluate the semi-classical limits of three-point functions in Liouville field theory. The explicit form of three-point function has been explained at §4.4, which we show again here:

$$\langle V_{\alpha_1}(z_1, \bar{z}_1) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_3}(z_3, \bar{z}_3) \rangle = \frac{C(\alpha_1, \alpha_2, \alpha_3)}{|z_{12}|^{2(h_1+h_2-h_3)} |z_{13}|^{2(h_1+h_3-h_2)} |z_{23}|^{2(h_2+h_3-h_1)}}, \quad (5.32)$$

$$\begin{aligned} C(\alpha_1, \alpha_2, \alpha_3) &= \left[\lambda \gamma(b^2) b^{-2b^2} \right]^{(Q - \sum_i \alpha_i)/b} \\ &\times \frac{\Upsilon'_b(0) \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3)}{\Upsilon_b(\sum_i \alpha_i - Q) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon_b(\alpha_2 + \alpha_3 - \alpha_1) \Upsilon_b(\alpha_3 + \alpha_1 - \alpha_2)}. \end{aligned} \quad (5.33)$$

As mentioned above, we consider the case where all the external operators are heavy, i.e. $\alpha_i = \eta_i/b$ with $b \rightarrow 0$ while η_i kept fixed. Here we will consider real η_i , related to the bulk energy E_i as $2\eta_i = 1 - \sqrt{1 - 8G_N E_i}$, which satisfies the Seiberg bound $0 < \eta_i < 1/2$. As argued in [36], we can further divide them into two possible regions, called Region I and Region II in that paper:

$$\text{Region I: } \sum_i \eta_i > 1, \quad (5.34)$$

$$\text{Region II: } \begin{cases} \sum_i \eta_i < 1, \\ \eta_i + \eta_j - \eta_k > 0. \end{cases} \quad (5.35)$$

The condition of Region I comes from requiring the convergence of path integral. On the other hand, the original path integral over real ϕ does not converge in Region II, which leads us to taking complex saddles into account. The second condition in (5.35) is also satisfied in Region I without any extra conditions. Further meanings of the conditions will be explained below. First we evaluate a three-point function for each region.

It is easily checked that the first term is universally behaved as

$$\left[\lambda \gamma(b^2) b^{-2b^2} \right]^{(Q - \sum_i \alpha_i)/b} \simeq \exp \left[\frac{1 - \sum_i \eta_i}{b^2} \log \lambda + \frac{1 - \sum_i \eta_i}{b^2} \log \frac{1}{b^2} \right]. \quad (5.36)$$

The semi-classical limits of the other terms are evaluated by using the formula described in appendix B. In particular, the behavior of $\Upsilon_b(\sum_i \alpha_i - Q)$ depends on which region η_i belong to.

Region I In this region, we can apply the asymptotic formula (B.7) to every Υ_b function in the DOZZ formula (5.33). The result is

$$\begin{aligned} C(\alpha_1, \alpha_2, \alpha_3) \sim \lambda^{(1 - \sum_i \eta_i)/b^2} \exp \left[\frac{1}{b^2} \left\{ 1 - \sum_i \eta_i + F(2\eta_1) + F(2\eta_2) + F(2\eta_3) + F(0) \right. \right. \\ \left. \left. - F(\sum_i \eta_i - 1) - F(\eta_1 + \eta_2 - \eta_3) - F(\eta_2 + \eta_3 - \eta_1) - F(\eta_3 + \eta_1 - \eta_2) \right\} \right], \end{aligned} \quad (5.37)$$

where $F(\eta)$ is defined through integral in (B.5). Since η_i are all real-valued while b^2 and conformal weights are purely imaginary, the absolute value is

$$| \langle V_{\alpha_1}(z_1, \bar{z}_1) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_3}(z_3, \bar{z}_3) \rangle |^2 \sim \mathcal{O}(1). \quad (5.38)$$

This result may be related to the fact that there is no bulk dual to the three-point functions in this regime. In other words, we cannot construct the geometry of two-sphere with three conical deficits whose deficit angles are given by $\pi\eta_i$.

Region II Since $\sum_i \eta_i - 1 < 0$, we have to use the recursion relation (B.1)

$$\Upsilon_b \left(\frac{\sum_i \eta_i - 1}{b} \right) \sim \frac{b^{1 - \frac{2(\sum_i \eta_i - 1)}{b^2}}}{\gamma \left(\frac{\sum_i \eta_i - 1}{b^2} \right)} \Upsilon_b \left(\frac{\sum_i \eta_i}{b} \right). \quad (5.39)$$

By using this, (5.33) approximates

$$\begin{aligned} C(\alpha_1, \alpha_2, \alpha_3) \sim \left(\lambda \frac{1}{b^4} \right)^{(1 - \sum_i \eta_i)/b^2} \gamma \left(\frac{\sum_i \eta_i - 1}{b^2} \right) \\ \times \frac{\Upsilon'_b(0) \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3)}{\Upsilon_b(\sum_i \alpha_i) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon_b(\alpha_2 + \alpha_3 - \alpha_1) \Upsilon_b(\alpha_3 + \alpha_1 - \alpha_2)}. \end{aligned} \quad (5.40)$$

Furthermore, since $\text{Re}[(\sum_i \eta_i - 1)/b^2] > 0$, we have

$$\gamma\left(\frac{\sum_i \eta_i - 1}{b^2}\right) \sim \left(e^{-\pi i \frac{1-\sum_i \eta_i}{b^2}} - e^{\pi i \frac{1-\sum_i \eta_i}{b^2}}\right) \times \exp\left[-\frac{2}{b^2}\left(1 - \sum_i \eta_i\right)\left(\log\left(1 - \sum_i \eta_i\right) + \log\frac{1}{b^2} + \pi i - 1\right)\right]. \quad (5.41)$$

The πi factor, which comes from $\log(-1/b^2) = \log(1/b^2) + \pi i$, is cancelled with

$$\left(\frac{1}{b^4}\right)^{(1-\sum_i \eta_i)/b^2} = \exp\left(\frac{1-\sum_i \eta_i}{b^2} \log\frac{1}{b^4}\right) = \exp\left[\frac{2(1-\sum_i \eta_i)}{b^2}\left(\log\frac{1}{b^2} + \pi i\right)\right]. \quad (5.42)$$

Therefore, by using the asymptotic formula (B.7), we find

$$\begin{aligned} C(\alpha_1, \alpha_2, \alpha_3) &\sim \left(e^{-\pi i \frac{1-\sum_i \eta_i}{b^2}} - e^{\pi i \frac{1-\sum_i \eta_i}{b^2}}\right) \lambda^{(1-\sum_i \eta_i)/b^2} \\ &\times \exp\left[\frac{1}{b^2}\left\{F(2\eta_1) + F(2\eta_2) + F(2\eta_3) + F(0) - F\left(\sum_i \eta_i\right) \right. \right. \\ &\quad \left. \left. - F(\eta_1 + \eta_2 - \eta_3) - F(\eta_2 + \eta_3 - \eta_1) - F(\eta_3 + \eta_1 - \eta_2) \right. \right. \\ &\quad \left. \left. + 2\left(1 - \sum_i \eta_i\right) \log\left(1 - \sum_i \eta_i\right) - 2\left(1 - \sum_i \eta_i\right)\right\}\right]. \end{aligned} \quad (5.43)$$

We can see that the two saddles $N = -1, 0$ of (4.47) contribute to the DOZZ formula (5.33). Thus the absolute value of the three-point function goes to

$$|\langle V_{\alpha_1}(z_1, \bar{z}_1) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_3}(z_3, \bar{z}_3) \rangle|^2 \sim \exp\left[\frac{\pi c^{(g)}}{3}\left(1 - \sum_i \eta_i\right)\right]. \quad (5.44)$$

Let us consider the bulk configurations dual to three-point functions, which amount to generalizations of the discussions in §5.1. The three insertions are parametrized by η_i with $i = 1, 2, 3$ as above. We are interested in type II region: $0 \leq \eta_i \leq 1/2$, $\sum_i \eta_i < 1$ and $0 < \eta_i + \eta_j - \eta_k$. The operator insertion with η_i is known to create a conical deficit with deficit angle $4\pi\eta_i$. The way to create three conical deficits on S^2 is summarized in subsection 4.2 of [36]. In particular, we can see that the area of the geometry is $(1 - \sum_i \eta_i)$ times that of S^2 . The dual geometry may be constructed with the metric of the form:

$$ds^2 = d\theta^2 + \cos^2 \theta ds_{\text{con}}^2, \quad (5.45)$$

where ds_{con}^2 is the metric of S^2 with three conical deficits.¹¹ From the construction, we can see that the volume of the geometry is $(1 - \sum_i \eta_i)$ times that of S^3 . This reproduces the result from Liouville three-point function.

Let us discuss generic cases with s insertions of vertex operators on S^2 in the Liouville field theory. The dual geometry in the Chern-Simons theory may be given by S^3 with s defect lines. Precisely speaking, the action of Chern-Simons theory is given by

$$I = I_{\text{CS}}[A] - I_{\text{CS}}[\tilde{A}] + \frac{\kappa}{4\pi} \int_{\partial\mathcal{M}} \text{Tr}(A \wedge \tilde{A}), \quad (5.46)$$

where the boundary term, ensuring the gauge invariance on the boundary, is needed to treat the defects as explained shortly. Using the topological nature of Chern-Simons theory, we may

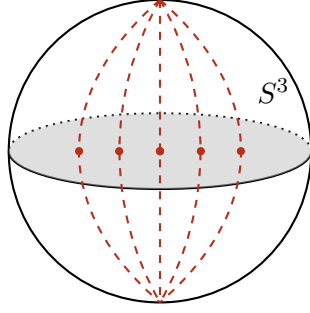


Figure 3: Utilizing the topological nature of Chern-Simons gravity, conical defect lines are set such as to start from (end on) the north (south) pole of S^3 .

put the starting (ending) points of the defect lines to the north (south) pole of S^3 , see fig. 3. For solutions to the equations of motion, only non-trivial contributions come from the boundary (defect) of the manifold and the topologically non-trivial configuration. For one insertion of defect line, the value of action is shifted by

$$\frac{\pi}{3}c^{(g)}m_i\eta_i \quad (5.47)$$

with integer m_i . Adding the topological contribution with the bulk winding number n , we have

$$\frac{\pi}{3}c^{(g)}\left(n - \sum_{i=1}^s m_i\eta_i\right). \quad (5.48)$$

See section 6 of [36] for more details. The results from Liouville three-point functions may be reproduced from setting

$$n = m_1 = m_2 = m_3 = \pm 1. \quad (5.49)$$

The leading contribution comes from the case with $+1$. Another consideration by using the mini-superspace method will be discussed later.

5.2.3 Four-point functions

In this subsection, we consider the following generic scalar four-point function in Liouville field theory:

$$G(z, \bar{z}) \equiv \langle V_1(0)V_2(z, \bar{z})V_3(1)V_4(\infty) \rangle, \quad (5.50)$$

where V_i ($i = 1, 2, 3, 4$) have the momenta $\alpha_i = \eta_i/b$ and the conformal weights h_i . From the Seiberg bound, every η_i satisfies $0 < \eta_i < 1/2$. As explained in §4.4, a four-point function can be decomposed by using the conformal blocks, and the expression highly depends on the parameters η_i, b . We will first discuss the region of η_i appropriate for our interest, and then evaluate the four-point functions.

We are interested in the insertions of four heavy operators $\alpha_i = \eta_i/b$ with real η_i such that the four-point function (5.50) has a dual geometry with four conical deficits. Let us discuss the condition on η_i in order to exist such a dual geometry. The dual geometry is given in the form of (5.45), where ds_{con}^2 is the metric of S^2 with four conical deficits in this case. The sphere

¹¹It would be nice if we can construct the metric ds_{con}^2 explicitly for any number of conical defects, see, e.g., [108].

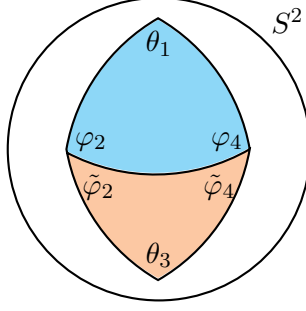


Figure 4: S^2 with four conical deficits can be embedded in S^2 as a spherical quadrangle with angles θ_i ($i = 1, 2, 3, 4$). This picture shows one way of triangulation by a geodesic that splits θ_2 and θ_4 into $\theta_2 = \varphi_2 + \tilde{\varphi}_2$ and $\theta_4 = \varphi_4 + \tilde{\varphi}_4$. We can also consider another way of triangulation by splitting θ_1 and θ_3 .

with conical deficits are embedded to a spherical quadrangle in a unit two-sphere. Assuming there exists a spherical quadrangle with four angles $\theta_i = \pi(1 - 2\eta_i)$ ($i = 1, 2, 3, 4$). The spherical quadrangle can be split into two spherical triangles in the same way as the ordinary triangulation. There are multiple ways to triangulate. Here we triangulate it by a geodesic that splits θ_2 and θ_4 , then we get two spherical triangles with angles $\theta_1, \varphi_2, \varphi_4$ and $\tilde{\varphi}_2, \tilde{\varphi}_4, \theta_3$, where $\varphi_2 + \tilde{\varphi}_2 = \theta_2$ and $\varphi_4 + \tilde{\varphi}_4 = \theta_4$, see fig. 4. Each spherical triangle has to satisfy the same condition as (5.35), therefore

$$\theta_1 + \varphi_2 + \varphi_4 > \pi, \quad \theta_1 + \varphi_2 - \varphi_4 < \pi, \quad \varphi_2 + \varphi_4 - \theta_1 < \pi, \quad \varphi_4 + \theta_1 - \varphi_2 < \pi, \quad (5.51)$$

$$\theta_3 + \tilde{\varphi}_2 + \tilde{\varphi}_4 > \pi, \quad \theta_3 + \tilde{\varphi}_2 - \tilde{\varphi}_4 < \pi, \quad \tilde{\varphi}_2 + \tilde{\varphi}_4 - \theta_3 < \pi, \quad \tilde{\varphi}_4 + \theta_3 - \tilde{\varphi}_2 < \pi. \quad (5.52)$$

Combining the two inequalities in each column, and repeating the same analysis for all ways of triangulation, we then obtain necessary conditions

$$\begin{aligned} \sum_i \eta_i &< 1, \\ 0 &< \eta_i + \eta_j + \eta_k - \eta_l < 1, \quad (i \neq j \neq k \neq l), \\ -1 &< \eta_i + \eta_j - \eta_k - \eta_l < 1, \quad (i \neq j \neq k \neq l). \end{aligned} \quad (5.53)$$

In particular, due to the first condition of (5.53), either $\eta_1 + \eta_2 > 1/2$ or $\eta_3 + \eta_4 > 1/2$ is broken. Hence α_i 's do not satisfy the condition for the conformal block decomposition to hold. Therefore we have to be careful for evaluating the four-point function.

First we consider the case where the operators satisfy $\eta_1 + \eta_2 > 1/2$ and $\eta_3 + \eta_4 < 1/2$ for simplicity. The calculation for this case is similar to that done in section 5 of [117], although the authors of the paper regard η_3 and η_4 as perturbatively small. When $\eta_3 + \eta_4 < 1/2$, some poles of the DOZZ coefficient $C(\alpha_3, \alpha_4, Q/2 + iP)$ cross the integral contour along the real axis in P -plane. In that case, the conformal block decomposition has contributions from the poles¹²

$$\begin{aligned} G(z, \bar{z}) &= i \sum_{\text{poles crossing } \mathbb{R}} C\left(\alpha_1, \alpha_2, \frac{Q}{2} - iP\right) \text{Res } C\left(\alpha_3, \alpha_4, \frac{Q}{2} + iP\right) \mathcal{F}_{34}^{12}(h_P|z) \mathcal{F}_{34}^{12}(h_P|\bar{z}) \\ &+ \frac{1}{2} \int_{-\infty}^{\infty} \frac{dP}{2\pi} C\left(\alpha_1, \alpha_2, \frac{Q}{2} - iP\right) C\left(\alpha_3, \alpha_4, \frac{Q}{2} + iP\right) \mathcal{F}_{34}^{12}(h_P|z) \mathcal{F}_{34}^{12}(h_P|\bar{z}). \end{aligned} \quad (5.54)$$

¹²In this paper, we focus on the contribution from the poles. We are currently examining whether the contribution from the integration over P can be interpreted as semi-classical saddles or not. See also arguments at the end of section 5 of [117].

Let us analyze the pole structure of $C(\alpha_3, \alpha_4, Q/2 + iP)$. $\Upsilon_b(x)$ has simple zeros at

$$x \in -b\mathbb{N} - \frac{1}{b}\mathbb{N} \quad \text{or} \quad x \in Q + b\mathbb{N} + \frac{1}{b}\mathbb{N} \quad (5.55)$$

($\mathbb{N}$ includes 0) and it has no poles. We can deduce that the factors in $C(\alpha_3, \alpha_4, Q/2 + iP)$ that produce the poles are $\Upsilon_b(\alpha_3 + \alpha_4 + iP - Q/2)$ and $\Upsilon_b(\alpha_3 + \alpha_4 - iP - Q/2)$ in the denominator, whose zero points are given by

$$\pm iP = -\alpha_3 - \alpha_4 + \frac{Q}{2} - mb - \frac{n}{b}, \quad (5.56)$$

$$\pm iP = -\alpha_3 - \alpha_4 + \frac{Q}{2} + (\tilde{m} + 1)b + \frac{\tilde{n} + 1}{b}, \quad (5.57)$$

with non-negative integers $m, n, \tilde{m}, \tilde{n}$. Since we are considering the range of $\text{Re } \alpha_3 + \text{Re } \alpha_4 < \text{Re}(Q/2)$, taking the semi-classical limit $b \rightarrow 0$, the poles that cross the real axis are [117]

$$\pm iP_m \equiv -\alpha_3 - \alpha_4 + \frac{Q}{2} - mb, \quad m = 0, 1, \dots, \quad (5.58)$$

i.e. only (5.56) with $n = 0$ contribute to the conformal block decomposition. Therefore we have

$$\begin{aligned} G(z, \bar{z}) &= i \sum_{m=0}^{\infty} C(\alpha_1, \alpha_2, Q - \alpha_3 - \alpha_4 - mb) \text{Res } C(\alpha_3, \alpha_4, \alpha_3 + \alpha_4 + mb) \mathcal{F}_{34}^{12}(h_{P_m}|z) \mathcal{F}_{34}^{12}(h_{P_m}|\bar{z}) \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \frac{dP}{2\pi} C\left(\alpha_1, \alpha_2, \frac{Q}{2} - iP\right) C\left(\alpha_3, \alpha_4, \frac{Q}{2} + iP\right) \mathcal{F}_{34}^{12}(h_P|z) \mathcal{F}_{34}^{12}(h_P|\bar{z}). \end{aligned} \quad (5.59)$$

Let us denote the term that comes from the residue at $\alpha_m = \alpha_3 + \alpha_4 + mb$ as

$$a_m \equiv i C(\alpha_1, \alpha_2, Q - \alpha_3 - \alpha_4 - mb) \text{Res } C(\alpha_3, \alpha_4, \alpha_3 + \alpha_4 + mb) \mathcal{F}_{34}^{12}(h_{P_m}|z) \mathcal{F}_{34}^{12}(h_{P_m}|\bar{z}). \quad (5.60)$$

First we consider the semi-classical limit of the $m = 0$ residue a_0 . Since $\Upsilon_b(x) \simeq \Upsilon'_b(0)x$ for $x \ll 1$, we have

$$\begin{aligned} i C(\alpha_1, \alpha_2, Q - \alpha_3 - \alpha_4) \text{Res } C(\alpha_3, \alpha_4, \alpha_3 + \alpha_4) &\simeq \left(\lambda \gamma(b^2) b^{-2b^2} \right)^{\frac{1}{b^2}(1 - \sum_i \eta_i)} \\ &\times \frac{\Upsilon'_b(0) \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3 + 2\alpha_4)}{\Upsilon_b(\sum_i \alpha_i - Q) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4) \Upsilon_b(Q + \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) \Upsilon_b(Q - \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)}. \end{aligned} \quad (5.61)$$

From Seiberg bound $0 < \eta_i < 1/2$, the conditions for the existence of the dual geometry (5.53), and the current assumption $\eta_1 + \eta_2 > 1/2$, $\eta_3 + \eta_4 < 1/2$, we can see that the arguments of all Upsilon functions but $\Upsilon_b(\sum_i \alpha_i - Q)$ are in the range $0 < \text{Re } x < \text{Re } Q$, which is the condition that the asymptotic formula (B.7) can be used. Due to $\text{Re}(\sum_i \alpha_i - Q) < 0$ from the first condition of (5.53), we have to use the defining recursion relation (B.1):

$$\Upsilon_b\left(\frac{\sum_i \eta_i - 1}{b}\right) = \gamma\left(\frac{\sum_i \eta_i - 1}{b^2}\right)^{-1} b^{1 + \frac{2}{b^2}(1 - \sum_i \eta_i)} \Upsilon_b\left(\frac{\sum_i \eta_i}{b}\right), \quad (5.62)$$

then the residue (5.61) becomes

$$iC(\alpha_1, \alpha_2, Q - \alpha_3 - \alpha_4) \text{Res } C(\alpha_3, \alpha_4, \alpha_3 + \alpha_4) \sim (\lambda b^{-4})^{\frac{1}{b^2}(1-\sum_i \eta_i)} \gamma\left(\frac{\sum_i \eta_i - 1}{b^2}\right) \\ \times \frac{\Upsilon'_b(0) \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3 + 2\alpha_4)}{\Upsilon_b(\sum_i \alpha_i) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4) \Upsilon_b(Q + \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) \Upsilon_b(Q - \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)}. \quad (5.63)$$

Due to $\text{Re } b^2 < 0$, we have

$$\gamma\left(\frac{\sum_i \eta_i - 1}{b^2}\right) \sim \left(e^{-i\pi(1-\sum_i \eta_i)/b^2} - e^{i\pi(1-\sum_i \eta_i)/b^2}\right) \\ \times \exp\left[-\frac{2}{b^2} \left(1 - \sum_i \eta_i\right) \left(\ln\left(1 - \sum_i \eta_i\right) + \ln\frac{1}{b^2} + \pi i - 1\right)\right]. \quad (5.64)$$

The πi factor is cancelled with

$$\left(\frac{1}{b^4}\right)^{\frac{1}{b^2}(1-\sum_i \eta_i)} = \exp\left[\frac{1}{b^2} \left(1 - \sum_i \eta_i\right) \log\left(\frac{1}{b^4}\right)\right] = \exp\left[\frac{2}{b^2} \left(1 - \sum_i \eta_i\right) \log\left(\frac{1}{b^2} + \pi i\right)\right]. \quad (5.65)$$

By using the formula for the semi-classical limit of the Υ_b function (B.7), we finally obtain

$$a_0 \sim \left(e^{-i\pi(1-\sum_i \eta_i)/b^2} - e^{i\pi(1-\sum_i \eta_i)/b^2}\right) \lambda^{(1-\sum_i \eta_i)/b^2} \\ \times \exp\left[\frac{1}{b^2} \left\{ F(2\eta_1) + F(2\eta_2) + F(2\eta_3 + 2\eta_4) + F(0) - F(\eta_1 + \eta_2 + \eta_3 + \eta_4) \right. \right. \\ \left. \left. - F(\eta_1 + \eta_2 - \eta_3 - \eta_4) - F(1 + \eta_1 - \eta_2 - \eta_3 - \eta_4) - F(1 - \eta_1 + \eta_2 - \eta_3 - \eta_4) \right. \right. \\ \left. \left. - 2 \left(1 - \sum_i \eta_i\right) \ln\left(1 - \sum_i \eta_i\right) + 2 \left(1 - \sum_i \eta_i\right) \right\} \right] \mathcal{F}_{34}^{12}(\alpha_3 + \alpha_4|z) \mathcal{F}_{34}^{12}(\alpha_3 + \alpha_4|\bar{z}). \quad (5.66)$$

Next let us consider all terms a_m . Note that the semi-classical conformal blocks behave as [118] (see [119] for a proof in that case where c, h_i, h_P are all real)

$$\mathcal{F}_{34}^{12}(h_P|z) \sim \exp\left[-\frac{c}{6} f\left(\frac{h_i}{c}, \frac{h_P}{c}, z\right)\right]. \quad (5.67)$$

Since c is pure imaginary and the ratios $h_i/c, h_P/c$ are real at the leading order in $1/c$, we can regard the semi-classical conformal block as a pure phase. In the semi-classical limit $b \rightarrow 0$, a ratio a_{m+1}/a_m behaves as

$$\frac{a_{m+1}}{a_m} \sim \frac{A}{(m+1)b^2}, \quad (5.68)$$

where A is a real constant given by

$$A = -\frac{\gamma(2\eta_3 + 2\eta_4)^2 \gamma(2\eta_3 + 2\eta_4 - 1)^2 \gamma(\eta_1 + \eta_2 - \eta_3 - \eta_4)}{\gamma(2\eta_3) \gamma(2\eta_4) \gamma(2\eta_3 + 2\eta_4 - 1) \gamma(-\eta_1 + \eta_2 + \eta_3 + \eta_4) \gamma(\eta_1 - \eta_2 + \eta_3 + \eta_4) \gamma(\sum_i \eta_i - 1)} \quad (5.69)$$

by using the recursion relations (B.1) and the asymptotic behavior of $\gamma(x) \sim x^{-2}$ for $x \ll 1$. Also note that $\gamma(1-x) = \gamma(x)^{-1}$. Each residue is then

$$a_m \simeq \frac{1}{m!} \left(\frac{A}{b^2} \right)^m a_0. \quad (5.70)$$

Therefore the semi-classical limit of the sum of the discrete terms is

$$\sum_{m=0}^{\infty} a_m \simeq \exp \left(\frac{A}{b^2} \right) a_0. \quad (5.71)$$

Thus we finally obtain the semi-classical limit of the four-point function:

$$\begin{aligned} G(z, \bar{z}) &\sim \left(e^{-i\pi(1-\sum_i \eta_i)/b^2} - e^{i\pi(1-\sum_i \eta_i)/b^2} \right) \lambda^{(1-\sum_i \eta_i)/b^2} e^{A/b^2} \\ &\times \exp \left[\frac{1}{b^2} \left\{ F(2\eta_1) + F(2\eta_2) + F(2\eta_3 + 2\eta_4) + F(0) - F(\eta_1 + \eta_2 + \eta_3 + \eta_4) \right. \right. \\ &\quad - F(\eta_1 + \eta_2 - \eta_3 - \eta_4) - F(1 + \eta_1 - \eta_2 - \eta_3 - \eta_4) - F(1 - \eta_1 + \eta_2 - \eta_3 - \eta_4) \\ &\quad \left. \left. - 2 \left(1 - \sum_i \eta_i \right) \ln \left(1 - \sum_i \eta_i \right) + 2 \left(1 - \sum_i \eta_i \right) \right\} \right] \mathcal{F}_{34}^{12}(\alpha_3 + \alpha_4|z) \mathcal{F}_{34}^{12}(\alpha_3 + \alpha_4|\bar{z}). \end{aligned} \quad (5.72)$$

We can see that two saddle points contribute. Since the conformal weights are purely imaginary in $b \sim 0$, the conformal block (5.67) just gives a phase. Therefore in the semi-classical limit $b^{-2} \sim -ic^{(g)}/6$, the absolute value of the four-point function is

$$\begin{aligned} |G(z, \bar{z})| &\sim \exp \left[\frac{\pi c^{(g)}}{6} \left(1 - \sum_i \eta_i \right) \right] - \exp \left[\frac{-\pi c^{(g)}}{6} \left(1 - \sum_i \eta_i \right) \right] \\ &\sim \sinh \left[\frac{\pi c^{(g)}}{6} \left(1 - \sum_i \eta_i \right) \right]. \end{aligned} \quad (5.73)$$

Even when $\eta_1 + \eta_2 < 1/2$ and $\eta_3 + \eta_4 < 1/2$, the calculation is essentially same to the above case. In this case both two DOZZ coefficients have poles, but the leading contribution comes from the “minimal” pole among $\alpha_1 + \alpha_2$ and $\alpha_3 + \alpha_4$. If $\eta_1 + \eta_2 > \eta_3 + \eta_4$, then the minimal pole is at $\alpha_3 + \alpha_4$ instead. Finally we have the same result as the above case.

5.2.4 Higher-point functions

We can extend the above calculations to higher-point functions. For simplicity, we only consider five-point functions

$$\langle V_1(0) V_2(z_2, \bar{z}_2) V_3(z_3, \bar{z}_3) V_4(1) V_5(\infty) \rangle \quad (5.74)$$

for Liouville momenta $\alpha_i = \eta_i/b$. The extension to general correlation functions is straightforward.

First of all, we would like to identify the regions of η_i from the condition for the dual geometry to exist. The dual geometry is expected to be S^3 with five conical defects with deficit angles $4\pi\eta_i$ with (5.45), each section of which can be mapped to a spherical pentagon with five angles $\theta_i = \pi(1 - 2\eta_i)$ in S^2 . A spherical pentagon can be split into a spherical triangle and a spherical

quadrangle. Applying conditions (5.35) and (5.53) to each way of splitting, we finally obtain necessary conditions

$$\begin{aligned} \sum_i \eta_i &< 1, \\ 0 &< \eta_i + \eta_j + \eta_k + \eta_l - \eta_m < 1, \quad (i \neq j \neq k \neq l \neq m), \\ -1 &< \eta_i + \eta_j + \eta_k - \eta_l - \eta_m < 1, \quad (i \neq j \neq k \neq l \neq m). \end{aligned} \quad (5.75)$$

Furthermore, we assume $\eta_1 \geq \eta_2 \geq \eta_3 \geq \eta_4 \geq \eta_5$ without loss of generality.

Let us perform the conformal-block expansion in a channel

$$\mathcal{F}_{25}^{134}(h_{P_1}, h_{P_2} | z_2, z_3) = \begin{array}{c} 1 \quad \quad 3 \quad \quad 4 \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \quad P_1 \quad P_2 \quad \quad \quad \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ 2 \quad \quad 5 \end{array}. \quad (5.76)$$

If η_i satisfied $\eta_1 + \eta_2 \geq 1/2$ and $\eta_3 + \eta_4 \geq 1/2$, the conformal block decomposition have the form

$$\begin{aligned} &\langle V_1(0) V_2(z_2, \bar{z}_2) V_3(z_3, \bar{z}_3) V_4(1) V_5(\infty) \rangle \\ &= \frac{1}{2} \int_{\mathbb{R}} \frac{dP_1}{2\pi} \frac{1}{2} \int_{\mathbb{R}} \frac{dP_2}{2\pi} C(\alpha_1, \alpha_2, \alpha_{P_1}) C(Q - \alpha_{P_1}, \alpha_3, \alpha_{P_2}) C(Q - \alpha_{P_2}, \alpha_4, \alpha_5) \\ &\quad \times \mathcal{F}_{25}^{134}(h_{P_1}, h_{P_2} | z_2, z_3) \mathcal{F}_{25}^{134}(h_{P_1}, h_{P_2} | \bar{z}_2, \bar{z}_3). \end{aligned} \quad (5.77)$$

However this does not hold for η_i 's in the region (5.75), so we have to discuss the analytic continuation in η_i . Let us assume $\eta_1 + \eta_2 \geq 1/2$, $\eta_3 + \eta_4 + \eta_5 \leq 1/2$. First, the integral contour in P_1 -plane picks up the poles

$$\alpha_{P_2} = \alpha_4 + \alpha_5 + mb, \quad m = 0, 1, \dots \quad (5.78)$$

of $C(Q - \alpha_{P_2}, \alpha_4, \alpha_5)$. As discussed in section 5.2.3, the contribution from summing over m finally gives just a constant phase. Therefore we only focus on the contribution from the $m = 0$ pole

$$\alpha_{P_2} = \alpha_4 + \alpha_5. \quad (5.79)$$

The residue for this pole is

$$\text{Res } C(Q - \alpha_4 - \alpha_5, \alpha_4, \alpha_5) = 1. \quad (5.80)$$

Next, the integral contour of P_1 picks up a pole

$$\alpha_{P_1} = \alpha_3 + \alpha_{P_2} = \alpha_3 + \alpha_4 + \alpha_5 \quad (5.81)$$

of $C(\alpha_{P_1}, \alpha_3, Q - \alpha_4 - \alpha_5)$. The residue for this pole is again

$$\text{Res } C(\alpha_3 + \alpha_4 + \alpha_5, \alpha_3, Q - \alpha_4 - \alpha_5) = 1. \quad (5.82)$$

Therefore the coefficients of the conformal block decomposition have the contributions from the first factor $C(\alpha_1, \alpha_2, \alpha_{P_1})$ at $\alpha_1 = \alpha_3 + \alpha_4 + \alpha_5$:

$$\begin{aligned} C(\alpha_1, \alpha_2, \alpha_3 + \alpha_4 + \alpha_5) &= \left[\lambda \gamma(b^2) b^{-2b^2} \right]^{\frac{1}{b}(Q - \sum_i \alpha_i)} \\ &\times \frac{\Upsilon'_b(0) \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3 + 2\alpha_4 + 2\alpha_5)}{\Upsilon_b(\sum_i \alpha_i - Q) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5) \Upsilon_b(\alpha_1 - \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \Upsilon_b(-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)}. \end{aligned} \quad (5.83)$$

From the conditions (5.75) and the assumption $\eta_1 + \eta_2 \geq 1/2$, $\eta_3 + \eta_4 + \eta_5 \leq 1/2$, we can see that only the factor $\Upsilon_b(\sum_i \alpha_i - Q)$ produces the γ function in the same way as four-point functions. Thus finally we have

$$C(\alpha_1, \alpha_2, \alpha_3 + \alpha_4 + \alpha_5) \sim \left(e^{-i\pi(1-\sum_i \eta_i)/b^2} - e^{i\pi(1-\sum_i \eta_i)/b^2} \right) \lambda^{(1-\sum_i \eta_i)/b^2} \\ \times \exp \left[\frac{1}{b^2} \left\{ F(2\eta_1) + F(2\eta_2) + F(2\eta_3 + 2\eta_4 + 2\eta_5) + F(0) - F(\eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5) \right. \right. \\ \left. \left. - F(\eta_1 + \eta_2 - \eta_3 - \eta_4 - \eta_5) - F(\eta_1 - \eta_2 + \eta_3 + \eta_4 + \eta_5) - F(-\eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5) \right. \right. \\ \left. \left. - 2 \left(1 - \sum_i \eta_i \right) \ln \left(1 - \sum_i \eta_i \right) + 2 \left(1 - \sum_i \eta_i \right) \right\} \right]. \quad (5.84)$$

In the semi-classical limit $b \rightarrow 0$, it is natural to expect that the five-point conformal block behaves as¹³

$$\mathcal{F}_{25}^{134}(h_{P_1}, h_{P_2} | z_2, z_3) \sim \exp \left[-\frac{c}{6} \tilde{f} \left(\frac{h_i}{c}, \frac{h_{P_1}}{c}, \frac{h_{P_2}}{c}; z_2; z_3 \right) \right] \quad (5.85)$$

as in the case of four-point conformal block in (5.67). Therefore the conformal blocks contribute to just a phase factor. Thus finally we obtain the semi-classical approximation

$$| \langle V_1(0) V_2(z_2, \bar{z}_2) V_3(z_3, \bar{z}_3) V_4(1) V_5(\infty) \rangle | \sim e^{\frac{\pi c(g)}{6}(1-\sum_i \eta_i)} - e^{-\frac{\pi c(g)}{6}(1-\sum_i \eta_i)}. \quad (5.86)$$

In the same way, we can calculate general correlation functions with any number of heavy operator insertions satisfying certain conditions coming from the existence of the dual geometry. Expanding correlation functions by using the linear channel s -point conformal block

$$\mathcal{F}^{(\text{linear})}(\{h_i\}_{1 \leq i \leq s}, \{h_{P_i}\}_{1 \leq i \leq s-3} | \{z_i\}_{2 \leq i \leq s-2}) = 1 \frac{\begin{array}{c} 2 \quad 3 \quad 4 \quad \cdots \quad s-1 \\ | \quad | \quad | \quad \quad | \quad | \\ P_1 \quad P_2 \quad \quad \quad P_{s-3} \end{array}}{s}, \quad (5.87)$$

the correlation function would give the same semi-classical approximation as above calculations.

As mentioned at the end of subsection 5.2.2, the saddle points of dual Chern-Simons gravity are classified by bulk winding number n and monodromies around the defects labeled m_i ($i = 1, 2, \dots, s$) in case with Liouville s -point function. Since the classical action for the saddle point with labels n, m_i is evaluated as (5.48), we conclude that the allowable saddles are given with

$$n = m_1 = \cdots = m_s = \pm 1. \quad (5.88)$$

The leading contribution comes from the case with $+1$.

5.2.5 Four-point functions dual to two linking Wilson lines

Originally, the model we are considering is the analytic continuation of a coset (2.45) introduced in §3.3. An important point in [14, 15] is that the $SU(2)_k$ WZW model part is dominant at the critical level $k \rightarrow -2$. Applying the CS/WZW correspondence [33], the WZW model can be described by $SU(2)$ Chern-Simons theory. In this way, we have a curious relation between

¹³See, e.g. [120] for a work on higher-point blocks.

dS₃ gravity and Chern-Simons theory. We should emphasize here that the relation is not the same as the Chern-Simons description of dS₃ gravity described in §2.2.1. For the Chern-Simons formulation of classical gravity, we take the level to infinity $k \rightarrow i\infty$, though the ordinary Chern-Simons description of WZW model uses the critical level $k \rightarrow -2$. Therefore, the two Chern-Simons descriptions are quite different. It is natural to suspect that they should be related as the triality relation of higher spin symmetry in the dual CFT [59]. In any cases, we can construct a geometry corresponding to Chern-Simons theory at the critical level with Wilson loop operators. In particular, the gravity solution was identified in [14, 15] corresponding to the configuration of two linked (unlinked) Wilson loops on S^3 in the Chern-Simons theory at the critical level. According to [33], the configuration corresponds to a four-point function of $SU(2)$ WZW model at the large central charge. Indeed, the partition function was computed from the both sides of the holographic duality and agreement was obtained.

In [14, 15], it was assumed that the expressions for the $SU(2)$ WZW model at an integer level k can be analytically continued to those at a complex k . In order to justify this, it was first argued that $SU(2)$ WZW model and the coset

$$\frac{SU(2)_k \times SU(2)_1}{SU(2)_{k+1}} \quad (5.89)$$

becomes the same at the critical level $k \rightarrow -2$ as the $SU(2)_k$ part of the coset dominates at the limit. Moreover, it was shown in [66] that correlation functions of the coset (5.89) and Liouville field theory are same. Therefore, it should be possible to describe the geometry corresponding to the two linked (unlinked) Wilson loops on S^3 in the Chern-Simons theory by a four-point function in Liouville field theory at the semi-classical limit $b \rightarrow 0$. Indeed, we show in this subsection that the results obtained from the native analytic continuation of S -matrix analysis lead to the correct answers even including sub-leading terms at the semi-classical limit.

We start by relating the CFT computations by four-point functions and those by modular S -matrices. In order to illustrate this, it is convenient to work with a rational CFT like $SU(N)_k$ WZW model. We will later consider a non-rational CFT, i.e. Liouville field theory. We examine the following four-point function

$$C_{ij}(z, \bar{z}) = \frac{\langle \mathcal{O}_i^\dagger(\infty) \mathcal{O}_i(1) \mathcal{O}_j^\dagger(z) \mathcal{O}_j(0) \rangle}{\langle \mathcal{O}_i^\dagger \mathcal{O}_i \rangle \langle \mathcal{O}_j^\dagger \mathcal{O}_j \rangle}, \quad (5.90)$$

where $\mathcal{O}_j$ is a CFT operator and $\mathcal{O}_j^\dagger$ is its conjugate. We define the function $C_{ij}(z, \bar{z})$ with a suitable normalization. Inserting a complete set $\mathbb{1} = \sum_p |p\rangle \langle p|$, we can decompose the four-point function by the sum of conformal blocks as

$$C_{ij}(z, \bar{z}) = \sum_p \mathcal{F}_{jj}^{ii}(p|z) \bar{\mathcal{F}}_{jj}^{ii}(p|\bar{z}). \quad (5.91)$$

We are interested in the region with large central charge. Then, we can argue that the identity block with $p = 0$ dominates, see [121]. When considering $z \sim 0$, then the function behaves as

$$C_{ij}(z, \bar{z}) \sim 1. \quad (5.92)$$

In the Chern-Simons formulation of WZW model, the four-point function (5.90) describes two unlinked Wilson loops. In order to obtain a configuration with two linked Wilson loops, we perform a move of the holomorphic coordinate z as in fig. 5. Namely, we start from $z \sim 0$, go

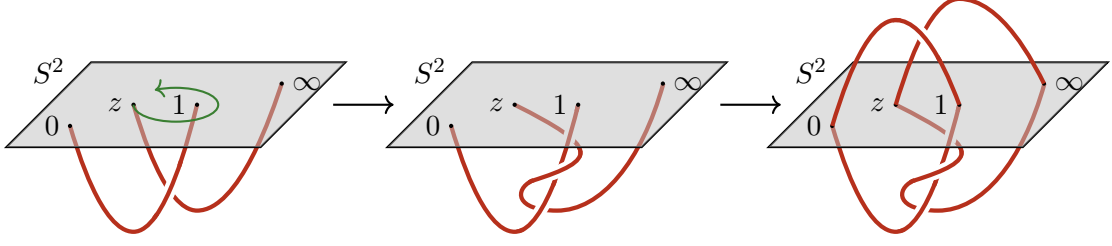


Figure 5: From the left to middle figures, we move the coordinate z around 1 anti-clockwise. From the middle to right figures, we glue the holomorphic and anti-holomorphic parts of four-point function. In the Chern-Simons description, there appears two Wilson loops and they are wrapped around each other.

around $z = 1$ anti-clockwise, then come back to $z \sim 0$.¹⁴ This move yields a monodromy matrix as

$$\mathcal{F}_{jj}^{ii}(p|z) \rightarrow \sum_q \mathcal{M}_{pq} \mathcal{F}_{jj}^{ii}(q|z). \quad (5.93)$$

Combining the anti-holomorphic part, where we do not perform the move of $\bar{z}$, we have

$$C_{ij}(z, \bar{z}) \sim \mathcal{M}_{00} \mathcal{F}_{jj}^{ii}(0|z) \bar{\mathcal{F}}_{jj}^{ii}(0|\bar{z}) \quad (5.94)$$

for $z \sim 0$. The monodromy matrix is [123, 124]

$$\mathcal{M}_{00} = \frac{S_{ij}^* S_{00} S_{00}}{S_{00} S_{0i} S_{0j}}. \quad (5.95)$$

As in fig. 5, we can identify the four-point function as the partition function of Chern-Simons theory with two linked Wilson loops for the Chern-Simons theory, see [125]. We thus find

$$\left| \left\langle \mathcal{O}_i^\dagger(\infty) \mathcal{O}_j^\dagger(1) \mathcal{O}_i(z) \mathcal{O}_j(0) \right\rangle \right| \sim \left| \frac{S_{ij}}{S_{00}} \right|, \quad (5.96)$$

which also implies that

$$\left| \left\langle \mathcal{O}_j^\dagger \mathcal{O}_j \right\rangle \right| \sim \left| \frac{S_{0j}}{S_{00}} \right| \quad (5.97)$$

with $i = 0$. Here the correlator with the insertion of the identity operator $\mathcal{O}_0 \equiv \mathbb{1}$ is normalized as one,

$$\langle \mathbb{1} \rangle = 1, \quad (5.98)$$

as usual. In [14, 15], unnormalized sphere correlators are used by adopting the convention of [33]. Specifically, the normalization of partition function,

$$\langle \mathbb{1} \rangle = S_{00}, \quad (5.99)$$

was used. Taking into account the normalization, we have

$$\left| \left\langle \mathcal{O}_j^\dagger \mathcal{O}_j \right\rangle \right| \sim |S_{0j}|, \quad (5.100)$$

¹⁴This procedure was used to compute out-of-time-order correlators in 2d CFTs [122].

and

$$\left| \left\langle \mathcal{O}_i^\dagger(\infty) \mathcal{O}_j^\dagger(1) \mathcal{O}_i(z) \mathcal{O}_j(0) \right\rangle \right| \sim |S_{ij}|. \quad (5.101)$$

These results reproduce those in [14, 15] from the modular S -matrix of WZW model. In case with two unlined Wilson loops, we should have (5.92). Using (5.100), we find

$$\left| \left\langle \mathcal{O}_i^\dagger(\infty) \mathcal{O}_j^\dagger(1) \mathcal{O}_i(z) \mathcal{O}_j(0) \right\rangle \right| \sim \left| \frac{S_{0i} S_{0j}}{S_{00}} \right|. \quad (5.102)$$

Here we should keep track of non-trivial normalization S_{00} . This also reproduces a finding in [14, 15].

Up to now we have assumed that the dual CFT is a rational one. In the following, we show that the same is true also for the case with Liouville field theory, which is a non-rational CFT. We start from the two-point function. In subsection 4, we have shown that the Liouville two-point function behaves as (5.25). The modular S -matrix of $SU(2)$ WZW model with level k is given by

$$S_j^l = \sqrt{\frac{2}{k+2}} \sin \left[\frac{\pi}{k+2} (2j+1)(2l+1) \right]. \quad (5.103)$$

Plugging $k \sim -2 + 6i/c^{(g)}$ and $2j+1 \sim \sqrt{1-8G_N E_j}$ into the above expression as in [14, 15], we find

$$|S_{0j}| \sim \left| e^{\frac{\pi}{6} c^{(g)} \sqrt{1-8G_N E_j}} - e^{-\frac{\pi}{6} c^{(g)} \sqrt{1-8G_N E_j}} \right|. \quad (5.104)$$

This reproduces the sub-leading non-perturbative corrections. We obtain the same conclusion even applying the modular S -matrix of Liouville field theory between the identity state $p=0$ and non-degenerate state given by [50]

$$S_{0j} = -2\sqrt{2} \sin 2\pi b(\alpha_j - Q/2) \sin 2\pi b^{-1}(\alpha_j - Q/2). \quad (5.105)$$

We then consider the four-point function corresponding to two linked Wilson loops in the Chern-Simons theory as in fig. 5. The modular S -matrix of Liouville theory among non-degenerate operators are given by [50]

$$S_{ij} = \sqrt{2} \cos 4\pi(\alpha_i - Q/2)(\alpha_j - Q/2). \quad (5.106)$$

Therefore, (5.101) leads to

$$\left| \left\langle \mathcal{O}_i^\dagger(\infty) \mathcal{O}_j^\dagger(1) \mathcal{O}_i(z) \mathcal{O}_j(0) \right\rangle \right| \sim \left| c_1 e^{\frac{\pi}{6} c^{(g)} \sqrt{1-8G_N E_i} \sqrt{1-8G_N E_j}} + c_2 e^{-\frac{\pi}{6} c^{(g)} \sqrt{1-8G_N E_i} \sqrt{1-8G_N E_j}} \right| \quad (5.107)$$

with real coefficients c_1, c_2 both for $SU(2)$ WZW model and Liouville field theory for the large central charge. In the following, we derive (5.107) directly from four-point functions of Liouville field theory, or more precisely speaking, four-point conformal blocks. We would like to read off the monodromy matrix of four-point conformal blocks around $z=1$. For this, we use the asymptotic behavior of conformal block $\mathcal{F}_{jj}^{ii}(p|z)$ near $z \sim 1$, which was obtained in [126] as

$$\mathcal{F}_{jj}^{ii}(p|z) \sim \sum_{n, \delta_\kappa = \pm} c_n^{\delta_\kappa} (1-z)^{\frac{c}{6} \kappa_n^{\delta_\kappa}} \quad (5.108)$$

with

$$\kappa_n^{\delta_\kappa} = n(1-n) - \frac{1}{2} + \left(\frac{1}{2} - n\right) ((1-2\eta_i) + \delta_\kappa(1-2\eta_j)) - \delta_\kappa \frac{(1-2\eta_i)(1-2\eta_j)}{2}. \quad (5.109)$$

We take $\delta_\kappa = \pm$ and $n \in \mathbb{Z}$. The monodromy can be read off as

$$\mathcal{F}_{jj}^{ii}(p|z) \rightarrow \exp\left(\frac{\pi i c}{3} \kappa_n^{\delta_\kappa}\right) \mathcal{F}_{jj}^{ii}(p|z). \quad (5.110)$$

The phase does not depend on the intermediate state p , but it is anyway set as $p = 0$. Performing the procedure in fig. 5, the absolute value of four-point function becomes

$$\left| \left\langle \mathcal{O}_i^\dagger(\infty) \mathcal{O}_j^\dagger(1) \mathcal{O}_i(z) \mathcal{O}_j(0) \right\rangle \right| \sim \sum_{\delta_i, \delta_j, \delta_\kappa = \pm, n} \left| \left\langle \mathcal{O}_i^\dagger \mathcal{O}_i \right\rangle_{\delta_i} \left\langle \mathcal{O}_j^\dagger \mathcal{O}_j \right\rangle_{\delta_j} \exp\left(-\frac{\pi c^{(g)} \kappa_n^{\delta_\kappa}}{3}\right) \right| \quad (5.111)$$

up to some coefficients. We already knew that the two-point functions behave as

$$\left\langle \mathcal{O}_j^\dagger \mathcal{O}_j \right\rangle = \sum_{\delta_j = \pm} c_{\delta_j} \left\langle \mathcal{O}_j^\dagger \mathcal{O}_j \right\rangle_{\delta_j}, \quad \left\langle \mathcal{O}_j^\dagger \mathcal{O}_j \right\rangle_{\delta_j} \sim \exp \delta_j \frac{\pi c^{(g)} (1-2\eta_j)}{6} \quad (5.112)$$

with real coefficients $c_\pm$. We are now interested in the identity block, which is defined by dividing the two-point functions as in (5.90). Therefore, it is natural to think that the terms linear in $(1-2\eta_i)$ and $(1-2\eta_j)$ come from the two-point functions. This means that we should choose $n = (\delta_i + 1)/2 = (\delta_j \delta_\kappa + 1)/2$. This choice indeed leads to (5.107).

5.3 Mini-superspace analysis

In the previous section, we determined the set of saddle points in three-dimensional gravity theory from dual CFT results. It is desired to see how such saddles are selected in the path integral for the gravity theory or equivalently which contours of path integral we should take. It is however difficult task to do full gravity analysis. In this section, we shall instead take mini-superspace approach to three-dimensional gravity with positive or negative cosmological constant.

We consider the Einstein-Hilbert action with a positive cosmological constant

$$I = -\frac{1}{16\pi G} \int d^3x \sqrt{g} \left(R - \frac{2}{\ell_{\text{dS}}^2} \right) + I_{\text{bdy}}. \quad (5.113)$$

Note that the boundary term I_{bdy} here is not restricted to the ordinary Gibbons-Hawking term. The data of boundary conditions and boundary terms will be discussed later. Even though we are interested in the Lorentzian dS_3 geometry, we adopt the Euclidean notation. Furthermore, the counter term

$$I_{\text{CT}} = \frac{1}{8\pi G} \int d^2x \sqrt{h} \sqrt{-\frac{1}{\ell_{\text{dS}}^2}} \quad (5.114)$$

at the outer boundary $\tau = 1$ will be needed to remove the divergence. In the mini-superspace approach, we often take an isotropic ansatz of metric

$$ds^2 = \ell_{\text{dS}}^2 [N(\tau)^2 d\tau^2 + a(\tau)^2 d\Omega_2^2], \quad (5.115)$$

in the normalized range $0 \leq \tau \leq 1$.

The quantity of our interest is the wave functional of universe:

$$\Psi = \int \mathcal{D}g_{\mu\nu} e^{-I - I_{\text{CT}}} . \quad (5.116)$$

The domain of the path integral should be taken as the set of metrics in which the diffeomorphism equivalent metrics are identified. It has been shown in [127] that after the gauge fixing in the path integral with the mini-superspace ansatz (5.115), the resulting path integral takes the form

$$\Psi = \int_{\mathcal{C}} dN \int \mathcal{D}a e^{-I[a;N] - I_{\text{CT}}} \quad (5.117)$$

where the action is¹⁵

$$I[a; N] = -\frac{\ell_{\text{dS}}}{2G} \int_0^1 d\tau N \left(\frac{1}{N^2} \left(\frac{da}{d\tau} \right)^2 - a^2 + 1 \right) + (\text{boundary contributions}) , \quad (5.118)$$

i.e. the τ -dependence of $N(\tau)$ is removed and the resulting form of Ψ includes the ordinary integral for N .

Here the remaining data to be input for the expression (5.117) are the boundary conditions of $a(\tau)$ at the outer boundary $\tau = 1$ and the inner boundary $\tau = 0$, and the integration contour $\mathcal{C}$ of N . On the outer boundary, we only consider the Dirichlet boundary condition $a(1) = a_1$ with a positive real constant $a_1 > 1$. The inner boundary is more non-trivial. Hence we will consider the Dirichlet-Dirichlet and Neumann-Dirichlet cases as the boundary conditions at the inner boundary and the outer boundary, respectively.

The second information, the integration contour $\mathcal{C}$, has also been discussed for a long time. When we take a contour asymptotically close to the positive imaginary axis, which seems to give an asymptotically dS spacetime, the integral (5.117) is not well-defined since the integrand does not converge along the contour. One way to define a well-defined integral is to extend the contour to a path in the complex plane $\mathbb{C}$ and apply the Picard-Lefschetz theory to find an appropriate contour. The Picard-Lefschetz theory is usually applied to an analytic continuation of an integral when a parameter involved in the integrand is extended to a complex value. In such situations, of course, the original integral before the continuation is well-defined. A peculiar point in the gravitational path integral is that there is no canonical way to choose $\mathcal{C}$ due to the lack of knowledge about quantum gravity, apart from the physical interpretations of the saddle point values of Ψ . Our novel point is that the contour can be determined such that the saddle point values match with the CFT results via dS holography.

5.3.1 Dirichlet-Dirichlet

Here we focus on the case where the Dirichlet boundary conditions are imposed both on $\tau = 0, 1$. Another boundary condition will be considered later. In this case, we do not need any additional boundary contributions to (5.118). Therefore we analyze the action:

$$I[a; N] = -\frac{\ell_{\text{dS}}}{2G} \int_0^1 d\tau N \left(\frac{1}{N^2} \left(\frac{da}{d\tau} \right)^2 - a^2 + 1 \right) . \quad (5.119)$$

The equation of motion with respect to a is given by

$$\frac{d^2 a}{d\tau^2} + N^2 a = 0 . \quad (5.120)$$

¹⁵Here we have an ambiguity of choosing a branch of the square root. Throughout this paper, we take it as satisfying $\sqrt{N^2} = N$.

A solution to this equation with the boundary conditions $a(0) = 0, a(1) = a_1$ imposed is

$$\bar{a}^{(N)}(\tau) = \frac{a_1}{\sin N} \sin(N\tau), \quad (5.121)$$

which also leads to

$$I[\bar{a}^{(N)}; N] = -\frac{\ell_{\text{dS}}}{2G} (N + a_1^2 \cot N). \quad (5.122)$$

Next we evaluate the perturbative contribution around the saddle $\bar{a}^{(N)}(r)$. Expanding $a(r)$ as $a(r) = \bar{a}^{(N)}(r) + A(r)$ with $A(0) = A(1) = 0$, we find

$$\begin{aligned} \Psi &= \int_{\mathcal{C}} dN e^{-I[\bar{a}^{(N)}; N] - I_{\text{CT}}} \int \mathcal{D}A \exp \left[\frac{\ell_{\text{dS}}}{2GN} \int_0^1 d\tau A(\tau) \left(-\frac{d^2}{d\tau^2} - N^2 \right) A(\tau) \right] \\ &= \int_{\mathcal{C}} dN e^{-I[\bar{a}^{(N)}; N] - I_{\text{CT}}} \text{Det} \left[\frac{\ell_{\text{dS}}}{2\pi GN} \left(\frac{d^2}{d\tau^2} + N^2 \right) \right]^{-\frac{1}{2}}. \end{aligned} \quad (5.123)$$

We would like to calculate the functional determinant. This determinant looks similar to the one-loop contribution of free scalar field, while the mass and the overall constant take complex values. See also [128] for more detailed discussion of such an analytic continuation. We solve the eigenvalue equation

$$\frac{\ell_{\text{dS}}}{2\pi GN} (A'' + N^2 A) = \lambda A, \quad (5.124)$$

under the boundary conditions $A(0) = A(1) = 0$. Then we find the spectrum

$$\lambda_m = \frac{N\ell_{\text{dS}}}{2\pi G} - \frac{m^2\pi\ell_{\text{dS}}}{2GN} \quad (m = 0, 1, \dots). \quad (5.125)$$

The functional determinant is formally expressed in terms of the eigenvalues as

$$\text{Det} \left[\frac{\ell_{\text{dS}}}{2\pi GN} \left(\frac{d^2}{dr^2} + N^2 \right) \right] = \prod_{m=0}^{\infty} \lambda_m = \frac{N\ell_{\text{dS}}}{2\pi G} \prod_{m=1}^{\infty} \left(\frac{N\ell_{\text{dS}}}{2\pi G} - \frac{m^2\pi\ell_{\text{dS}}}{2GN} \right). \quad (5.126)$$

To evaluate the infinite product, which needs to be regularized due to the divergence, it is useful to consider the ratio

$$\frac{\prod_{m=1}^{\infty} \left(\frac{N\ell_{\text{dS}}}{2\pi G} - \frac{m^2\pi\ell_{\text{dS}}}{2GN} \right)}{\prod_{m=1}^{\infty} \left(-\frac{m^2\pi\ell_{\text{dS}}}{2GN} \right)} = \prod_{m=1}^{\infty} \left(1 - \frac{N^2}{\pi^2 m^2} \right) = \frac{1}{N} \sin N \quad (5.127)$$

where we have used a formula $\sinh z = z \prod_{m=1}^{\infty} (1 + \frac{z^2}{m^2\pi^2})$. To evaluate the denominator, we use the zeta function regularization

$$\prod_{m=1}^{\infty} \left(-\frac{m^2\pi\ell_{\text{dS}}}{2GN} \right) = e^{-\zeta'_{\lambda}(0)}, \quad (5.128)$$

where

$$\zeta_{\lambda}(s) := \sum_{m=1}^{\infty} \left(-\frac{m^2\pi\ell_{\text{dS}}}{2GN} \right)^{-s} = \left(-\frac{\pi\ell_{\text{dS}}}{2GN} \right)^{-s} \zeta(2s) \quad (5.129)$$

with the Riemann zeta function $\zeta(s)$. By using the identity (5.129), we have

$$\zeta'_\lambda(0) = \log\left(-\frac{2GN}{\pi\ell_{\text{dS}}}\right) \zeta(0) + 2\zeta'(0) = -\log\left(-2\sqrt{\frac{2\pi GN}{\ell_{\text{dS}}}}\right). \quad (5.130)$$

Summarizing the above calculations, we obtain the regularized functional determinant

$$\text{Det}\left[\frac{\ell_{\text{dS}}}{2\pi GN}\left(\frac{d^2}{dr^2} + N^2\right)\right] = -2\sqrt{\frac{\ell N}{2\pi G}} \sin N. \quad (5.131)$$

Ignoring the irrelevant overall constant, we arrive at the expression of the partition function

$$\Psi = \int_{\mathcal{C}} dN \left(\frac{1}{\sqrt{N} \sin N}\right)^{\frac{1}{2}} e^{-I[\bar{a}^{(N)}; N] - I_{\text{CT}}}. \quad (5.132)$$

Though the pre-factor in front of the exponential will not give any effects on the saddle points, it may contain information about singularities or branch cuts. In the current case, the pre-factor does not have any poles, but $N = \pi\mathbb{Z}$ are the branch points coming from $\sqrt{\sin N}$.

Now we are at the stage to perform the saddle point approximation. From the equation $\partial I[\bar{a}^{(N)}; N]/\partial N = 0$, the saddle points should satisfy

$$\sin N = \pm a_1. \quad (5.133)$$

Since we are restricting to the region $a_1 > 1$, the saddle points are listed as

$$N_m^+ = \left(m + \frac{1}{2}\right) \pi + i \log\left(a_1 + \sqrt{a_1^2 - 1}\right), \quad (5.134)$$

$$N_m^- = \left(m + \frac{1}{2}\right) \pi - i \log\left(a_1 + \sqrt{a_1^2 - 1}\right), \quad (5.135)$$

and the on-shell action on each saddle point is

$$I[\bar{a}^{(N_m^+)}; N_m^+] = -\frac{\ell_{\text{dS}}}{2G} \left(m\pi - ia_1 \sqrt{a_1^2 - 1} + i \log\left(a_1 + \sqrt{a_1^2 - 1}\right)\right), \quad (5.136)$$

$$I[\bar{a}^{(N_m^-)}; N_m^-] = -\frac{\ell_{\text{dS}}}{2G} \left(m\pi + ia_1 \sqrt{a_1^2 - 1} - i \log\left(a_1 + \sqrt{a_1^2 - 1}\right)\right). \quad (5.137)$$

The saddle points and the steepest descents are depicted in figure 6. We denote $\mathcal{J}_m^+$ and $\mathcal{J}_m^-$ as the steepest descents associated with N_m^+ and N_m^- , respectively. The orientations of steepest descents are defined to be positive in the positive directions of the real and imaginary axes. We can see that the steepest descent paths in figure 6 show a curious behavior: A steepest descent path that emanates from a saddle point can end at another saddle point. This is actually a critical behavior in the sense that the theory exactly lies on the Stokes line. Precisely speaking, the saddle point approximation is not well-defined in such a situation (see e.g. [35, 111]). This is actually a specific feature in three dimensions, which cannot be observed in four dimensions [129]. As we can find from (5.132), there are branch points at $N = \pi\mathbb{Z}$ coming from $\sqrt{\sin N}$. We choose the branch cuts as connecting $N = 2n\mathbb{Z}$ and $N = (2n + 1)\mathbb{Z}$ as depicted in figure 6.

To avoid this problem, we try to slightly shift the parameter ℓ_{dS} in the imaginary direction $\ell_{\text{dS}} \rightarrow \ell_{\text{dS}} \pm i\epsilon$, which are depicted in figure 7. The configurations of steepest descent paths highly depend on the sign of the shift. When we only focus on the upper half plane, the right panel in figure 7 is in the same phase as the AdS₃ case as we will see in the next section, while the left

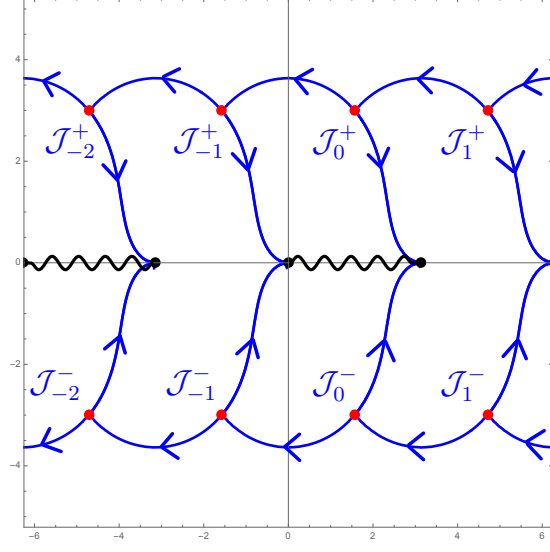


Figure 6: The steepest descent paths for the dS_3 case with Dirichlet-Dirichlet boundary conditions. The red dots denote the saddle points and the wavy lines correspond to branch cuts.

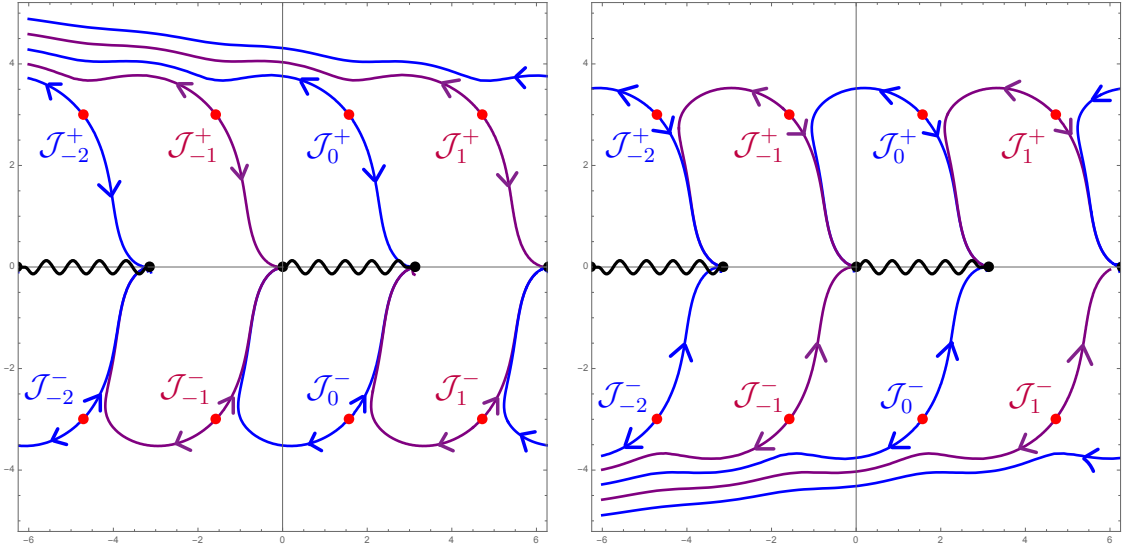


Figure 7: The left panel shows the plot when $\ell_{dS} \rightarrow \ell_{dS} + i\epsilon$. The right panel shows the plot when $\ell_{dS} \rightarrow \ell_{dS} - i\epsilon$.

panel is very different. The transition between these phases causes the Stokes phenomenon. In fact, this Stokes phenomenon looks very similar to that for Gamma function, see e.g. Appendix C of [36].

Now we have a question: Which direction of the shift, $\ell_{\text{dS}} \rightarrow \ell_{\text{dS}} \pm i\epsilon$, is physically appropriate for the dS₃ case? One answer is given by the dual CFT point of view via the dS/CFT correspondence, where the central charge of the dual CFT is

$$c = -i \frac{3\ell_{\text{dS}}}{2G} + 13 + \mathcal{O}(G). \quad (5.138)$$

In the gravity interpretation, the constant term “13” comes from the one-loop contributions. This contribution can also be realized as a shift $\ell_{\text{dS}} \rightarrow \ell_{\text{dS}} + i\epsilon$, which corresponds to the left panel of figure 7 (Note that we adopt the notation $\ell_{\text{AdS}} = -i\ell_{\text{dS}}$).

Finally let us discuss the contour $\mathcal{C}$ for the N integral. Our goal is to find a contour that reproduces the CFT calculation (5.27). Each saddle point $N_m^\pm$ gives the leading contribution to the wave functional of universe

$$\Psi_m^\pm \sim e^{\frac{(2m+1)\ell_{\text{dS}}\pi}{4G}} (2a_1)^{\mp i \frac{\ell_{\text{dS}}}{2G} \pm \frac{\epsilon}{2G}}. \quad (5.139)$$

Due to ϵ in the exponent, Ψ_m^- from the saddles in the lower half plane are suppressed in the large a_1 approximation. Naively to preserve future time direction, one would choose the integration contour as $i\mathbb{R}_+$. Though the integral along this contour diverges, it can be deformed to coincide $\mathcal{J}_{-1}^+$. However, this contour does not seem to be physical because Ψ has only the suppressing saddle. To match with the CFT result (5.27), we have to pick up two saddles $\mathcal{J}_0^+, \mathcal{J}_{-1}^+$ from the upper half plane. Therefore we propose the contour that comes from the positive imaginary infinity, rounds the branch cut lying between $N = 0$ and $N = \pi$, and goes back to the positive imaginary infinity. This contour can be deformed to

$$-\mathcal{J}_{-1}^+ + \mathcal{J}_0^- + \mathcal{J}_0^+, \quad (5.140)$$

where the signatures take -1 if the orientations of the steepest descent and the contour are opposite. Since the contribution from $\mathcal{J}_0^-$ is suppressed as explained above, the semi-classical approximation of the wave functional of universe is obtained as

$$\Psi \sim \left(e^{\frac{\pi\ell_{\text{dS}}}{4G}} - e^{-\frac{\pi\ell_{\text{dS}}}{4G}} \right) (2a_1)^{i \frac{\ell_{\text{dS}}}{2G}}. \quad (5.141)$$

Identifying $c^{(g)} = 3\ell_{\text{dS}}/2G$, this actually reproduces (5.27) with $\eta = 0$.

Here we simply show that the complex geometry realized as a saddle can be put into the form of the one discussed in §5.1. As in [130] we may consider a coordinate system which interpolate between the two types of metrics as

$$T(\tau) = - \left(n + \frac{1}{2} \right) \pi (1 - \tau)^q + i \ln(2a_1) \tau^q. \quad (5.142)$$

Due to the Cauchy’s theorem, the integral does not depends on the path integral contour over the metrics. We can see that the time coordinate reduces to the one realized by a saddle with $q = 1$ up to a constant shift and approaches the one used for the no-boundary proposal of Hartle and Hawking [131] for $q \rightarrow \infty$.

The properties of mini-superspace analysis for three-dimensional gravity are highly different from the four dimensional case summarized in [111]. The most distinct point is the infinite

number of saddle points. The four dimensional model has finite number of saddle points that can be regarded as saddle points of Airy functions. As described in [9], such a behavior of the number of saddle points depend on whether the spacetime dimension is even or odd. When we analytically continue one coordinate of D -dimensional sphere S^D to the no-boundary geometries, there exist infinite saddle points for odd D . For even D , precisely speaking, the infinite saddles degenerate in the sense that the on-shell actions take the same value among those saddles. This observation is consistent with our analysis. Therefore the extension to higher dimensions are expected to have similar structure to our analysis in odd-dimensional mini-superspace model.

5.3.2 Neumann boundary condition

So far we have only considered the Dirichlet boundary condition at the initial boundary $\tau = 0$. This is the most straightforward boundary condition in order to construct the no-boundary saddle points. Nevertheless, it is also possible to impose Neumann boundary condition at $\tau = 0$. Although the initial value of the scaling factor $a(0)$ then does not always take zero in the path integral, we might adjust the boundary condition so that there exists a saddle point we want. Indeed, such attempts have been tried in [132, 133], where the authors construct certain no-boundary saddle points from the other boundary conditions in four dimensions. Our purpose to match the mini-superspace calculations with the CFT results has already been accomplished from the Dirichlet-Dirichlet setup in the previous section. In this subsection, however, we will try another boundary condition; a Neumann boundary condition $a'(0) = N$. The value of $a'(0)$ has been chosen so that the theory has the physically relevant saddle $\mathcal{J}_0^+$ as in Dirichlet case. The Neumann-Dirichlet condition is, of course, not the only possibility of other boundary conditions. There are various choices, e.g. Ribon boundary condition, as studied in detail in [133]. We leave the study of such other boundary conditions as a future work.

When we impose a Neumann boundary condition, the boundary term at $\tau = 0$ should be modified. In terms of a and N , the action becomes

$$I[a; N] = -\frac{\ell_{\text{dS}}}{2G} \int_0^1 d\tau N \left(\frac{1}{N^2} \left(\frac{da}{d\tau} \right)^2 - a^2 + 1 \right) - \frac{\ell_{\text{dS}}}{GN} a \frac{da}{d\tau} \Big|_{\tau=0}. \quad (5.143)$$

A solution to the equation of motion

$$\frac{d^2 a}{d\tau^2} + N^2 a = 0 \quad (5.144)$$

with boundary conditions $\frac{da}{d\tau} \Big|_{\tau=0} = N$ and $a(1) = a_1$ is

$$\bar{a}^{(N)}(\tau) = \frac{a_1}{\cos N} \cos(N\tau) + \frac{\sin(N(\tau - 1))}{\cos N}. \quad (5.145)$$

The on-shell action is then

$$I[\bar{a}^{(N)}; N] = -\frac{\ell_{\text{dS}}}{2G} \left[N + \frac{2a_1}{\cos N} - (1 + a_1^2) \tan N \right]. \quad (5.146)$$

The derivative with respect to N is

$$\frac{\partial I[\bar{a}^{(N)}; N]}{\partial N} = \frac{\ell_{\text{dS}}}{2G} \frac{(\sin N - a_1)^2}{\cos^2 N}. \quad (5.147)$$

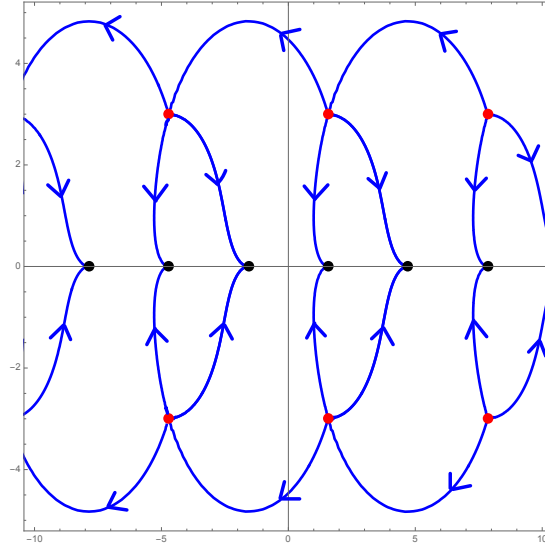


Figure 8: The steepest descent paths for the dS_3 case with Neumann-Dirichlet boundary conditions. The red dots denote the saddle points.

Differing from the previous case, it turns out that the solutions to $\sin N = a_1$ are the second order saddle points rather than the first ones. Namely, the saddle points have the vanishing second order derivative as well:

$$\frac{\partial^2 I[\bar{a}^{(N)}; N]}{\partial N^2} = 0. \quad (5.148)$$

In terms of the analysis of steepest ascend/descend paths, we consider the fluctuations around each of these saddle points. This requires us to expand to the third order derivatives, which leads to three steepest ascending/descending directions respectively. Such paths are depicted in figure 8. In this case, the saddle point that corresponds to $\mathcal{J}_{-1}^+$ we encountered in Dirichlet-Dirichlet case does not exist. Therefore the current boundary condition does not reproduce the CFT result.

5.4 Relation with Lorentzian quantum cosmology

Recently, the similar approach to the discussions so far has been well studied for four-dimensional gravity, starting with [112, 113]. The authors of the paper start with the Lorentzian contour $\mathcal{C} = i\mathbb{R}_+$ and apply the Picard-Lefschetz theory to discuss the saddle point approximation. This approach is often called “Lorentzian quantum cosmology”. In four dimensions, the gravitational path integral has only two types of saddle points; the Hartle-Hawking saddle [5] and the tunneling saddle [134, 135] by Vilenkin, which are the counterparts of $\mathcal{J}_0^+$ and $\mathcal{J}_{-1}^+$ in the previous section, respectively. As the result of [112, 113], the gravitational path integral with $\mathcal{C} = i\mathbb{R}_+$ only picks up the tunneling saddle, which leads to the conclusion that the Hartle-Hawking saddle is not relevant to the Lorentzian quantum cosmology. As a modification of the theory, the authors of [132, 133] tried other boundary conditions at the beginning of the universe $\tau = 0$ than the Dirichlet boundary condition. They found that the Hartle-Hawking saddle can be relevant by choosing the boundary condition properly.

Let us compare our approach with them. First of all, our motivation is to make it consistent with the CFT results by properly choosing the integration contour. On the other hand, the

approach explained above assumes the integration contour to be $\mathcal{C} = \mathbb{R}_+$ and discusses the results from the given integral. Therefore our motivation is different from the Lorentzian quantum cosmology. If we follow the same procedure with [112, 113], we obtain the similar answer with them: The integration contour $\mathcal{C} = \mathbb{R}_+$ only picks up $\mathcal{J}_{-1}^+$ as one can see from the left panel of figure 7. However, the result is not consistent with the CFT result. As discussed in §5.3.2, even if the Neumann boundary condition is imposed, the result does not become consistent. Thus we conjecture that choosing the contour is needed to accomplish the holographic correspondence of saddle points.

Besides, [136] proposed another possible choice of the contour in the context of Lorentzian quantum cosmology based on [137]. In that paper, the authors proposed the contour along the whole real axis $\mathcal{C} = \mathbb{R}$. Here, the contour has to avoid a singularity at the origin of N -plane by deforming the theory in some way. As a result, the gravitational path integral has contributions from both the Hartle-Hawking and the tunneling saddle points, similarly to our result. It is pointed out by [138] that this procedure cannot be essentially thought of as the Lorentzian path integral because the integral cannot be expressed as the summation over the Lorentzian metrics due to the presence of the singularity $N = 0$. The correct procedure for the integration contour seems to be still controversial, though the later one [136] would be relevant to our procedure.

6 Holographic study of semi-classical saddles in AdS gravity

Next we would like to apply the discussion in the previous section to the negative cosmological constant case. The strategy for analyses is similar to the previous section, but we will find interesting observations because Euclidean AdS_3 gravity have richer structures such as black holes or geometries with various topologies.

6.1 CFT calculations

Let us consider the ordinary spacelike Liouville theory with $b > 0$. Compared with the dS/CFT case, we can simply neglect the $\mathcal{O}(1)$ contribution in the leading order approximation:

$$\frac{1}{b^2} \sim \frac{c}{6} = \frac{\ell_{\text{AdS}}}{4G}. \quad (6.1)$$

When $b \in \mathbb{R}$, the dictionary (2.15) for a heavy operator $\alpha = \eta/b$ is given by

$$\ell_{\text{AdS}} E \sim \frac{2\eta(1-\eta)}{b^2} \quad (6.2)$$

in the leading order in b . For the dual energy E to be real-valued, η has to be

$$\eta \in \frac{1}{2} + i\mathbb{R}, \quad \text{or} \quad \eta \in (0, 1/2). \quad (6.3)$$

An operator of the first form (6.3) has the conformal weight larger than the black hole threshold $(c-1)/24$, which corresponds to a BTZ black hole. On the other hand, an operator of the second form of (6.3) is below the threshold, which corresponds to a conical defect geometry whose deficit angle is given by $2\pi(1 - \sqrt{1 - 8GE})$. In this thesis, we only focus on the latter, leaving the black hole case as a future work.

Let us consider the two-point functions

$$\langle V_\alpha(z_1) V_\alpha(z_2) \rangle = |z_{12}|^{-4\alpha(Q-\alpha)} \frac{2\pi}{b^2} [\pi \mu \gamma(b^2)]^{(Q-2\alpha)/b} \gamma \left(\frac{2\alpha}{b} - 1 - \frac{1}{b^2} \right) \gamma(2b\alpha - b^2) \delta(0). \quad (6.4)$$

In the current case, we have $\text{Re}(1/b^2) > 0$, which is opposite to the dS/CFT case. This is a big difference between AdS/CFT and dS/CFT. When $\text{Re}(1/b^2) > 0$, the semi-classical approximation of (6.4) is given by

$$\langle V_\alpha(z_1)V_\alpha(z_2) \rangle \sim \frac{1}{e^{-\frac{\pi i}{b^2}(1-2\eta)} - e^{\frac{\pi i}{b^2}(1-2\eta)}} |z_{12}|^{-\frac{4}{b^2}\eta(1-\eta)} e^{-\frac{2}{b^2}[(1-2\eta)\ln(1-2\eta)-(1-2\eta)]} \quad (6.5)$$

Let us focus on the phase factor and clarify how many saddle points are contributing. In order to express (6.5) as a sum of the saddle point contributions, we would like to perform the polynomial expansion for the phase factor. For our purpose, it is convenient to introduce a regulator by replacing ℓ_{AdS} by $\ell_{\text{AdS}} + i\epsilon$ with $\epsilon > 0$. This implies that

$$\text{Im } b^{-2} \sim \text{Im } \frac{c}{6} \sim \text{Im } \frac{\ell_{\text{AdS}} + i\epsilon}{4G} > 0. \quad (6.6)$$

With the regulator, we can rewrite

$$\frac{1}{e^{-\frac{\pi i}{b^2}(1-2\eta)} - e^{\frac{\pi i}{b^2}(1-2\eta)}} \sim e^{\pi i(1-2\eta)\frac{\ell_{\text{AdS}}}{4G}} \sum_{n=0}^{\infty} e^{n\pi i(1-2\eta)\frac{\ell_{\text{AdS}}}{2G}}. \quad (6.7)$$

The two-point function can be decomposed as

$$\begin{aligned} \langle V_\alpha(z_1)V_\alpha(z_2) \rangle &\sim \sum_{n=0}^{\infty} e^{\frac{\ell_{\text{AdS}}}{4G}(2n+1)\pi i(1-2\eta)} \\ &\times |z_{12}|^{-\frac{\ell_{\text{AdS}}}{2G}\eta(1-\eta)} \lambda^{\frac{\ell_{\text{AdS}}}{4G}(1-2\eta)} e^{-\frac{\ell_{\text{AdS}}}{2G}[(1-2\eta)\ln(1-2\eta)-(1-2\eta)]}. \end{aligned} \quad (6.8)$$

By taking the limit $\eta \rightarrow 0$, we obtain the semi-classical approximation of the partition function

$$\mathcal{Z}_{\text{AdS}} \sim \sum_{n=0}^{\infty} e^{\frac{\ell_{\text{AdS}}}{4G}(2n+1)\pi i} (e^2 \lambda)^{\frac{\ell_{\text{AdS}}}{4G}}. \quad (6.9)$$

The expression suggests that there are infinitely many saddles labeled by $n = 0, 1, 2, \dots$. The dependence on the label n appears in the pure phase factor. We may regularize the expression with the opposite sign $\ell_{\text{AdS}} - i\epsilon$. In the case, the series expansion becomes the sum of negative $n = -1, -2, \dots$ in (6.7). One might think that those regularizations $\ell_{\text{AdS}} \pm i\epsilon$ are not natural in the sense that the resulting partition function does not seem to give a real value. Of course, this complex nature is nothing but the result of artificially taking a special regularization. In fact, there is an ambiguity on how to regularize the divergence, in analogous sense with the various regularizations (advanced, retarded and Feynman prescriptions) of propagators in ordinary quantum field theories. Another possible regularization is explained in C, where the saddle point approximation can be expressed as a sum of real terms as (C.10).

As we explained in §5.2.2, three-point functions have a richer structure than two-point functions. The DOZZ formula (5.33) has different form of approximation in the following two regions [36]

$$\text{Region I: } \sum_i \text{Re } \eta_i > 1, \quad (6.10)$$

$$\text{Region II: } \begin{cases} \sum_i \text{Re } \eta_i < 1, \\ \text{Re } \eta_i + \text{Re } \eta_j - \text{Re } \eta_k > 0. \end{cases} \quad (6.11)$$

along with the Seiberg bound $0 < \text{Re } \eta_i < 1/2$. For region I, the semi-classical limit of three-point function is

$$C(\alpha_1, \alpha_2, \alpha_3) \sim \lambda^{(1-\sum_i \eta_i)/b^2} \exp \left[\frac{1}{b^2} \left\{ 1 - \sum_i \eta_i + F(2\eta_1) + F(2\eta_2) + F(2\eta_3) + F(0) \right. \right. \\ \left. \left. - F\left(\sum_i \eta_i - 1\right) - F(\eta_1 + \eta_2 - \eta_3) - F(\eta_2 + \eta_3 - \eta_1) - F(\eta_3 + \eta_1 - \eta_2) \right\} \right]. \quad (6.12)$$

For region II, we have

$$C(\alpha_1, \alpha_2, \alpha_3) \sim \frac{1}{e^{-\pi i \frac{1-\sum_i \eta_i}{b^2}} - e^{\pi i \frac{1-\sum_i \eta_i}{b^2}}} \lambda^{(1-\sum_i \eta_i)/b^2} \\ \times \exp \left[\frac{1}{b^2} \left\{ F(2\eta_1) + F(2\eta_2) + F(2\eta_3) + F(0) - F\left(\sum_i \eta_i\right) \right. \right. \\ \left. \left. - F(\eta_1 + \eta_2 - \eta_3) - F(\eta_2 + \eta_3 - \eta_1) - F(\eta_3 + \eta_1 - \eta_2) \right. \right. \\ \left. \left. + 2 \left(1 - \sum_i \eta_i \right) \log \left(1 - \sum_i \eta_i \right) - 2 \left(1 - \sum_i \eta_i \right) \right\} \right]. \quad (6.13)$$

These different forms of approximation can be understood as the Stokes phenomenon in the parameter space of η_i . The Stokes line is along $\sum_i \text{Re } \eta_i = 1$. One can find that the DOZZ coefficients have contributions from complex saddles only for region II, leading to the prediction that the complex geometries emerge when “not too heavy” operators are inserted.

Apart from Liouville theory, the DOZZ formula also appears in more general CFT as the universal OPE coefficients with heavy operators [114]. In this context, the saddle points for region II and its dual geometries are discussed in [139], and for region I recently investigated in [18]. The latter theory is interpreted as the average CFT, and the dual geometries to the CFT saddles have been found to be real wormhole geometries such as [140]. Our discussions in this section can straightforwardly be applied to such contexts.

In the remainder of this section, we present the calculations in the gravity side and attempt to answer the following questions:

- What are the saddle point geometries that correspond to the infinite saddles in (6.9)?
- How does the Stokes phenomenon between region I and II look like in the gravity side?

6.2 Complex saddle points in AdS_3 gravity

We would like to construct geometries that correspond to terms in the expansion (6.9) of partition function for Euclidean AdS_3 gravity. The discussion is very similar to the dS counterpart in §5.1. Since the boundary CFT we calculated in the previous subsection is defined on the sphere S^2 , we consider an ansatz for the metric of complexified Euclidean AdS_3 ,

$$ds^2 = \ell_{\text{AdS}}^2 \left[\left(\frac{d\theta(u)}{du} \right)^2 du^2 + \sinh^2 \theta(u) d\Sigma^2 \right]. \quad (6.14)$$

We set $d\Sigma^2$ as the metric of S^2 without singularities, though it is straightforward to include conical defects corresponding to insertions of heavy operators with $\alpha_i = \eta_i/b$ as long as they satisfy the conditions (6.11), which guarantee for $d\Sigma^2$ to have a positive curvature. As before, we only complexify one direction represented by a holomorphic function $\theta(u)$. We choose the

range of parameter u as $0 \leq u < \infty$. We assign the boundary condition of asymptotic Euclidean AdS_3 , which means that $\theta = u$ for $u \rightarrow \infty$. Further, we assign the condition where the geometry truncates at $u = 0$. We thus require that $\theta = n\pi i$ at $u = 0$. We may glue the two geometries by setting

$$\theta = n\pi i(1 - u) \quad (0 \leq u \leq 1), \quad \theta = (u - 1) \quad (u > 1). \quad (6.15)$$

The geometry is given by the usual Euclidean AdS_3 for $u > 1$. However, the geometry becomes multiply wrapped $\mathbb{S}^3$ with an imaginary radius $i\ell_{\text{AdS}}$ or with three Lorentzian time directions for $0 \leq u \leq 1$. The two geometries may be connected at a point and the resultant geometry should be unphysical for non-zero n .

In case of pure gravity, the partition function is written as

$$\mathcal{Z}_{\text{AdS}}[h] = \int \mathcal{D}g \exp(-I_{\text{EH}} - I_{\text{GH}} - I_{\text{CT}}). \quad (6.16)$$

As in (6.8), we can expand the gravity partition function by the contributions from the saddles labeled by an integer parameter n . The region $0 \leq u \leq 1$ leads to the continuation of n three-spheres with radius $i\ell_{\text{AdS}}$. The phase factor contribution from the saddle with n to the gravity partition function is

$$e^{\frac{n\pi i \ell_{\text{AdS}}}{2G}}. \quad (6.17)$$

The CFT result can be reproduced when the geometry is summed over $n = 0, 1, \dots$ up to an overall factor.

In Chern-Simons formalism, the imaginary pieces again come from the large gauge transformation, which do not make the theory invariant for the Lorentzian AdS case. Therefore the integer n could be regarded as a winding number in some sense.

6.3 Mini-superspace analysis: Spherical slices

We consider the three-dimensional Einstein gravity with negative cosmological constant $\Lambda = -1/\ell_{\text{AdS}}^2$, whose action is given by

$$I = -\frac{1}{16\pi G} \int d^3x \sqrt{g} \left(R + \frac{2}{\ell_{\text{AdS}}^2} \right) + I_{\text{bdy}}. \quad (6.18)$$

We assume the almost same mini-superspace ansatz as in the dS_3 case:

$$ds^2 = \ell_{\text{AdS}}^2 [N(r)^2 dr^2 + a(r)^2 d\Omega_2^2], \quad (6.19)$$

where $d\Omega_2^2$ denotes the metric of the unit two-sphere $\mathbb{S}^2$, and r is the radial coordinate normalized as taking the values in the range $0 \leq r \leq 1$. This again has two boundaries at $r = 0$ and $r = 1$. We will need the counter term at $r = 1$ [141]

$$I_{\text{CT}} = \pm \frac{\ell_{\text{AdS}}}{8\pi G} \int_{r=1} d^2x \sqrt{\gamma} = \pm \frac{\ell_{\text{AdS}}}{2G} a(1)^2 \quad (6.20)$$

in order to compensate the infinite volume divergence. The signature in front is determined by the orientation of the boundary, which is identical to the signature of $\text{Re } N$ for our ansatz.

We are interested in the following gravitational partition function

$$\mathcal{Z} = \int \mathcal{D}g_{\mu\nu} e^{-I - I_{\text{CT}}}, \quad (6.21)$$

which again reduces to

$$\mathcal{Z} = \int_{\mathcal{C}} dN \int \mathcal{D}a e^{-I[a;N]-I_{\text{CT}}} \quad (6.22)$$

after the gauge fixing. The action takes the following form

$$I[a; N] = -\frac{\ell_{\text{AdS}}}{2G} \int_0^1 dr N \left(\frac{1}{N^2} \left(\frac{da}{dr} \right)^2 + a^2 + 1 \right) + (\text{boundary contributions}). \quad (6.23)$$

In the Dirichlet-Dirichlet case, we do not need to add any additional boundary terms to the expression (6.23), therefore we have

$$I[a; N] = -\frac{\ell_{\text{AdS}}}{2G} \int_0^1 dr N \left(\frac{1}{N^2} \left(\frac{da}{dr} \right)^2 + a^2 + 1 \right). \quad (6.24)$$

The equation of motion for $a(r)$ is given by

$$a'' - N^2 a = 0. \quad (6.25)$$

We require that the manifold truncates at $r = 0$. We thus impose the following Dirichlet boundary conditions:

$$a(0) = 0, \quad a(1) = a_1. \quad (6.26)$$

Under these boundary conditions, a solution $\bar{a}(r)$ to the EOM is given by

$$\bar{a}^{(N)}(r) = \frac{a_1}{\sinh N} \sinh(Nr). \quad (6.27)$$

Inserting this solution to the action, we have

$$I[\bar{a}^{(N)}; N] = -\frac{\ell_{\text{AdS}}}{2G} (N + a_1^2 \coth N). \quad (6.28)$$

Next we evaluate the perturbative contribution around the saddle $\bar{a}^{(N)}(r)$. Expanding $a(r)$ as $a(r) = \bar{a}^{(N)}(r) + A(r)$ with boundary conditions $A(0) = A(1) = 0$, we find

$$\begin{aligned} \mathcal{Z} &= \int_{\mathcal{C}} dN e^{-I[\bar{a}^{(N)};N]-I_{\text{CT}}} \int \mathcal{D}A \exp \left[\frac{\ell_{\text{AdS}}}{2GN} \int_0^1 dr A(r) \left(-\frac{d^2}{dr^2} + N^2 \right) A(r) \right] \\ &= \int_{\mathcal{C}} dN e^{-I[\bar{a}^{(N)};N]-I_{\text{CT}}} \text{Det} \left[\frac{\ell_{\text{AdS}}}{2\pi GN} \left(\frac{d^2}{dr^2} - N^2 \right) \right]^{-\frac{1}{2}}. \end{aligned} \quad (6.29)$$

The calculation of the functional determinant is almost same as the positive cosmological constant case. The result is identical to the analytic continuation $\ell_{\text{dS}} \rightarrow i\ell_{\text{AdS}}$, $N \rightarrow iN$ of (5.131), that is,

$$\text{Det} \left[\frac{\ell_{\text{AdS}}}{2\pi GN} \left(\frac{d^2}{dr^2} - N^2 \right) \right] = 2\sqrt{\frac{\ell N}{2\pi G}} \sinh N \quad (6.30)$$

up to the overall sign. Ignoring the irrelevant overall constant, we obtain the expression of the partition function

$$\mathcal{Z} = \int_{\mathcal{C}} dN \left(\frac{1}{\sqrt{N} \sinh N} \right)^{\frac{1}{2}} e^{-I[\bar{a}^{(N)};N]-I_{\text{CT}}}. \quad (6.31)$$

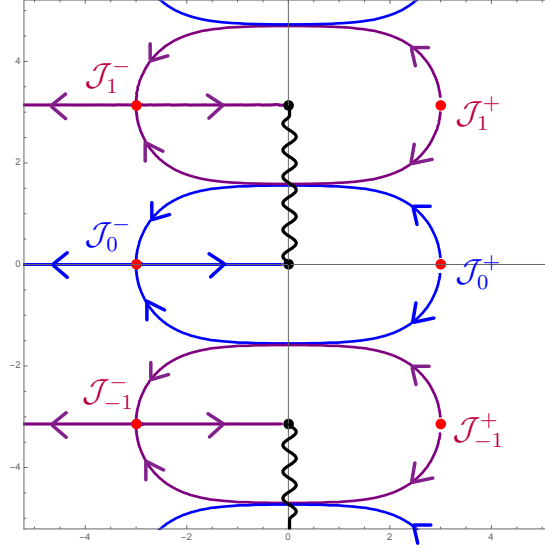


Figure 9: The steepest descent paths for the AdS_3 case with Dirichlet-Dirichlet boundary conditions. The red dots denote the saddle points and the wavy lines correspond to branch cuts.

We can see from the prefactor in the integral that $N = \pi i \mathbb{Z}$ are branch points.

Let us discuss the saddle point structure of our integral. The stationary points satisfying $\partial I[\bar{a}^{(N)}; N]/\partial N = 0$ are listed as

$$N_m^+ = \sinh^{-1} a_1 + \pi i m \quad (m \in \mathbb{Z}), \quad (6.32)$$

$$N_m^- = -\sinh^{-1} a_1 + \pi i m \quad (m \in \mathbb{Z}), \quad (6.33)$$

and the on-shell action for each saddle point is

$$I[\bar{a}^{(N_m^+)}; N_m^+] = -\frac{\ell_{\text{AdS}}}{2G} \left(i m \pi + a_1 \sqrt{a_1^2 + 1} + \sinh^{-1} a_1 \right), \quad (6.34)$$

$$I[\bar{a}^{(N_m^-)}; N_m^-] = -\frac{\ell_{\text{AdS}}}{2G} \left(i m \pi - a_1 \sqrt{a_1^2 + 1} - \sinh^{-1} a_1 \right). \quad (6.35)$$

The configurations of the saddle points and the steepest descent paths are shown in figure 9. In a way similar to the positive cosmological constant case, some steepest descents flow into other saddle points. Therefore the present theory lies on the Stokes line, which makes the integral ill-defined. Furthermore, there is another problem in figure 9: Each plus thimble $\mathcal{J}_m^+$ crosses a branch cut. Since each branch point is surrounded by a thimble $\mathcal{J}_m^+$, the branch cut emanating from the branch point necessarily crosses with $\mathcal{J}_m^+$. Although this does not lead to any ill-definedness of the integral, this would make the analysis highly complicated.

Again, we slightly shift the AdS radius in the imaginary direction as $\ell_{\text{AdS}} \rightarrow \ell_{\text{AdS}} \pm i\epsilon$. The steepest descent paths after the regularization are depicted in figure 10. After the regularization, the theory leaves the Stokes line. Furthermore, we can take the branch cut from each branch point so as not to cross any steepest descent paths. In contrast to the positive cosmological constant case, the signature in front of $i\epsilon$ does not change the final result, as we will see explicitly.

The leading contributions from the saddle points N_m^+, N_m^- are

$$\mathcal{Z}_m^\pm \sim e^{\frac{i m \pi \ell_{\text{AdS}}}{2G}} (2a_1)^{\pm \frac{\ell_{\text{AdS}}}{2G}} \quad (6.36)$$

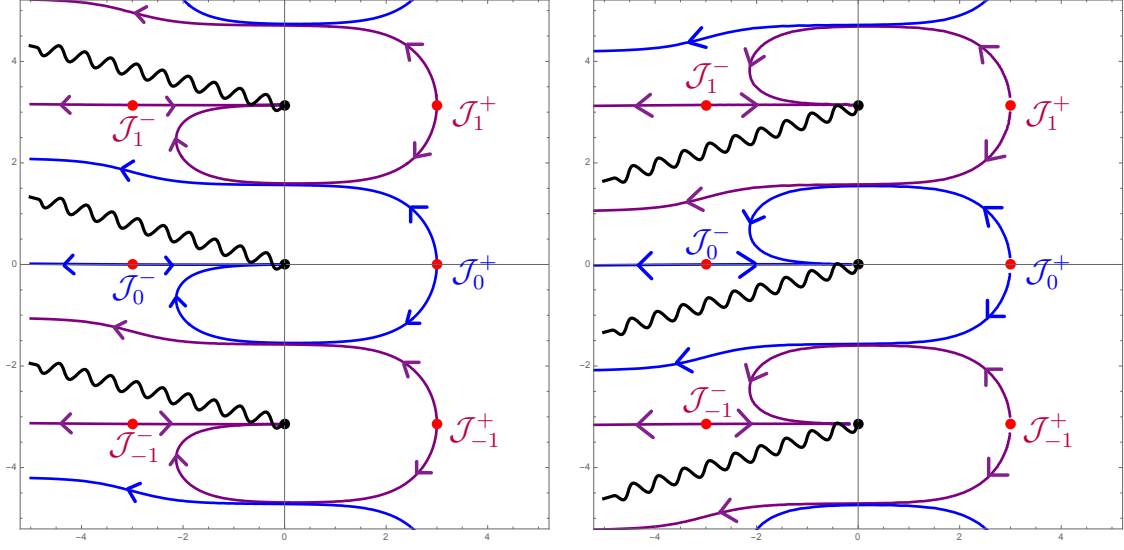


Figure 10: The left panel shows the plot when $\ell_{\text{AdS}} \rightarrow \ell_{\text{AdS}} + i\epsilon$. The right panel shows the plot when $\ell_{\text{AdS}} \rightarrow \ell_{\text{AdS}} - i\epsilon$.

in the large a_1 regime. We can easily see that the contributions from the minus saddles N_m^- are suppressed. A naive contour is the positive real axis $\mathcal{C} = \mathbb{R}_+$. Since the path integral along this contour diverges, we have to deform the contour to be expressed as a summation of some steepest descents. The proper deformation depends on the regularization of ℓ_{AdS} . For $\ell_{\text{AdS}} + i\epsilon$, we deform the contour upwards such that it can be expressed as

$$\mathcal{C} \rightarrow \sum_{m=0}^{\infty} \mathcal{J}_m^+ - \sum_{m=1}^{\infty} \mathcal{J}_m^-, \quad (6.37)$$

along which the semi-classical limit of the partition function is

$$\mathcal{Z} \sim \sum_{m=0}^{\infty} e^{\frac{i\pi m(\ell_{\text{AdS}} + i\epsilon)}{2G}} (2a_1)^{\frac{\ell_{\text{AdS}} + i\epsilon}{2G}} = -\frac{2ie^{-\frac{i\pi\ell_{\text{AdS}}}{4G}}}{\sin\left(\frac{\ell_{\text{AdS}}\pi}{4G}\right)} (2a_1)^{\frac{\ell_{\text{AdS}}}{2G}}. \quad (6.38)$$

Thanks to the $i\epsilon$ regularization, the series is indeed convergent. On the other hand, for $\ell_{\text{AdS}} - i\epsilon$, we deform the contour downwards:

$$\mathcal{C} \rightarrow -\sum_{m=1}^{\infty} \mathcal{J}_{-m}^+ + \sum_{m=0}^{\infty} \mathcal{J}_{-m}^-. \quad (6.39)$$

Along this contour, the partition function approximates

$$\mathcal{Z} \sim -\sum_{m=-\infty}^0 e^{\frac{i\pi m(\ell_{\text{AdS}} - i\epsilon)}{2G}} (2a_1)^{\frac{\ell_{\text{AdS}} - i\epsilon}{2G}} = -\frac{2ie^{\frac{i\pi\ell_{\text{AdS}}}{4G}}}{\sin\left(\frac{\ell_{\text{AdS}}\pi}{4G}\right)} (2a_1)^{\frac{\ell_{\text{AdS}}}{2G}}. \quad (6.40)$$

Up to the overall phase factor, the leading contribution is identical to (6.38). Therefore the different regularizations $\ell_{\text{AdS}} \pm i\epsilon$ pick up different saddles, but finally lead to the almost same result. Furthermore our results are consistent with the CFT result (6.8). We then conclude that the naive contour $\mathcal{C} = \mathbb{R}_+$ is the appropriate one consistent in our perspective via the AdS/CFT. Another possible choice of contour is the whole real axis $\mathcal{C} = \mathbb{R}$, with avoiding the singularity

at the origin by slightly deforming the contour to the lower direction. The only difference from the previous case is just a saddle $\mathcal{J}_0^-$, which does not affect the large a_1 approximation (6.38).

We can assign the Neumann boundary condition at $r = 0$. The analysis is similar to the dS₃ case, and the CFT results are not expected to be reproduced. Therefore, we do not analyze this case in details. We can also introduce a new radial coordinate [130]:

$$R(r) = -n\pi i(1-r)^q + \ln(2a_1)r^q, \quad (6.41)$$

which interpolates between the complex geometry realized by a saddle and the one discussed in §6.2. Namely, the radial coordinate is the one used for a saddle point for $q = 1$ and approaches to the one discussed below (6.14) for $q \rightarrow \infty$.

6.4 Mini-superspace analysis: Hyperbolic slices

So far we have discussed the dual geometries to the sphere partition function without any operator insertions. Similarly to the dS/CFT case, it is straightforward to extend the analysis to the correlation functions with heavy operator insertions as long as the Liouville momenta $\alpha_i = \eta_i/b$ satisfy $\sum_i \eta_i < 1$. The next question we ask in this subsection is the following: What happens when we leave this region? We still assume the Seiberg bound $0 < \eta_i < 1/2$, where the operator insertion corresponds to a conical deficit, then another region of interest will be $\sum_i \eta_i > 1$. This is nothing but the region I (6.10). In the case with three operators, the three-point functions behave as (6.12) without any phase factors which appear in Region II. The transition of the non-perturbative contributions is interpreted as the Stokes phenomenon. In this subsection, we aim to understand this phenomenon in the gravity side. This accounts to considering the metric with hyperbolic slices because the Riemann surface with operator insertions with $\sum_i \eta_i > 1$ has a negative curvature.

6.4.1 Half wormhole ansatz

We consider the minisuperspace ansatz with hyperbolic slices

$$ds^2 = \ell_{\text{AdS}}^2 [N^2 dr^2 + a(r)^2 d\Sigma^2], \quad (6.42)$$

with $d\Sigma^2$ being the metric for a compact Riemann surface with a negative curvature ${}^{(2)}R = -2$. First, we consider geometries with a single boundary, normalizing r as $0 < r < 1$. When $g = 0$, such a geometry requires the existence of enough conical defects in order to make the curvature negative. Precisely speaking, the deficit angles $\delta_i \equiv 4\pi\eta_i$ have to satisfy

$$\sum_i \eta_i > 1. \quad (6.43)$$

Henceforth we consider this type of Riemann surface, though it is straightforward to extend to a general Riemann surface $\Sigma_{g,n}$ with negative curvature.

In this ansatz, the bulk Einstein-Hilbert action can be written as

$$\begin{aligned} I_{\text{EH}} &= -\frac{\ell_{\text{AdS}}}{16\pi G} \int d^3x \sqrt{g} (R + 2) \\ &= -\frac{\ell_{\text{AdS}}}{8\pi G} \int_0^1 dr d\Sigma \sqrt{h} \frac{1}{N} (a'^2 + N^2 a^2 + N^2 {}^{(2)}R) \\ &= -\frac{\ell_{\text{AdS}} V_\Sigma}{8\pi G N} \int_0^1 dr (a'^2 + N^2 a^2) - \frac{\ell_{\text{AdS}} N \chi}{2G}. \end{aligned} \quad (6.44)$$

In the current case the volume V_Σ and the Euler characteristic χ for the surface Σ are

$$V_\Sigma = 4\pi \left(\sum_i \eta_i - 1 \right), \quad (6.45)$$

$$\chi = \frac{1}{4\pi} \int d\Sigma \sqrt{h} {}^{(2)}R = 2 - 2 \sum_i \eta_i. \quad (6.46)$$

The counterterm is given by

$$I_{\text{CT}} = \pm \frac{\ell_{\text{AdS}} a_1^2}{8\pi G} \int d\Sigma \sqrt{h} = \pm \frac{\ell_{\text{AdS}} \delta a_1^2}{2G}, \quad (6.47)$$

where we have defined $\delta \equiv \sum_i \eta_i - 1$ (> 0). The equation of motion for a is

$$a'' - N^2 a = 0. \quad (6.48)$$

We impose the same boundary conditions $a(0) = 0, a(1) = a_1$ as we considered for the spherical slices. The solution to the equation of motion is

$$a^{(N)}(r) = \frac{a_1}{\sinh N} \sinh(Nr), \quad (6.49)$$

on which the action takes

$$I[a^{(N)}; N] = -\frac{\ell_{\text{AdS}} \delta a_1^2}{2G} \coth N + \frac{\ell_{\text{AdS}} \delta}{G} N. \quad (6.50)$$

Path integration over a leads to

$$\mathcal{Z} = \int_{\mathcal{C}} dN \left(\frac{1}{N \sinh N} \right)^{\frac{1}{2}} e^{\frac{\ell_{\text{AdS}} \delta}{2G} (a_1^2 \coth N - 2N) - I_{\text{CT}}}. \quad (6.51)$$

Note that the minus sign in front of the term linear in N comes from the fact that the Euler characteristic (6.46) is negative, which makes the integral convergent at along the positive real axis of N . This is an important difference from the case with spherical slices.

The solutions to the saddle point equation are

$$N_n^\pm = \pm \sinh^{-1} \left(\frac{a_1}{\sqrt{2}} \right) + \left(n + \frac{1}{2} \right) \pi i. \quad (6.52)$$

The values of the on-shell action on the saddles are

$$\begin{aligned} I[\bar{a}^{(N_n^\pm)}] &= \frac{i(2n+1)\pi \ell_{\text{AdS}} \delta}{G} \mp \frac{\ell_{\text{AdS}} \delta}{G} \left[\frac{a_1}{\sqrt{2}} \sqrt{\frac{a_1^2}{2} - 1} - \log \left(\frac{a_1}{\sqrt{2}} + \sqrt{\frac{a_1^2}{2} - 1} \right) \right] \\ &\sim \frac{i(2n+1)\pi \ell_{\text{AdS}} \delta}{G} \mp \frac{\ell_{\text{AdS}} \delta}{G} \left[\frac{a_1^2}{2} - \log(\sqrt{2} a_1) \right] \end{aligned}$$

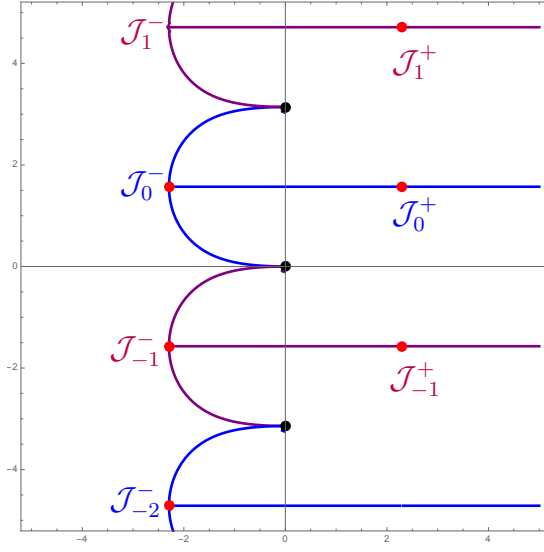


Figure 11: The saddle points and the steepest descents are shown for the half wormhole ansatz with hyperbolic slices.

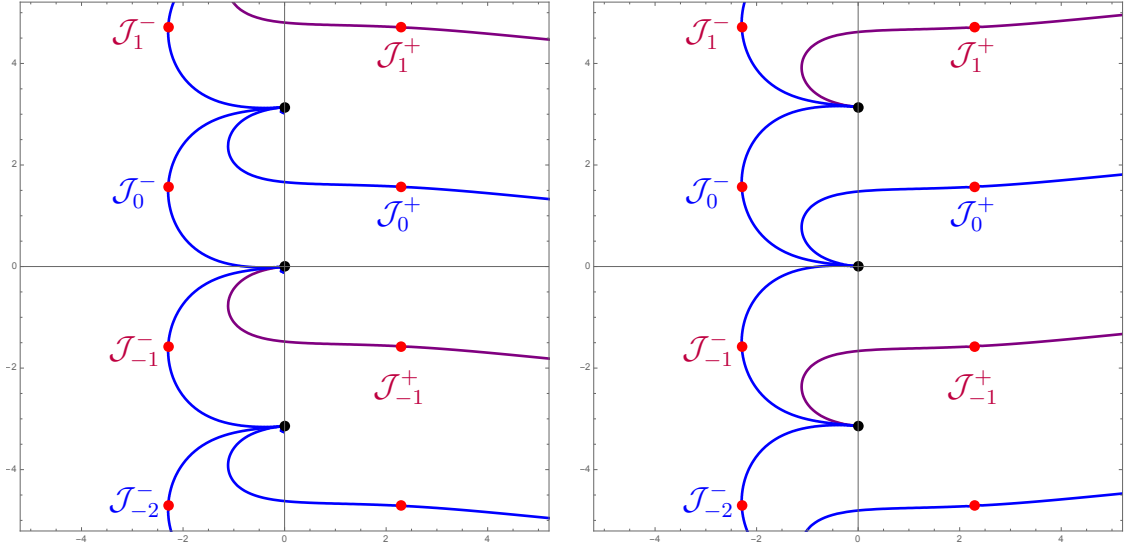


Figure 12: Left panel: $\ell_{\text{AdS}} \rightarrow \ell_{\text{AdS}} + i\epsilon$. Right panel: $\ell_{\text{AdS}} \rightarrow \ell_{\text{AdS}} - i\epsilon$.

The saddle points and the steepest descents are shown in figure 11. The theory is again lying on the Stokes line since some steepest descents end on other saddle points, and the integration along the real axis $\mathcal{C} = \mathbb{R}_+$ does not converge. Thus we need some regularization such as $\ell_{\text{AdS}} \rightarrow \ell_{\text{AdS}} \pm i\epsilon$. The regularized versions are shown in figure 12. Let us assume that the original contour is given as the positive real axis $\mathcal{C} = \mathbb{R}_+$. For the plus regularization $\ell_{\text{AdS}} \rightarrow \ell_{\text{AdS}} + i\epsilon$, two saddle points $\mathcal{J}_0^+$ and $\mathcal{J}_0^-$ contribute. On the other hand, for the other regularization $\ell_{\text{AdS}} \rightarrow \ell_{\text{AdS}} - i\epsilon$, other two $\mathcal{J}_{-1}^+$ and $\mathcal{J}_{-1}^-$ become relevant. The saddle point approximation then takes the form

$$\mathcal{Z} \sim e^{\mp i \frac{\pi \ell_{\text{AdS}} \delta}{G}} \left(a_1^{\frac{\ell_{\text{AdS}} \delta}{G}} + a_1^{-\frac{\ell_{\text{AdS}} \delta}{G}} \right), \quad (6.53)$$

for the two types of regularization $\ell_{\text{AdS}} \rightarrow \ell_{\text{AdS}} \pm i\epsilon$, respectively. Therefore one can find that the result does not depend on the regularization up to the overall phase factor. A remarkable point compared with the case with spherical slices is that the dominant saddle comes from the $\text{Re } N < 0$ region no matter how we regularize.

In either regularization, the partition function (6.53) only has one saddle point contribution in the leading order of a_1 . This is consistent with the CFT result. Recalling that in the previous section the same contour $\mathcal{C} = \mathbb{R}_+$ gave the consistent answer with the CFT side when $\sum_i \eta_i < 1$, the difference from the case $\sum_i \eta_i < 1$ can be interpreted as the Stokes phenomenon. Therefore, the gravitational path integral with the contour $\mathcal{C} = \mathbb{R}_+$ shows the desired behaviors, including the Stokes phenomenon, expected from CFT results. Thus we have been able to check that the half real axis $\mathcal{C} = \mathbb{R}_+$ gives the correct integration contour for the gravitational path integral corresponding to the Liouville correlation functions.

6.4.2 Wormhole ansatz

In the half wormhole ansatz, the relevant saddle point geometry is a complex geometry. To get the real metric with hyperbolic slices, we need to have two disjoint boundaries. Therefore, we next consider the wormhole ansatz by normalizing the r -coordinate in (6.42) as $-1 < r < 1$ and imposing the boundary conditions $a(1) = a(-1) = a_1$. Such a wormhole geometry is similar to what is considered in [140], and recently proposed to be the dual geometry to a correlation function of the averaged holographic CFT [18]. In that paper, the OPE coefficients of the averaged CFT is given by the square of the DOZZ formula

$$|\overline{c_{ijk}}|^2 \sim |C_{\text{DOZZ}}(P_i, P_j, P_k)|^2, \quad (6.54)$$

so we would be able to apply the discussions of Liouville field theory to this theory. The purpose of this section is to discuss the relation of our method and the averaged CFT. Here we show some calculations in which wormhole geometries seem to be relevant, although the discussions in this section are not yet complete currently.

The only difference on integrating a from the half wormhole case is the boundary condition on a . The equation of motion for a is

$$a'' - N^2 a = 0, \quad (6.55)$$

which is solved by

$$a^{(N)}(r) = \frac{a_1}{\cosh N} \cosh(Nr), \quad (6.56)$$

under the boundary conditions $a(1) = a(-1) = a_1$. The on-shell action on the solution $a^{(N)}$ becomes

$$I[a^{(N)}; N] = -\frac{\ell_{\text{AdS}} \delta a_1^2}{G} \tanh N + \frac{2\ell_{\text{AdS}} \delta}{G} N, \quad (6.57)$$

where we have defined $\delta \equiv \sum_i \eta_i - 1$ (> 0). Taking the one-loop correction into account, the integral we would like to analyse is

$$\mathcal{Z} = \int_{\mathcal{C}} dN \left(\frac{1}{\cosh N} \right)^{\frac{1}{2}} e^{\frac{\ell_{\text{AdS}} \delta}{G} (a_1^2 \tanh N - 2N) - I_{\text{CT}}}. \quad (6.58)$$

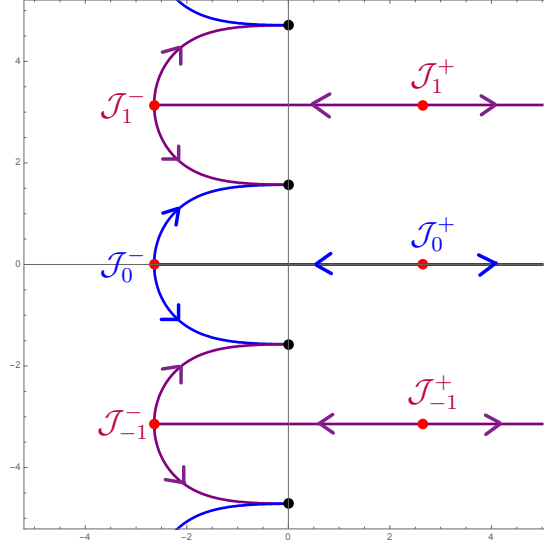


Figure 13: Saddle points (red dots) and Steepest descents for AdS_3 for the wormhole ansatz.

The saddle points in the N -plane are at

$$N_m^\pm = \pm \cosh^{-1} \left(\frac{a_1}{\sqrt{2}} \right) + \pi i m. \quad (6.59)$$

For each saddle point, the on-shell action takes the form

$$I[\bar{a}^{(N_m^+)}; N] = \frac{\ell_{\text{AdS}} \delta}{G} \left(2\pi i m + 2 \cosh^{-1} \left(\frac{a_1}{\sqrt{2}} \right) - a_1 \sqrt{a_1^2 - 2} \right), \quad (6.60)$$

$$I[\bar{a}^{(N_m^-)}; N] = \frac{\ell_{\text{AdS}} \delta}{G} \left(2\pi i m - 2 \cosh^{-1} \left(\frac{a_1}{\sqrt{2}} \right) + a_1 \sqrt{a_1^2 - 2} \right). \quad (6.61)$$

The configurations of saddle points and steepest descents are depicted in figure 13. Since $\mathcal{J}_m^-$ saddles can be ignored in the large a_1 approximation, the result does not depend on which regularization is chosen.

Let us discuss the integration contour. Similarly to the half wormhole case, $\mathcal{J}_m^-$ saddles are dominant in the large a_1 approximation. A difference is that there is no singularity at $N = 0$, so the contour cannot start with the origin. A natural contour is then the whole real axis $\mathcal{C} = \mathbb{R}$. In this case, the contour deformed into upper or lower direction depending on which regularization $\ell_{\text{AdS}} \rightarrow \ell_{\text{AdS}} \pm i\epsilon$ is chosen, although the final results will be the same in the similar way to the case of spherical ansatz discussed in §6.3. Therefore, with this contour $\mathcal{C} = \mathbb{R}$, the infinitely many saddle points contribute to the semiclassical approximation as

$$\mathcal{Z} \sim \sum_{m=0}^{\infty} e^{i \frac{2\pi m \ell_{\text{AdS}} \delta}{G} a_1 \frac{2\ell_{\text{AdS}} \delta}{G}}. \quad (6.62)$$

One observation from this form is that the real factor of each factor is indeed the square of the dominant term in (6.53). This is consistent with the expectation from the fact that the current ansatz has two boundaries at $r = -1, 1$, on each of which a CFT is defined. The most important question is about the presence of the infinite terms. It would be natural to think that the wormhole contribution cannot be simply written as the square of a single term (6.53). In

this sense, the summation in (6.62) might be interpreted as inserting a completeness relation like $\sum_m |m\rangle \langle m| = \mathbf{1}$. Each saddle associated with m can be interpreted as a complex metric in the same way as the spherical slices discussed in §6.3. Of course, the discussion here is too naive to be justified because there is no precise relation between the half wormhole ansatz and the wormhole ansatz with different boundary conditions on a . Nevertheless, it will be interesting if we can interpret the information about m in the dual CFT perspective. We leave this as a future problem.

7 Conclusion and Discussion

In this thesis, we have reviewed the recent studies on the gravitational saddle points from the holographic point of view. After summarizing the basics and the model of the AdS/CFT and dS/CFT we are focusing on, we discussed the saddle points and the integration contour for both three-dimensional dS and AdS gravity theories.

We started with dS₃ gravity in §5. As a concrete model, we have considered the analytic continuation $c \rightarrow -ic^{(g)}$ from a concrete model of the AdS/CFT, where the dual CFT is described by the Liouville field theory. We found that the correlation functions have two non-perturbative contributions from the complex saddle points, whose leading contributions are consistent with those of the Hartle-Hawking saddle and the tunneling saddle. In particular, the most dominant saddle gives the correct de Sitter entropy. After that, we have moved on to the gravity side. We have utilized the mini-superspace method to analyze the saddle point approximation of the gravitational partition function. The appropriate contour which would pick up the desired saddle points are determined. In this procedure, we often encounter the subtlety that the theory under consideration lies at the “critical point,” where the Stokes phenomenon just occurs, which means that the usual saddle point approximation cannot be applied to the theory. To treat properly, we have to regularize the theory by slightly shift a parameter to leave the critical point, which in our situation corresponds to regularizing the dS radius $\ell_{\text{dS}} \rightarrow \ell_{\text{dS}} \pm i\epsilon$. A difficulty is that the result may depend on which regularization we choose. In the literatures, people have not taken care of this problem so seriously because we cannot determine the correct regularization just from the gravity discussions. In this thesis, we have also been able to determine the correct regularization from the holographic discussion, so that the dual central charge takes the desired form.

In §6, we repeated the same procedure for the AdS₃ gravity. In the Liouville calculations, we can observe that the Stokes phenomenon occurs under the analytic continuation from the positive real central charge to the imaginary central charge. From this observation, it would be natural to expect that the dual gravity theory includes quite different saddle point geometries in the semi-classical approximation. We have again used the mini-superspace method with the metric ansatz with spherical slices and found that an integration contour $\mathcal{C} = \mathbb{R}_+$ gives the consistent result with the CFT calculation. This contour would be natural in the sense that it goes through the real hyperbolic metrics. For AdS₃, we have additional motivations to consider richer geometries with various topology; higher genus, wormholes, and so on. In this thesis, we also considered the higher genus slices where each slice has a negative curvature, which would be related with wormhole geometries. By analyzing the mini-superspace model in the metric ansatz with hyperbolic slices, we found that the same contour $\mathcal{C} = \mathbb{R}_+$ shows quite different saddle point structures. This implies that a Stokes phenomenon occurs when we deform the parameter of conical defects to be large enough to make the curvature of slices negative from positive. Remarkably, this Stokes phenomenon aligns with that observed in the CFT side when the conformal dimensions of operators are made sufficiently large. This observation would provide

an evidence convincing us of the choice $\mathcal{C} = \mathbb{R}_+$ of the integration contour. However, it remains unclear whether the mini-superspace analysis can be directly related to wormhole geometries. While we examined the wormhole ansatz and confirmed that its leading contribution aligns with predictions, a direct connection is difficult to establish because the boundary conditions imposed on the metric differ between the two ansatz. Understanding the connection between the different ansatz and the recent studies on the averaged CFT [18] is an interesting future work.

In this thesis, we focused on the ansatz with spherical slices and hyperbolic slices, the curvatures of which are positive and negative, respectively. As explained above, the transition between them is interpreted as the Stokes phenomenon. Of course, we are also interested in the torus slices, which have zero curvature. There are many holographic works on this setup, e.g. [29]. In our approach, this case exactly lies on the critical point where the Stokes phenomenon occurs, which makes the analysis complicated in the same reason with the necessity of regularization in figure 6 or figure 9. In the torus case, there would not be any criterion to decide how to regularize it. A precise way to regularize such a case is given by the resurgence technique. Applying the resurgence to this problem would be interesting.

Recently, another condition for the relevant saddle point geometries has been proposed by Witten [9] based on [10]. This so-called “allowable” condition comes from the well-definedness of the path integral of every p -form fields ($p = 0, 1, \dots, D$) on the manifold. As also commented in [9], this condition has not yet been proved necessary nor sufficient for whether the geometry contributes as a saddle point or not. Nevertheless, the condition has been checked to be consistent with some known examples, so it would be worth for making comments on the relation of our results with the condition. For dS case, we found two geometries to be relevant as saddle points. Only these two saddles are actually allowable also in the sense of Witten. This is a surprising result coming from two completely different discussions of allowability. However, in AdS case, our results are actually not consistent with the allowable condition. Among the contributing saddle points, only the dominant one is allowable in the sense of Witten. Does it mean that the allowable condition is not correct in our setup? If so, is there any alternative condition for the allowability? Answering these questions are also important future works.

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A Notations of Lie algebras

In this thesis, we use the same letter J_a ($a = 0, 1, 2$) for both $SU(2)$ generators and $SL(2, \mathbb{R})$ generators used in (2.23) and (2.25) for Chern-Simons formulation of three-dimensional gravity.

$SU(2)$ generators

For the fundamental representation of $\mathfrak{su}(2)$, we use the following notations

$$J_0 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (\text{A.1})$$

These are related to the Pauli matrices as $J_0 = \frac{i}{2}\sigma_z$, $J_1 = \frac{i}{2}\sigma_x$, $J_2 = -\frac{i}{2}\sigma_y$. J_a satisfy the commutation relations

$$[J_a, J_b] = \epsilon_{abc} J_c, \quad (\text{A.2})$$

and the trace formulas are

$$\text{tr}_f[J_a J_b] = -\frac{1}{2}\delta_{ab}, \quad \text{tr}_f[J_a J_b J_c] = -\frac{1}{4}\epsilon_{abc}. \quad (\text{A.3})$$

$SL(2, \mathbb{R})$ generators

For $\mathfrak{sl}(2, \mathbb{R})$, there are two conventions for the generators. First let us define the generators L_a ($a = 0, \pm 1$) as

$$L_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (\text{A.4})$$

which satisfy

$$[L_0, L_{\pm 1}] = \pm L_{\pm 1}, \quad [L_1, L_{-1}] = 2L_0. \quad (\text{A.5})$$

The trace formulas for products in the fundamental representation are given by

$$\text{tr}_f[L_0 L_0] = \frac{1}{2}, \quad \text{tr}_f[L_1 L_{-1}] = -1 \quad (\text{A.6})$$

and otherwise 0.

In order to make it clear the relation with $SU(2)$ generators (A.1), we define another generators J_a so as to satisfy

$$L_0 = J_0, \quad L_1 = J_1 + J_2, \quad L_{-1} = -J_1 + J_2. \quad (\text{A.7})$$

The fundamental representations for new generators are then

$$J_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_1 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\text{A.8})$$

which satisfy

$$[J_0, J_1] = -J_2, \quad [J_1, J_2] = J_0, \quad [J_2, J_0] = J_1. \quad (\text{A.9})$$

By preparing a Minkowski metric $\eta := \text{diag}(1, 1, -1)$ and raising the $\mathfrak{sl}(2, \mathbb{R})$ index as $J^a := \eta^{ab} J_b$, the commutation relations can be written as

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad (\text{A.10})$$

and trace formulas are also summarized as

$$\text{tr}_f[J_a J_b] = \frac{1}{2}\eta_{ab}. \quad (\text{A.11})$$

Note that the “time” direction is identified with J_2 , not J_0 as usual. We can relate this commutation relations with those (A.2) of $\mathfrak{su}(2)$ by replacing

$$J_0 \rightarrow iJ_0, \quad J_1 \rightarrow -iJ_1, \quad J_2 \rightarrow -J_2. \quad (\text{A.12})$$

B Υ_b function

Here we summarize some properties of the Upsilon function $\Upsilon_b(x)$, especially on the asymptotic behavior in the semi-classical limit. The basic properties of the Upsilon function are summarized for real b in, e.g. appendix A of [36]. Since our interests include an imaginary central charge, we extend to the case with complex b . See also [62] for more detailed discussions on analytic continuation of the Upsilon function.

The Upsilon function $\Upsilon_b(x)$ is defined as a unique solution to the recursion relations

$$\begin{aligned}\Upsilon_b(x+b) &= \gamma(bx)b^{1-2bx}\Upsilon_b(x), \\ \Upsilon_b\left(x+\frac{1}{b}\right) &= \gamma\left(\frac{x}{b}\right)b^{\frac{2x}{b}-1}\Upsilon_b(x),\end{aligned}\tag{B.1}$$

where $\gamma(x)$ is defined as $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. The solution to (B.1) is unique for $b \in \mathbb{R}$, while it is not unique otherwise. For $b \notin \mathbb{R}$, we promise to always consider the analytic continuation from the solution for $b \in \mathbb{R}$. When $\text{Re } b > 0$, the solution has an integral form of the Υ_b function:

$$\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x\right)^2 e^{-t} - \frac{\sinh^2\left(\left(\frac{Q}{2} - x\right)\frac{t}{2}\right)}{\sinh \frac{tb}{2} \sinh \frac{t}{2b}} \right], \quad 0 < \text{Re } x < \text{Re } Q, \tag{B.2}$$

where $Q = b + 1/b$. By using the defining recursion relations (B.1), we can extend the range of x to the whole complex plane.

For applications to CFT, it is useful to introduce the central charge $c = 1 + 6Q^2 \in \mathbb{C}$. Note that we can express it as $c = 13 + 6(b^2 + b^{-2})$. For any $c \notin (-\infty, 1]$, we can choose the branch of b -plane such that it satisfies $\text{Re } b > 0$. Therefore we can appropriately define $\Upsilon_b(x)$ for any CFT whose the central charge is in the region $c \notin (-\infty, 1]$ in $\mathbb{C}$. On the other hand, for $c \in (-\infty, 1]$, b should be purely imaginary. Therefore the integral form (B.2) cannot be defined. In order to consider CFTs that have $c \in (-\infty, 1]$, we have to use another function $\hat{\Upsilon}_b(x)$ defined by the modified recursion relations [109]

$$\begin{aligned}\hat{\Upsilon}_b(x+b) &= \gamma(bx)(ib)^{1-2bx}\hat{\Upsilon}_b(x), \\ \hat{\Upsilon}_b\left(x+\frac{1}{b}\right) &= \gamma\left(\frac{x}{b}\right)(ib)^{\frac{2x}{b}-1}\hat{\Upsilon}_b(x).\end{aligned}\tag{B.3}$$

The function $\hat{\Upsilon}_b(x)$ can be defined for $\text{Im } b < 0$, equally $c \notin [25, \infty)$. Therefore we can use either $\Upsilon_b(x)$ or $\hat{\Upsilon}_b(x)$ for CFTs with $c \notin (-\infty, 1] \cup [25, \infty)$. Because the theory of our interest has real or purely imaginary c , henceforth we will discuss only the unhatted Upsilon function $\Upsilon_b(x)$ by promising we always take a branch $\text{Re } b > 0$.

Let us discuss the semi-classical limits $b \rightarrow 0$ of $\Upsilon_b(x)$ assuming that x scales as $x = \eta/b$. For simplicity, we restrict ourselves to real η ,¹⁶ which is the situation we are interested in. We first consider $x = \frac{\eta}{b} + \frac{b}{2}$, keeping η fixed under $b \rightarrow 0$. When $0 < \eta < 1$, we find

$$\begin{aligned}b^2 \log \Upsilon_b\left(\frac{\eta}{b} + \frac{b}{2}\right) &= -\left(\eta - \frac{1}{2}\right)^2 \log b \\ &\quad + \int_0^\infty \frac{dt}{t} \left[\left(\eta - \frac{1}{2}\right)^2 e^{-t} - \frac{2 \sinh^2\left((\eta - 1/2)\frac{t}{2}\right)}{\sinh \frac{t}{2}} \right] + \mathcal{O}(b^4).\end{aligned}\tag{B.4}$$

¹⁶When both b and η are imaginary, the condition $0 < \text{Re } \eta < 1$ does not imply $0 < \text{Re } (\eta/b) < \text{Re } Q$. We then would need some additional condition for $\text{Im } \eta$.

If we define:

$$F(\eta) \equiv \int_0^\infty \frac{dt}{t} \left[\left(\eta - \frac{1}{2} \right)^2 e^{-t} - \frac{2 \sinh^2 \left((\eta - 1/2) \frac{t}{2} \right)}{t \sinh \frac{t}{2}} \right] \quad (\text{B.5})$$

and use the integral representation (B.2) together with an identity

$$\log x = \int_0^\infty \frac{dt}{t} (e^{-t} - e^{-xt}) , \quad \text{Re } x > 0. \quad (\text{B.6})$$

We find the asymptotic formula for Υ_b :

$$\Upsilon_b \left(\frac{\eta}{b} \right) = \exp \left(\frac{1}{b^2} [-(\eta - 1/2)^2 \log b + F(\eta) + \mathcal{O}(b \log b)] \right). \quad (\text{B.7})$$

Here we again emphasize that this formula is applicable for any b with $\text{Re } b > 0$, equally $c \notin (-\infty, 1]$, but only for $0 < \eta < 1$. We can obtain the asymptotic formula for other ranges of η by applying the recursion relations (B.1). For example, let us consider $-1 < \eta < 0$, and use inversely the second equation in (B.1),

$$\Upsilon_b \left(\frac{\eta}{b} + \frac{b}{2} \right) = \frac{b^{1-\frac{2\eta}{b^2}}}{\gamma \left(\frac{\eta}{b^2} \right)} \Upsilon_b \left(\frac{\eta+1}{b} + \frac{b}{2} \right), \quad (\text{B.8})$$

we obtain:

$$\Upsilon_b \left(\frac{\eta}{b} \right) = \frac{b^{1-\frac{2\eta}{b^2}}}{\gamma \left(\frac{\eta}{b^2} \right)} \exp \left(\frac{1}{b^2} [-(\eta+1/2)^2 \log b + F(\eta+1) + \mathcal{O}(b \log b)] \right). \quad (\text{B.9})$$

C Revisiting the series expansion of the phase factor

Here we consider the CFT dual of AdS geometries with conical deficits, i.e. $b, \eta_i \in \mathbb{R}$. Following [36], the semi-classical approximation of the two-point function with $\alpha = b/2 + \eta/b$ is given by

$$\begin{aligned} \langle V_\alpha(z_1) V_\alpha(z_2) \rangle &\sim \frac{1}{e^{i\pi(1-2\eta)/b^2} - e^{-i\pi(1-2\eta)/b^2}} \\ &\times e^{-\frac{2}{b^2} [(1-2\eta) \log(1-2\eta) - (1-2\eta)]} \lambda^{(1-2\eta)/b^2} |z_{12}|^{-4\eta(1-\eta)/b^2} \delta(0). \end{aligned} \quad (\text{C.1})$$

Let us carefully discuss the expansion of the phase factor. If η were not real-valued, the expansion can be easily obtained as

$$\frac{1}{e^{i\pi(1-2\eta)/b^2} - e^{-i\pi(1-2\eta)/b^2}} = \pm \sum_{n=0}^{\infty} e^{\mp 2\pi i(n+1/2)(1-2\eta)/b^2}, \quad (\text{C.2})$$

where the two signatures take the upper one if $\text{Im } \eta > 0$ and the lower one if $\text{Im } \eta < 0$ since the radius of convergence of the series expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (\text{C.3})$$

is 1. However, if η has no imaginary part, we cannot expand as (C.2) in the usual sense. Nevertheless, we may be able to define the expansion of the phase factor by using the notion of hyperfunction.

To define a hyperfunction $f(x)$ for a real variable x , we prepare a holomorphic function $F(z)$ defined on the complex plane except for the real axis and define $f(x)$ as

$$f(x) := F(x + i0) - F(x - i0) . \quad (\text{C.4})$$

For example, the defining function F for the delta function $\delta(a)$ is given by

$$F(z) = -\frac{1}{2\pi i} \frac{1}{z - a} , \quad z \in \mathbb{C} \setminus \mathbb{R} . \quad (\text{C.5})$$

For our case, the appropriate $F(\eta)$ for $\eta \in \mathbb{C} \setminus \mathbb{R}$ may be

$$F(\eta) = \begin{cases} \sum_{n=0}^{\infty} e^{-2\pi i(n+1/2)(1-2\eta)/b^2} , & \text{Im } \eta > 0 , \\ \sum_{n=0}^{\infty} e^{2\pi i(n+1/2)(1-2\eta)/b^2} , & \text{Im } \eta < 0 . \end{cases} \quad (\text{C.6})$$

We define the series that can be thought of as the series expansion of the phase factor by using this $F(\eta)$. Therefore the precise expression of the expansion could be

$$\frac{1}{e^{i\pi(1-2\eta)/b^2} - e^{-i\pi(1-2\eta)/b^2}} = F(\eta + i0) - F(\eta - i0) \quad (\text{C.7})$$

for real η . Roughly speaking, one may understand this result as follows: The two-point function in the current case has the contributions from infinite saddles (i.e. labeled by integers $\mathbb{Z}$), not semi-infinite saddles. Naively treating the hyperfunction, we write the series expansion as

$$\frac{1}{e^{i\pi(1-2\eta)/b^2} - e^{-i\pi(1-2\eta)/b^2}} = 2i \sum_{n=0}^{\infty} \sin \left[\pi (2n+1) (1-2\eta) \frac{1}{b^2} \right] , \quad (\text{C.8})$$

by pairing the terms with same n in $F(\eta + i0)$ and $F(\eta - i0)$, although of course the right-hand side is literally divergent. A justification to use this naive notation is given by computing the sum inversely with adopting a simple regularization

$$\begin{aligned} \sum_{n=0}^{\infty} \sin [(2n+1)x] &\rightarrow \lim_{\epsilon \rightarrow 0} \sum_{n=0}^{\infty} e^{-\epsilon(2n+1)} \sin [(2n+1)x] \\ &= \lim_{\epsilon \rightarrow 0} \frac{\sin x + \mathcal{O}(\epsilon^2)}{2 \sin^2 x + \mathcal{O}(\epsilon^2)} \\ &= \frac{1}{2 \sin x} . \end{aligned} \quad (\text{C.9})$$

Finally we obtain the two-point function

$$\begin{aligned} \langle V_\alpha(z_1) V_\alpha(z_2) \rangle &\sim \sum_{n=0}^{\infty} \sin \left[\pi (2n+1) (1-2\eta) \frac{1}{b^2} \right] \\ &\times e^{-\frac{2}{b^2} [(1-2\eta) \log(1-2\eta) - (1-2\eta)]} \lambda^{(1-2\eta)/b^2} |z_{12}|^{-4\eta(1-\eta)/b^2} \delta(0) . \end{aligned} \quad (\text{C.10})$$

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