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The action of isotropy subgroups of the modular groups on infinite dimensional Teichmüller spaces

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For a compact Riemann surface R of genus greater than one, it is well known that the Teichmüller modular group (or mapping class group) $\mathrm{Mod}(R)$ acts on the finite dimensional Teichmüller space $T(R)$ isometrically and properly discontinuously. In more details, although $\mathrm{Mod}(R)$ has fixed points on $T(R)$, the isotropy subgroup $\mathrm{Stab}(p)$ at any $p \in R$ is a finite group. However, this is not always the case for non-compact Riemann surfaces such as R of infinite genus or of the infinite number of punctures, for which the Teichmüller space $T(R)$ is infinite dimensional. In this case, the orbit of a point in $T(R)$ under $\mathrm{Mod}(R)$ may be non-discrete and the isotropy subgroup $\mathrm{Stab}(p)$ may be infinite. In this note, we consider the action of isotropy subgroups more closely. Teichmüller spaces are always assumed to be infinite dimensional hereafter.

Let R be a Riemann surface and $\mathrm{Aut}(R)$ the group of all conformal automorphisms of R. The isotropy subgroup at the origin of the Teichmüller space $T(R)$ is identified with $\mathrm{Aut}(R)$. Let $B(R)$ be the complex Banach space of the holomorphic quadratic differentials φ on R with the hyperbolic L^{∞}-norm $\|\varphi\|$ finite. By the Bers embedding, the Teichmüller space $T(R)$ can be identified with a bounded contractible domain in $B(R)$. Then the action of $\mathrm{Aut}(R)$ on $T(R)$ is nothing but the restriction of the action on $B(R)$ to $T(R)$, which is defined by $\varphi \mapsto g^*(\varphi) := \varphi(g)\cdot(g')^2$ for $\varphi \in B(R)$ and $g \in \mathrm{Aut}(R)$. For a subgroup G of $\mathrm{Aut}(R)$, we set

$$B(R/G) = \{\varphi \in B(R) \mid g^*(\varphi) = \varphi \ for\ \forall g \in G\}.$$

This is a Banach subspace of $B(R)$, whose intersection with $T(R)$ corresponds to the Bers embedding of the Teichmüller space of the orbifold R/G.

For a subset X of $B(R)$, the limit set of X is defined as $L(X) := \overline{X} - X$. For a subgroup $G \subset \mathrm{Aut}(R)$ and a point $\varphi \in B(R)$, the orbit of φ under G is defined as

$$G(\varphi) := \{g^*(\varphi) \in B(R) \mid g \in G\}.$$

We say that the orbit $G(\varphi)$ is discrete if it has no accumulation points in $B(R)$.

Proposition. *Let G be a subgroup of* $\mathrm{Aut}(R)$ *and φ a point in $B(R)$. The orbit $G(\varphi)$ is discrete if and only if the limit set of the orbit $L(G(\varphi))$ is empty.*

Proof. If the orbit $G(\varphi)$ is discrete, then $G(\varphi)$ is closed and hence the limit set $L(G(\varphi))$ is empty. Conversely, suppose that $G(\varphi)$ is not discrete. Then there exists a sequence $\{g_n\}$ of elements in G such that $g_n^*(\varphi)$ converges to some point in $B(R)$. We may assume that this point is φ itself by replacing g_n with $g_{n+1}^{-1} \circ g_n$. Moreover, for each point $g^*(\varphi)$ in $G(\varphi)$, a sequence $\{(g \circ g_n)^*(\varphi)\} \subset G(\varphi)$ converges to $g^*(\varphi)$. If $G(\varphi)$ is closed, then this implies that $G(\varphi)$ is a closed perfect set. In a complete metric space in general, every closed perfect set contains uncountably many points. However this contradicts the fact that $G(\varphi)$ is countable. Hence $G(\varphi)$ is not closed, that is, $L(G(\varphi))$ is not empty. □

We announce the following two results in this note. These are prototypes of our further investigation of the action of the modular groups on infinite dimensional Teichmüller spaces.

Theorem 1. *If φ belongs to the limit set $L(\cup B(R/G_n))$ for some infinite sequence of subgroups $\{G_n\}_{n=1}^{\infty}$ of $G =$ $\mathrm{Aut}(R)$, then the orbit $G(\varphi)$ is not discrete. Such an orbit always exists whenever G contains an element of infinite order.*

Proof. Take a sequence $\{\varphi_n\}$ such that $\varphi_n \in B(R/G_n)$ and $\|\varphi_n - \varphi\| \to 0$. Take an element $g_n \in G_n$ for each n and consider a sequence $\{g_n^*(\varphi)\}$. Since $g_n^*(\varphi_n) = \varphi_n$, we have

$$\begin{aligned} \|g_n^*(\varphi) - \varphi\| &= \|g_n^*(\varphi) - g_n^*(\varphi_n)\| + \|\varphi_n - \varphi\| \\ &= 2\|\varphi_n - \varphi\| \to 0, \end{aligned}$$

which means that $g_n^*(\varphi)$ converge to φ. Here $g_n^*(\varphi) \neq \varphi$ for every n because φ does not belong to any $B(R/G_n)$. Hence the orbit $G(\varphi)$ is not discrete.

Next suppose that G contains an element g of infinite order and set $G_n = \langle g^{2^{(n-1)}} \rangle$. Consider the normal covering $R/G_{n+1} \to R/G_n$ for each n. Then $G_n/G_{n+1} \cong \mathbb{Z}_2$ acts on R/G_{n+1} as the covering transformation group and thus acts on $B(R/G_{n+1})$ with the fixed point set $B(R/G_n)$. Excluding a few exceptional surfaces which do not appear in our present case, we know that the action of the Teichmüller modular group is faithful. (This was first proved in [1]. Another proof was given in [2].) This implies that the containment $B(R/G_n) \subset B(R/G_{n+1})$ is proper. Therefore we have a strictly increasing sequence of closed subspaces

$$B(R/G_1) \subsetneq B(R/G_2) \subsetneq \cdots \subsetneq B(R/G_n) \subsetneq \cdots \subset B(R).$$

Then $L(\cup B(R/G_n))$ is not empty by the Baire category theorem. □

Theorem 2. *Suppose that the orders of the elements of $G = \mathrm{Aut}(R)$ is uniformly bounded. If φ does not belong to the limit set $L(\cup B(R/G_n))$ for any infinite sequence of subgroups $\{G_n\}_{n=1}^{\infty}$ of G, then $G(\varphi)$ is discrete.*

Proof. Assume that $G(\varphi)$ is not discrete. Then there exists a sequence $\{g_n\}$ of elements in G such that $g_n^*(\varphi)$ converges to φ as in the proof of Proposition. Also we may assume that none of $\{g_n\}$ fixes φ. For $G_n = \langle g_n \rangle$, this means that φ does not belong to $\cup B(R/G_n)$. Let $k(n)$ be the order of g_n. The average of the orbit of φ under G_n is defined as

$$P_{G_n}(\varphi) := \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} (g_n^i)^*(\varphi).$$

Then $\psi_n = P_{G_n}(\varphi)$ satisfies $g_n^*(\psi_n) = \psi_n$, which means that $\psi_n \in B(R/G_n)$.

We prove that ψ_n converge to φ. The difference is estimated by

$$\begin{aligned} \|\psi_n - \varphi\| &\leq \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \|(g_n^i)^*(\varphi) - \varphi\| \\ &\leq \frac{\sum_{i=0}^{k(n)-1} i}{k(n)} \|(g_n)^*(\varphi) - \varphi\| \\ &= \frac{k(n)-1}{2} \|(g_n)^*(\varphi) - \varphi\|. \end{aligned}$$

Since $(g_n)^*(\varphi)$ converge to φ and since $k(n)$ is uniformly bounded, we see that this converges to 0 as $n \to \infty$. This implies that φ belongs to $L(\cup B(R/G_n))$. □

Remark 1. Concrete examples of the point φ for which the orbit $G(\varphi)$ is not discrete was given in [3]. Theorem 1 asserts that such points always exist if G has an element of infinite order.

An infinite group the orders of whose elements are bounded is known to exist as a counterexample to the Burnside problem in the group theory. Hence, due to the uniformization theorem, we can see that there exists a Riemann surface R such that $G = \mathrm{Aut}(R)$ satisfies the assumption of Theorem 2.

The remaining case where the orders of the elements of G are finite but not bounded seems more difficult to treat.

Remark 2. In the proof of Theorem 1, we have used the fact that if a holomorphic normal covering of non-exceptional Riemann surfaces $R \to R'$ is not trivial, then the containment $B(R) \supset B(R')$ is proper. In [4], this result is extended to any covering $R \to R'$, not necessarily normal.

References

1. C. Earle, F. Gardiner and N. Lakic, *Teichmüller spaces with asymptotic conformal equivalence* (preprint).
2. A. Epstein, *Effectiveness of Teichmüller modular groups*, In the tradition of Ahlfors and Bers, Contemporary Math. 256, AMS, 2000, pp. 69–74.
3. E. Fujikawa, H. Shiga and M. Taniguchi, *On the action of the mapping class group for Riemann surfaces of infinite type*, J. Math. Soc. Japan (to appear).
4. K. Matsuzaki, *Isomorphisms between the Bers embeddings of infinite dimensional Teichmüller spaces* (preprint).

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