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Algebraic Independence and Transcendental Numbers

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We introduce 4 methods for algebraic independence, and mention recent progress concerning problems on large transcendence degrees.

§1. Algebraic independence.

We begin with recalling what means algebraic independence.

DEFINITION 1. Let $\theta_1,\cdots,\theta_t$ be complex numbers. One says that $\theta_1,\cdots,\theta_t$ are *algebraically independent* if for all non-zero polynomials P in $X_1,\cdots,X_t$ with rational integral coefficients, we have $P(\theta_1,\cdots,\theta_t) \neq 0$.

DEFINITION 2. Let E be a subfield of $\mathbf{C}$ and t be a non-negative integer. We define the *transcendence degree* of E by

$$\mathrm{trdeg}_{\mathbf{Q}} E \geqq t$$

if there are t elements of E which are algebraically independent.

REMARK. If E is a finitely generated extension of **Q**, then $\mathrm{trdeg}_{\mathbf{Q}} E$ is the largest number of algebraically independent elements of E.

Now, the question is to find concretely some numbers which are algebraically independent. For this purpose, we have 4 methods.

First method: Liouville's method (1844).

The first example of a transcendental number was given by J.Liouville, by using some series which converges very quickly. The same type of argument yields some examples of algebraically independent numbers; one can even construct some subsets of **C** with large transcendence degree (power of continuum). Recent progress with this method has been achieved by K.Nishioka (see these proceedings).

Second method: Mahler's method (1929).

K.Mahler got several examples of algebraically independent numbers; for example, some series $\sum_n z^{-n^2}$ or some functions like this which take algebraically independent values.

Third method: Lindemann-Weierstrass method (1882).

C.L.F.Lindemann and K.Weierstrass proved the following:

THEOREM 1 (Lindemann-Weierstrass). *Let* $\alpha_1,\cdots,\alpha_n$ *be algebraic numbers which are linearly independent over* **Q**. *Then the numbers* $e^{\alpha_1},\cdots,e^{\alpha_n}$ *are algebraically independent.*

The method of proof of this result uses very deeply the properties of the exponential functions. A development of this method was given by C.L.Siegel in 1929, which generalizes the result to more general functions which he called E functions. These functions satisfy differential equations and also have some nice properties. This theory has been developed by the russian school (A.B.Shidlovsky etc.), and they work for certain generalizations of the exponential function which are entire functions of order one.

Fourth method: Gel'fond's method (1948).

Th.Schneider gave another proof of the Lindemann-Weierstrass theorem for the case n = 1, by using "Gel'fond's method".

THEOREM 2 (Hermite-Lindemann). *Let* α *be a non-zero algebraic number, then* e^{α} *is transcendental.*

This method arose from Hilbert's seventh problem which was solved in 1934 by A.O.Gel'fond and Schneider.

THEOREM 3 (Gel'fond-Schneider). *Let* α *and* β *be algebraic numbers where* $\alpha \neq 0$, $\alpha \neq 1$ *and* β *is not rational. Let* log α *be any determination of the logarithm of* α. *Then the number* $\alpha^{\beta} = \exp(\beta \log \alpha)$ *is transcendental.*

The proof involves several tools. The first one is the construction of an auxiliary function. Such a construction ocurred also in the work of C.Hermite, but here one uses a lemma which deals with systems of linear homogeneous equations; this lemma was introduced by A.Thue and Siegel. Another tool is connected with Gel'fond's studies of integral valued entire functions. This work of Gel'fond is a development of previous works by G.Pólya in 1914 and S.Fukasawa in 1924.

Now, let us consider the question whether it is possible to prove the complete version of the Lindemann-Weierstrass theorem by using Gel'fond's method. The first step was provided by G.V.Chudnovsky in 1978. He proved in that way the Lindemann-Weierstrass theorem for $n = 2,3$ and he showed also an elliptic analog. The elliptic function which shall be considered is Weierstrass' elliptic function, denoted by $\wp$, satisfying $\wp'^2 = 4\wp^3 - g_2\wp - g_3$ where g_2, g_3 are algebraic and $g_2^3 \neq 27g_3^2$. It is known that there is only one meromorphic function which satisfies such a differential equation and has a double pole at the origin. This function is periodic in the plane with two periods ω_1, ω_2 which are linearly

independent over the field of real numbers. Schneider, in 1937, proved that if α is a non-zero algebraic number, then α is not a pole of $\wp$ and that $\wp(\alpha)$ is transcendental. Further results on the algebraic independence of the values of $\wp$ are given by Chudnovsky in 1978.

THEOREM 4 (Chudnovsky). *Let α_1, α_2, α_3 be three algebraic numbers which are linearly independent over* **Q**. *Then at least two of the numbers $\wp(\alpha_1)$, $\wp(\alpha_2)$, $\wp(\alpha_3)$ are algebraically independent.*

§2. Large transcendence degree.

Now, the following more general question arises: if $\alpha_1,\cdots,\alpha_n$ are linearly independent over **Q**, then are the values $\wp(\alpha_1),\cdots,\wp(\alpha_n)$ algebraically independent? In fact, this is not true because of the complex multiplication. We define $k = \mathbf{Q}(\omega_2/\omega_1)$ if ω_2/ω_1 is algebraic ("with complex multiplication") and $k = \mathbf{Q}$ otherwise ("without complex multiplication"). It is known that if ω_2/ω_1 is algebraic, then k is of degree 2 over **Q**.

The conjecture is the following:

CONJECTURE 1. *If $\alpha_1,\cdots,\alpha_n$ are algebraic numbers which are linearly independent over* k, *then $\wp(\alpha_1),\cdots,\wp(\alpha_n)$ are algebraically independent.*

Theorem 4 shows that for $n = 2$ or 3,

$$\mathrm{trdeg}_{\mathbf{Q}}\, \mathbf{Q}(\wp(\alpha_1),\cdots,\wp(\alpha_n)) \geqq \frac{n}{2}\ ,$$

which is the half of what is required. Recently, in 1984, P.Philippon and G.Wüstholz proved the conjecture in the case $k \neq \mathbf{Q}$, and later, Philippon and E.M.Jabbouri proved the half of the conjecture for all n in the case $k = \mathbf{Q}$.

THEOREM 5 (Philippon-Wüstholz). *The conjecture 1 is true if* $k \neq \mathbf{Q}$.

THEOREM 6 (Philippon-Jabbouri). *Under the assumption of the conjecture 1, we have*

$$\mathrm{trdeg}_{\mathbf{Q}}\, \mathbf{Q}(\wp(\alpha_1),\cdots,\wp(\alpha_n)) \geqq \frac{n}{2}\ .$$

Hence the conjecture is not yet completely settled. We now state another problem on algebraic independence.

CONJECTURE 2. *Let* α *be a non-zero algebraic number with* $\alpha \neq 1$. *Let* $\log \alpha$ *be any determination of its logarithm and let* β *be an algebraic number of degree* $d \geqq 2$ *over* $\mathbf{Q}$. *Then the numbers* α^{β}, α^{β^2},$\cdots$, $\alpha^{\beta^{d-1}}$ *are algebraically independent, where* $\alpha^{\beta} = e^{\beta \log \alpha}$,$\cdots$, $\alpha^{\beta^{d-1}} = e^{\beta^{d-1}\log \alpha}$.

This conjecture 2 means that the transcendence degree over

$\mathbf{Q}$ of the field $\mathbf{Q}(\alpha^{\beta},\alpha^{\beta^2},\cdots,\alpha^{\beta^{d-1}})$ is $d-1$. We know already that if $d \geq 2$, then

$$t = \mathrm{trdeg}_{\mathbf{Q}}\, \mathbf{Q}(\alpha^{\beta},\alpha^{\beta^2},\cdots,\alpha^{\beta^{d-1}}) \geq 1$$

because each number α^{β^j} $(1 \leq j \leq d-1)$ is transcendental by the Gel'fond-Schneider theorem. Gel'fond showed in 1949 that if $d \geq 3$, then $t \geq 2$. For instance, if β is cubic, then α^{β} and α^{β^2} are algebraically independent.

This problem has been studied by A. A. Shmelev, W. D. Brownawell and Chudnovsky and others, who showed that there exists an explicit constant d_0 such that if $d \geq d_0$ then $t \geq 3$. In fact, the value found for d_0 was 19, then 15, then 7, and may be 5. A fundamental breakthrough was provided by Chudnovsky in 1974.

THEOREM 7 (Chudnovsky, 1974, preprint of Kiev Univ.). *Under the assumption of the conjecture 2, we have*

$$\mathrm{trdeg}_{\mathbf{Q}}\, \mathbf{Q}(\alpha^{\beta},\alpha^{\beta^2},\cdots,\alpha^{\beta^{d-1}}) \geq \left[\frac{\log\,(d+1)}{\log\,2}\right].$$

Until 1984, this preprint was not published and some parts of the proof were not clear. P.Warkentin, Philippon, E. Reyssat, R.Endell and Yu.V.Nesterenko have tried to understand it and they succeeded to get the result with essentially the same kind of estimates. Nesterenko's proof involved some quite

different ideas using commutative algebra and he obtained also a measure of algebraic independence. A very important progress has then been achieved by Philippon:

THEOREM 8 (Philippon, 1984). *Under the assumption of the conjecture 2, we have*

$$\mathrm{trdeg}_{\mathbf{Q}}\ \mathbf{Q}(\alpha^{\beta},\alpha^{\beta^2},\cdots,\alpha^{\beta^{d-1}}) \geq \left[\frac{d}{2}\right].$$

Philippon's proof is completely different from that of Chudnovsky, but it relies on some of Nesterenko's ideas. Also it enables one to get a complete version of the Lindemann-Weierstrass theorem by using Gel'fond's method.

Next, we shall mention the elliptic analog of the conjecture 2.

CONJECTURE 3. *Let $\wp$ be a Weierstrass elliptic function with algebraic invariants g_2,g_3. Let u be a complex number which is not a pole of $\wp$, such that $\wp(u)$ is algebraic. Further let β be an algebraic number of degree $d \geq 2$ over k. Then the numbers $\wp(\beta u),\wp(\beta^2 u),\cdots,\wp(\beta^{d-1}u)$ are algebraically independent.*

Define $t_{\wp} = \mathrm{trdeg}_{\mathbf{Q}}\ \mathbf{Q}(\wp(\beta u),\wp(\beta^2 u),\cdots,\wp(\beta^{d-1}u))$, so the conjecture 3 is $t_{\wp} = d - 1$. It has been proved by Schneider in 1937 that $d \geq 2$ implies $t_{\wp} \geq 1$, and the work by D.W.Masser

and Wüstholz in 1981 gives that $d \geq 3$ implies $t_\wp \geq 2$. Chudnovsky's proof cannot be adapted to this elliptic case. However, Philippon succeeded to prove the following result:

THEOREM 9 (Philippon, 1982). *Under the assumption of the conjecture 3, we have*

$$t_\wp \geq \left[\frac{d-1}{2}\right] \quad \text{if} \quad k \neq \mathbf{Q},$$

and

$$t_\wp \geq \left[\frac{d-1}{3}\right] \quad \text{if} \quad k = \mathbf{Q}.$$

His result was proved in the case of any abelian variety, and for this, it is important that the abelian variety is compact and complete. However, the multiplicative group is not compact, so the multiplicative case is more difficult and was proved later.

§3. Transcendence criterion.

The main tool in the proof of the above mentioned results is a transcendence criterion. For a polynomial $P(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0$, we define $H(P) = \max_{0 \leq i \leq n} |a_i|$.

THEOREM 10 (Gel'fond's criterion, 1949). *Let θ be a complex number. If, for all $N \gg 1$, there exists $P_N \in \mathbf{Z}[X]$, $P_N \neq 0$, such that $\deg P_N \leq N$, $H(P) \leq e^N$ and*

$|P_N(\theta)| < e^{-10N^2}$, *then we have* $P_N(\theta) = 0$ *for all sufficiently large* N.

The idea of this proof is the following: if $P_N(\theta)$ is very small, then θ is close to one of the roots of $P_N(X)$, denoted by α_N. If we compare α_N and α_{N+1}, then we can find that they are very close. For two algebraic numbers α_N and α_{N+1}, Liouville's theorem shows that they are equal when they are very close. Then we have $\alpha_N = \alpha_{N+1} = \cdots$ which converges to θ, that means $\alpha_N = \alpha_{N+1} = \cdots = \theta$, therefore we get $P_N(\theta) = 0$.

All the difficult points to get some results for large transcendence degrees, are to generalize this criterion for algebraic independence.

Reference

[1] M. Waldschmidt, Algebraic independence of transcendental numbers. Gel'fond's method and its developments, in *Perspectives in Math.*, Ann. of Oberwolfach 1984, Birkhäuser Verlag, pp.551-571.