

Computing point residues for a shape basis case via differential operators

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1 Introduction

In this paper, we study computational aspects of point residues. We concentrate on a shape basis case and we present algorithms which compute point residues for this generic case.

In 1987, Gianni and Mora ([2]) proved the following result:

(Shape lemma) *Let I be a radical 0-dimensional ideal in $\mathbb{Q}[z]$, regular in z_1 . Then there are $g_1(z_1), \dots, g_n(z_1) \in \mathbb{Q}[z_1]$ such that g_1 is squarefree, $\deg(g_i) < \deg(g_1)$ for $i > 1$ and the Gröbner basis of the ideal I w.r.t. the lexicographical order $\succ$ with $z_1 \succ \dots \succ z_n$ is of the form*

$$\{g_1(z_1), z_2 - g_2(z_1), \dots, z_n - g_n(z_1)\}. \quad (1.1)$$

On the other hand, if the reduced Gröbner basis of I w.r.t. $\succ$ is of this form, then I is a radical 0-dimensional ideal.

Furthermore, it is known that for "almost every" system of algebraic equations with finitely many solutions, after a suitable linear coordinate transformation, the reduced Gröbner basis of the transformed ideal will be in this simple form even though the system does not coincide with its radical ([5], [6], [7], [15]). The basis of the form (1.1) is called the shape basis of I .

We study the algebraic local cohomology class associated with the shape basis of a given 0-dimensional ideal I . We explicitly construct the holonomic system of linear partial differential equations for the algebraic local cohomology class. By making use of this holonomic system, we derive algorithms for computing point residues.

2 Notation and a former result

Let $X = \mathbb{C}^n$ and fix a coordinate system $z = (z_1, \dots, z_n)$ of X . We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X . Denote by $\mathcal{I}$ the zero dimensional ideal in $\mathcal{O}_X$ generated by holomorphic functions $f_1, \dots, f_n$ of z .

Put $Y = \{z \in X \mid f_1 = \dots = f_n = 0\}$. The algebraic local cohomology group $\mathcal{H}_{[Y]}^n(\mathcal{O}_X)$ which satisfies $\mathcal{H}_{[Y]}^n(\mathcal{O}_X) = \lim \operatorname{ind}_k \operatorname{Ext}_{\mathcal{O}_X}^n(\mathcal{O}_X/\mathcal{I}^k, \mathcal{O}_X)$, has a structure of a left $\mathcal{D}_X$ -module, where $\mathcal{D}_X$ is the sheaf of linear partial differential operators on

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X . Let $\left[\begin{smallmatrix} h \\ f_1 \cdots f_n \end{smallmatrix} \right]$ be a class in $\mathcal{E}xt_{\mathcal{O}_X}^n(\mathcal{O}_X/\mathcal{I}, \mathcal{O}_X)$ for $h \in \mathcal{O}_X$. Denote by η the algebraic local cohomology class $\left[\frac{h}{f_1 \cdots f_n} \right]$ defined by the image of $\left[\begin{smallmatrix} h \\ f_1 \cdots f_n \end{smallmatrix} \right]$ by the canonical mapping

$$\mathcal{E}xt_{\mathcal{O}_X}^n(\mathcal{O}_X/\mathcal{I}, \mathcal{O}_X) \rightarrow \mathcal{H}_{[Y]}^n(\mathcal{O}_X). \quad (2.1)$$

Denote by Ann the ideal in $\mathcal{D}_X$ consisting of annihilators of η . Then we have $\mathcal{H}_{[Y]}^n(\mathcal{O}_X) \cong \mathcal{D}_X/Ann$. For the Weyl algebra, it is possible to compute a Gröbner basis of Ann by using the computer algebra system Kan ([8], [9], [14]).

We have the canonical pairing

$$\begin{aligned} \text{Res}_\alpha : \Omega_X \times \mathcal{H}_{[\alpha]}^n(\mathcal{O}_X) &\rightarrow \mathbb{C} \\ (\psi dz, \eta) &\mapsto \text{Res}_\alpha \langle \psi dz, \eta \rangle \end{aligned}$$

defined by the point residue $\text{Res}_\alpha((h\psi)dz/f_1 \cdots f_n)$ of a meromorphic differential form $(h\psi)dz/f_1 \cdots f_n$ at $\alpha \in Y$.

The sheaf of holomorphic differential forms Ω_X is naturally endowed with a structure of a right $\mathcal{D}_X$ -module by setting $(\phi(z)dz)R = ((R^*\phi)(z))dz$ for a differential operator $R \in \mathcal{D}_X$, where R^* stands for the formal adjoint operator of R . Then we have, for any $R \in Ann$,

$$\text{Res}_\alpha \langle (R^*\phi(z))dz, \eta \rangle = \text{Res}_\alpha \langle \phi(z)dz, R\eta \rangle = 0, \quad \alpha \in Y.$$

Theorem 2.1 ([10], [11]) *Put $\mathcal{K} = \{\phi(z)dz \in \Omega_X \mid \text{Res}_\alpha \langle \phi(z)dz, \eta \rangle = 0, \forall \alpha \in Y\}$. Then we have*

$$\mathcal{K} = \{(R^*\psi(z))dz \mid R \in Ann, \psi(z)dz \in \Omega_X\}.$$

3 Construction of the holonomic system in the shape basis case

Let us consider the system

$$(S) \begin{cases} f_1 = g_1(z_1), \\ f_2 = z_2 - g_2(z_1), \\ \dots\dots\dots, \\ f_n = z_n - g_n(z_1), \end{cases}$$

where $g_i(z_1) \in \mathbb{Q}[z_1]$. Denote by Y the set of common zeros of the system (S) , i.e., $Y = \{z = (z_1, \dots, z_n) \in X \mid f_1 = \cdots = f_n = 0\}$. Put $\eta = [h/f_1 \cdots f_n] \in \mathcal{H}_{[Y]}^n(\mathcal{O}_X)$ for $h \in \mathcal{O}_X$ with $h(\alpha) \neq 0$, $\alpha \in Y$. Since η depends on the modulo class of h in $\mathcal{O}_X/\mathcal{I}$, the numerator h of the cohomology class η can be expressed as an univariate function of the variable z_1 .

Let $P, F_1, \dots, F_n$ be differential operators defined by following forms:

$$(A) \begin{cases} P &= \text{sf}(g_1)\partial_1 + \sum_{i=2}^n \text{sf}(g_1)g'_i(z_1)\partial_i + \frac{g'_1(z_1)}{\gcd(g_1(z_1), g'_1(z_1))} - \frac{h'(z_1)}{h(z_1)}\text{sf}(g_1), \\ F_1 &= g_1(z_1), \\ F_2 &= z_2 - g_2(z_1), \\ \dots &\dots \dots, \\ F_n &= z_n - g_n(z_1), \end{cases}$$

where $\text{sf}(g_1)$ is the square free part $g_1(z_1)/\gcd(g_1(z_1), g'_1(z_1))$ of $g_1(z_1)$, $g'_i(z_1) := \partial g_i / \partial z_1$, and $\partial_i := \partial / \partial z_i$, $i = 1, \dots, n$. Then we have the next theorem.

Theorem 3.1 *Let Ann be the left ideal in $\mathcal{D}_X$ consisting of annihilators of η . Then Ann is generated by P and F_i , $i = 1, \dots, n$ in (A) .*

Proof. Recall the isomorphism

$$\mathcal{H}_{[Y]}^n(\mathcal{O}_X) \cong \frac{\mathcal{O}_X[* (Z_1 \cup \dots \cup Z_n)]}{\sum_{i=1}^n \mathcal{O}_X[* (Z_1 \cup \dots \cup \widehat{Z_i} \cup \dots \cup Z_n)]}, \quad (3.1)$$

where $Z_i = \{z \in X \mid f_i(z) = 0\}$ and $\mathcal{O}_X[*Z]$ stands for a sheaf of meromorphic functions with poles at Z . By this isomorphism, we can readily see that operators in (A) annihilate η . Let $g_1 = \prod_{\iota=1}^{\nu} (z_1 - \alpha_{1,\iota})^{m_\iota}$ be the factorization of g_1 over $\mathbb{C}$. Then we have $\eta_\iota \in \mathcal{H}_{[\alpha_\iota]}^n(\mathcal{O}_X)$ such that $\eta = \eta_1 + \dots + \eta_\nu$, where $\alpha_\iota = (\alpha_{1,\iota}, g_2(\alpha_{1,\iota}), \dots, g_n(\alpha_{1,\iota})) \in Y$, $\iota = 1, \dots, \nu$. Let U_k be a sufficiently small neighborhood of a point $\alpha_k \in Y$ and assume that $U_k \cap Y = \{\alpha_k\}$. Let us find the annihilators of η on U_k . Denote by $g_{i,k}$ the modulo class of g_i in $\mathcal{O}_X / \langle (z_1 - \alpha_{1,k})^{m_k} \rangle$. Put $f_{i,k}(z_1) = z_i - g_{i,k}(z_1)$. If we set $h_k = h / \prod_{\iota \neq k} (z_1 - \alpha_{1,\iota})^{m_\iota}$, we have

$$\eta_k = \left[\frac{h_k}{(z_1 - \alpha_{1,k})^{m_k} f_{2,k} \cdots f_{n,k}} \right].$$

Then we have

$$P_k = (z_1 - \alpha_{1,k})\partial_1 + (z_1 - \alpha_{1,k}) \sum_{i \neq k} g'_{i,k} \partial_i + m_k - \frac{h'_k}{h_k} (z_1 - \alpha_{1,k}), \quad (3.2)$$

$$F_{1,k} = (z_1 - \alpha_{1,k})^{m_k}, \quad (3.3)$$

and

$$F_{i,k} = z_i - g_{i,k}(z_1), \quad i = 2, \dots, n \quad (3.4)$$

as annihilators of η on U_k . Note that the annihilator P_k can be rewritten as

$$P_k = (z_1 - \alpha_{1,k})\partial_1 + (z_1 - \alpha_{1,k}) \sum_{i \neq k} g'_{i,k} \partial_i + \sum_{\iota=1}^{\nu} m_\iota \frac{z_1 - \alpha_{1,k}}{z_1 - \alpha_{1,\iota}} - \frac{h'}{h} (z_1 - \alpha_{1,k}). \quad (3.5)$$

We set $\text{Ann}_k = \{R \in \mathcal{D}_X \mid R\eta_k = 0\}$. Since $\langle P_k, F_{1,k}, \dots, F_{n,k} \rangle \subset \text{Ann}_k$, we have a surjective morphism $\mathcal{D}_X / \langle P_k, F_{1,k}, \dots, F_{n,k} \rangle \rightarrow \mathcal{D}_X / \text{Ann}_k \rightarrow 0$. Recall that $\mathcal{D}_X / \text{Ann}_k$ is a simple holonomic system, the multiplicity of $\mathcal{D}_X / \text{Ann}_k$ is equal to 1. We can see that the multiplicity of $\mathcal{D}_X / \langle P_k, F_{1,k}, \dots, F_{n,k} \rangle$ is also equal to 1. Thus $\mathcal{D}_X / \text{Ann}_k = \mathcal{D}_X / \langle P_k, F_{1,k}, \dots, F_{n,k} \rangle$ and finally we have $\langle P_k, F_{1,k}, \dots, F_{n,k} \rangle = \text{Ann}_k$. On the other hand, the localization of P and F_i , $i = 1, \dots, n$ to U_k have the following forms:

$$\begin{aligned} P|_{\alpha_k} &= \frac{1}{(z_1 - \alpha_{1,1}) \cdots (z_1 - \alpha_{1,k-1})(z_1 - \alpha_{1,k+1}) \cdots (z_1 - \alpha_{1,n})} P \\ &= (z_1 - \alpha_{1,k})\partial_1 + (z_1 - \alpha_{1,k}) \sum_{i \neq k} g'_{i,k} \partial_i \\ &\quad + \sum_{\iota=1}^{\nu} m_\iota \frac{\prod_{\ell \neq \iota} (z_1 - \alpha_{1,\ell})}{(z_1 - \alpha_{1,1}) \cdots (z_1 - \alpha_{1,k-1})(z_1 - \alpha_{1,k+1}) \cdots (z_1 - \alpha_{1,n})} \\ &\quad - \frac{h'}{h} (z_1 - \alpha_{1,k}) \\ &= (z_1 - \alpha_{1,k})\partial_1 + (z_1 - \alpha_{1,k}) \sum_{i \neq k} g'_{i,k} \partial_i + \sum_{\iota=1}^{\nu} m_\iota \frac{1}{z_1 - \alpha_{1,\iota}} (z_1 - \alpha_{1,k}) \\ &\quad - \frac{h'}{h} (z_1 - \alpha_{1,k}), \end{aligned} \quad (3.6)$$

$$F_1|_{\alpha_k} = (z_1 - \alpha_{1,k})^{m_k}, \quad (3.7)$$

$$F_i|_{\alpha_k} = z_i - g_{i,k}(z_1), \quad i = 2, \dots, n. \quad (3.8)$$

According to the formulas from (3.3) to (3.8), we have $P|_{U_k} = P_k$, $F_i|_{U_k} = F_{i,k}$. Then we have $\text{Ann}_k = \langle P|_{U_k}, F_1|_{U_k}, \dots, F_n|_{U_k} \rangle$. If we denote by $\text{Ann}|_{U_k}$ the restriction of the ideal Ann to U_k , we have $\text{Ann}|_{U_k} = \text{Ann}_k$. Thus, we obtain that $\text{Ann}|_{U_k} = \langle P|_{U_k}, F_1|_{U_k}, \dots, F_n|_{U_k} \rangle$. Consequently, $\text{Ann} = \langle P, F_1, \dots, F_n \rangle$. $\square$

3.1 Properties of P^*

The following relations between operators P and F_i , $i = 1, \dots, n$ hold:

Corollary 3.1

$$[P^*, F_i^*] = \begin{cases} -\frac{g_1'(z_1)}{\gcd(g_1(z_1), g_1'(z_1))} F_1, & i = 1, \\ 0, & i = 2, 3, \dots, n. \end{cases}$$

Proof. Since g_1 is a univariate polynomial of z_1 , we have

$$\begin{aligned} [P^*, F_1^*] &= -\text{sf}(g_1) \cdot g_1' \\ &= -\frac{g_1'}{\gcd(g_1, g_1')} F_1. \end{aligned}$$

For $i = 2, 3, \dots, n$, we have

$$[P^*, F_i^*] = -\text{sf}(g_1)g_i' + \text{sf}(g_1)g_i' = 0.$$

$\square$

This corollary implies that, if $\varphi \in \mathcal{I}$, then $P^*\varphi \in \mathcal{I}$ holds. Thus, we have the next proposition.

Proposition 3.1 P^* acts on the sheaf $\mathcal{O}_X/\mathcal{I}$, i.e.,

$$P^* : \mathcal{O}_X/\mathcal{I} \rightarrow \mathcal{O}_X/\mathcal{I}.$$

Let $\tilde{\mathcal{I}}$ be the ideal generated by $\gcd(g_1(z_1), g_1'(z_1)), z_2 - g_2(z_1), \dots, z_n - g_n(z_1)$ in $\mathcal{O}_X$. Then P^* has the following property:

Theorem 3.2 A necessary and sufficient condition for $P^*\varphi(z) \in \mathcal{I}$ is $\varphi(z) \in \tilde{\mathcal{I}}$.

Proof. We prove first that the condition is sufficient. Since $F_j^* = F_j = f_j$, we have $P^*(\chi f_i) = (P^*\chi)f_i$ for any $\chi \in \mathcal{O}_X$ by Corollary 3.1. Since the operator P^* can be written in the form

$$P^* = -\frac{g_1(z_1)}{h(z_1)} \partial_1 \frac{h(z_1)}{\gcd(g_1(z_1), g_1'(z_1))} - \sum_{i=2}^n \frac{g_1(z_1)}{\gcd(g_1(z_1), g_1'(z_1))} g_i'(z_1) \partial_i, \quad (3.9)$$

we have

$$\begin{aligned} P^*(\gcd(g_1, g_1')\varphi) &= -\frac{g_1}{h} \partial_1 \left(\frac{h}{\gcd(g_1, g_1')} \gcd(g_1, g_1') \varphi \right) \\ &\quad - \sum_{i=2}^n \frac{g_1}{\gcd(g_1, g_1')} g_i' \partial_i (\gcd(g_1, g_1') \varphi) \\ &= -\left(\frac{1}{h} \partial_1 h \varphi + \sum_{i=2}^n g_1 g_i' \partial_i \varphi \right) g_1 \\ &= -\left(\frac{1}{h} \partial_1 h \varphi + \sum_{i=2}^n g_1 g_i' \partial_i \varphi \right) f_1. \end{aligned}$$

These formulas imply the sufficiency. In order to prove the necessity, we set

$$\varphi(z) = \varphi_1(z)\gcd(g_1(z_1), g'_1(z_1)) + \varphi_2(z)f_2(z) + \cdots + \varphi_n(z)f_n(z) + \varphi_0(z_1),$$

where $\varphi_0, \varphi_1, \dots, \varphi_n \in \mathcal{O}_X$ and φ_0 is an univariate polynomial of z_1 with $\deg \varphi_0(z_1) < \deg \gcd(g_1(z_1), g'_1(z_1))$. Since $P^*\varphi \in I$ by Corollary 3.1, there is an univariate polynomial $\psi(z_1)$ of z_1 such that $P^*\varphi_0(z_1) = \psi(z_1)f_1$. On the other hand, we have

$$P^*\varphi_0 = -\frac{g_1}{h}\partial_1 \frac{h}{\gcd(g_1, g'_1)}\varphi_0.$$

Thus we have

$$\begin{aligned} -\frac{g_1}{h}\partial_1 \frac{h}{\gcd(g_1, g'_1)}\varphi_0 &= \psi f_1 \\ \frac{h}{\gcd(g_1, g'_1)}\varphi_0 &= -\int^{z_1} \frac{h(t)}{g_1(t)}\psi(t)f_1(t)dt \\ \varphi_0 &= \left(-\frac{1}{h}\int^{z_1} \frac{h(t)}{g_1(t)}\psi(t)f_1(t)dt\right)\gcd(g_1, g'_1). \end{aligned}$$

Since $\varphi_0 \notin \tilde{I}$, we have $\varphi_0 = 0$. This completes the proof. $\square$

From the exact sequence $0 \rightarrow \tilde{I}/I \rightarrow \mathcal{O}_X/I \rightarrow \mathcal{O}_X/\tilde{I} \rightarrow 0$, we have that $\dim \Gamma(X, \tilde{I}/I) = \dim \Gamma(X, \mathcal{O}_X/I) - \dim \Gamma(X, \mathcal{O}_X/\tilde{I}) = \nu$. Put $d = \deg g_1(z_1)$. Then, we have the following corollary:

Corollary 3.2

- (i) $\dim \Gamma(X, \text{Im}(P^* : \mathcal{O}_X/I \rightarrow \mathcal{O}_X/I)) = d - \nu$.
- (ii) $\dim \Gamma(X, \text{Ker}(P^* : \mathcal{O}_X/I \rightarrow \mathcal{O}_X/I)) = \nu$.

Let $v_j(z_1)$ be the image of z_1^j by P^* in $\Gamma(X, \mathcal{O}_X/I)$ for $j = 0, \dots, d - \nu - 1$. Put $\mathcal{K} = \{v(z) \in \mathcal{O}_X \mid \text{Res}_\alpha \langle v(z)dz, \eta \rangle = 0, \alpha \in Y\}$.

Corollary 3.3

$$\Gamma(X, \mathcal{K}/I) \cong \text{Span}\{v_0(z_1), \dots, v_{d-\nu-1}(z_1)\}.$$

That is, any $v(z_1)$ which satisfies $\text{Res}_\alpha \langle v(z_1)dz, \eta \rangle = 0$ for $\alpha \in Y$ and $\deg v(z_1) \leq d - 1$ can be expressed as a linear combination of $v_0(z_1), \dots, v_{d-\nu-1}(z_1)$.

3.2 Localization

Let $g_1(z_1) = g_{1,1}^{\mu_1}(z_1) \cdots g_{1,N}^{\mu_N}(z_1)$ be the factorization of $g_1(z_1)$ over $\mathbb{Q}$. Let $g_{i,k}(z_1)$ be the remainder of division of $g_i(z_1)$ by $g_{1,k}^{\mu_k}(z_1)$. Put $f_{1,k}(z) = g_{1,k}^{\mu_k}(z_1)$ and $f_{i,k}(z) = z_i - g_{i,k}(z_1)$ for $k = 1, \dots, N$ and $i = 2, \dots, n$. Denote by I_k the ideal in $\mathbb{Q}[z]$ generated by $f_{1,k}(z), \dots, f_{n,k}(z)$. Let $F_{i,k}$ be the differential operator of order zero defined by $F_{i,k} = f_{i,k}$. From Corollary 3.1, we have the following formulas:

Corollary 3.4

$$[P^*, F_{i,k}^*] = \begin{cases} -((\prod_{j \neq i} g_{1,j})g'_{1,k})g_{1,k}^{\mu_k}, & i = 1, \\ 0, & i = 2, 3, \dots, n. \end{cases} \quad (3.10)$$

These formulas imply the next result.

Lemma 3.1 P^* acts on the vector space $\mathcal{O}_X/I_k$, i.e.,

$$P^* : \mathcal{O}_X/I_k \rightarrow \mathcal{O}_X/I_k.$$

Thus we can localize results in Section 3.1 to $\mathcal{I}_k$. Put $\nu_k = \deg g_{1,k}(z_1)$ and $d_k = \nu_k \mu_k$. Then we have the following:

Corollary 3.5

$$(i) \dim \Gamma(X, \text{Im}(P^* : \mathcal{O}_X/\mathcal{I}_k \rightarrow \mathcal{O}_X/\mathcal{I}_k)) = d_k - \nu_k.$$

$$(ii) \dim \Gamma(X, \text{Ker}(P^* : \mathcal{O}_X/\mathcal{I}_k \rightarrow \mathcal{O}_X/\mathcal{I}_k)) = \nu_k.$$

Let $v_{k,j}(z_1)$ be the image of z_1^j by P^* in $\Gamma(X, \mathcal{O}_X/\mathcal{I}_k)$ for $j = 0, \dots, d_k - \nu_k - 1$. Denote by Y_k the set of common zeros of $f_{1,k}, \dots, f_{n,k}$. Put $\mathcal{K}_k = \{v(z) \in \mathcal{O}_X \mid \text{Res}_\alpha \langle v(z)dz, \eta_k \rangle = 0, \alpha \in Y_k\}$.

Corollary 3.6

$$\Gamma(X, \mathcal{K}_k/\mathcal{I}_k) \cong \text{Span}\{v_{k,0}(z_1), \dots, v_{k,d_k-\nu_k-1}(z_1)\}.$$

That is, any $v(z_1)$ which satisfies $\text{Res}_{\alpha \in Y_k} \langle v(z_1)dz, \eta_k \rangle = 0$ and $\deg v(z_1) \leq d_k - 1$ can be expressed as a linear combination of $v_{k,0}(z_1), \dots, v_{k,d_k-\nu_k-1}(z_1)$.

4 Algorithm

We describe algorithms for computing point residues. Let $f_1(z), \dots, f_n(z)$ be polynomials in $\mathbb{Q}[z_1, \dots, z_n]$ of the form (S) and $dz = dz_1 \wedge \dots \wedge dz_n$. Let us consider a meromorphic differential form $\theta(z)dz/f_1(z) \cdots f_n(z)$ with a polynomial $\theta(z) \in \mathbb{Q}[z]$. Denote by $\underline{\theta}$ the remainder of θ by I . Now we introduce three vector spaces

$$U = \{u(z_1) \in \mathbb{Q}[z_1] \mid \deg u(z_1) \leq d - 1\}, \quad (4.1)$$

$$V = \{v(z_1) \in \mathbb{Q}[z_1] \mid \deg v(z_1) \leq d - 1, \text{Res}_\alpha \left(v(z_1)dz, \left[\frac{1}{f_1 \cdots f_n} \right] \right) = 0, \alpha \in Y\}, \quad (4.2)$$

and

$$W = \{w(z_1) \in \mathbb{Q}[z_1] \mid \deg w(z_1) \leq d - 1, \frac{w(z_1)}{f_1 \cdots f_n} \text{ has at most simple poles}\}. \quad (4.3)$$

The dimensions of these vector spaces are $\dim U = d$, $\dim V = d - \nu$ and $\dim W = \nu$, respectively. Let P be the annihilator of the cohomology class $[1/f_1 \cdots f_n]$ defined in (A), i.e.,

$$P = \text{sf}(g_1)\partial_1 + \sum_{i=2}^n \text{sf}(g_i)g'_i(z_1)\partial_i + \frac{g'_1(z_1)}{\gcd(g_1(z_1), g'_1(z_1))}.$$

Denote by $v_j(z_1)$ the remainder of $P^* z_1^j$ by $g_1(z)$, $j = 1, \dots, d - \nu - 1$. Let Jac be Jacobian of $f_1, \dots, f_n$. In this case, $\text{Jac} = g'_1(z_1)$. Let $w_j(z_1)$ be the remainder of $\text{Jac} \cdot z_1^j$ by $g_1(z_1)$ for $j = 0, \dots, \nu - 1$.

Proposition 4.1

- (i) $U = V \oplus W$
- (ii) $V = \text{Span}\{v_0(z_1), \dots, v_{d-\nu-1}(z_1)\}$
- (iii) $W = \text{Span}\{w_0(z_1), \dots, w_{\nu-1}(z_1)\}$

For computing the residues, we write

$$\underline{\theta}(z_1) = \sum_{j=0}^{d-\nu-1} a_j v_j(z_1) + \sum_{\ell=0}^{\nu-1} b_\ell w_\ell(z_1).$$

Then we have

$$\begin{aligned} \text{Res}_{\alpha \in Y} \left(\frac{\theta(z_1)}{f_1 \dots f_n} dz \right) &= \text{Res}_{\alpha \in Y} \left(\frac{\sum_{\ell=0}^{\nu-1} b_\ell w_\ell}{f_1 \dots f_n} dz \right) \\ &= \text{Res}_{\alpha \in Y} \left(\left(\frac{\text{Jac}}{f_1 \dots f_n} \sum_{\ell=0}^{\nu-1} b_\ell z_1^\ell \right) dz \right). \end{aligned}$$

Since $\text{Jac} \sum_{\ell=0}^{\nu-1} b_\ell z_1^\ell dz / f_1 \dots f_n$ is a meromorphic n -form with only simple poles, we can proceed as follows:

Let $g_1(z_1) = g_{1,1}^{\mu_1}(z_1) \dots g_{1,N}^{\mu_N}(z_1)$ be the factorization of $g_1(z_1)$ over $\mathbb{Q}$. Denote by $g_{j,k}$ the remainder of g_j by $g_{1,k}^{\mu_k}$ and σ_k the remainder of $\sum_{\ell=0}^{\nu-1} b_\ell z_1^\ell$ by $g_{1,k}$. Let J_k be the ideal of $\mathbb{Q}[z, t]$ generated by $g_{1,k}, z_2 - g_{2,k}, \dots, z_n - g_{n,k}$ and $\mu_k \sigma_k - t$. We obtain a univariate polynomial $\varrho_k(t)$ of t as the generator of $J_k \cap \mathbb{Q}[t]$. Then $\varrho_k(t) = 0$ is the equation for residues of $\theta dz / f_1 \dots f_n$ at Y_k .

Algorithm 1 (point residues for shape basis case)

Input $g_1(z_1), z_2 - g_2(z_1), \dots, z_n - g_n(z_1)$: the shape basis, $\theta(z) \in \mathbb{Q}[z]$
 $\underline{\theta}(z_1) \leftarrow$ the remainder of $\theta(z)$ by $\langle g_1(z_1), z_2 - g_2(z_1), \dots, z_n - g_n(z_1) \rangle$
 $\text{sf}(g_1) \leftarrow g_1 / \gcd(g_1, g_1')$
 $\nu \leftarrow \deg \text{sf}(g_1)$
 $d \leftarrow \deg g_1$
for j **from** 0 **to** $d - \nu - 1$
 $v_j \leftarrow$ the remainder of $-\frac{g_1}{\gcd(g_1, g_1')} j z_1^{j-1} + \frac{g_1}{\gcd(g_1, g_1')} \frac{\gcd(g_1, g_1')'}{\gcd(g_1, g_1')} z_1^j$ by $f_{1,k}$
for ℓ **from** 0 **to** $\nu - 1$
 $w_\ell \leftarrow$ the remainder of $g_1' z_1^\ell$ of g_1
 $\vartheta \leftarrow \underline{\theta} - \sum_{j=0}^{d-\nu-1} a_j v_j - \sum_{\ell=0}^{\nu-1} b_\ell w_\ell$
 $(a_0, \dots, a_{d-\nu-1}, b_0, \dots, b_{\nu-1}) \leftarrow$ the coefficients s.t. $\vartheta = 0$
 $g_{1,1}^{\mu_1} \dots g_{1,N}^{\mu_N} \leftarrow$ the squarefree factorization of g_1
for k **from** 1 **to** N
for i **from** 2 **to** n
 $g_{i,k} \leftarrow$ the remainder of g_i by $g_{1,k}^{\mu_k}$
 $\sigma_k \leftarrow$ the remainder of $\sum_{\ell=0}^{\nu-1} b_\ell z_1^\ell$ by $g_{1,k}$
 $J_k \leftarrow \langle g_{1,k}, z_2 - g_{2,k}, \dots, z_n - g_{n,k}, \mu_k \sigma_k - t \rangle$
 $G_k \leftarrow$ Gröbner basis of J_k w.r.t. the lexicographical order $z \succ t$
Output $\{G_1, \dots, G_N\}$

Example 1 Put $z = (x, y)$. Let us consider $f_1 = x^4(2x^2 - 1)^3$, $f_2 = y - (x^3 + 1)$ and $\theta = 35xy^3 - x^2y + y - 1$. The annihilator P of the cohomology class $[1/f_1 f_2]$ is $P = (2x^3 - x)\partial_x + (6x^5 - 3x^3)\partial_y + 20x^2 - 4$. Then we have $V = \text{Span}\{v_0, \dots, v_6\}$, where $v_0 = 14x^2 - 3$, $v_1 = 12x^3 - 2x$, $v_2 = 10x^4 - x^2$, $v_3 = 8x^5$, $v_4 = 6x^6 + x^4$, $v_5 = 4x^7 + 2x^5$, $v_6 = 2x^8 + 3x^6$ and $W = \text{Span}\{w_0, w_1, w_2\}$, where
 $w_0 = 80x^9 - 96x^7 + 36x^5 - 4x^3$,
 $w_1 = 24x^8 - 24x^6 + 6x^4$,
 $w_2 = 24x^9 - 24x^7 + 6x^5$.

The remainder $\underline{\theta}$ of θ by $\langle f_1, f_2 \rangle$ is $\underline{\theta} = (105/2)x^8 + 105x^7 - (105/4)x^6 - x^5 + (875/8)x^4 + x^3 - x^2 + 35x$. It can be written as $\underline{\theta} = -(35/2)v_1 + v_2 + (2425/16)v_3 +$

$(3555/64)v_4 - (739/4)v_5 - (1965/32)v_6 - (211/4)w_0 + (935/128)w_1 + (1055/6)w_2$. Put $\sigma = -(211/4) + (935/128)x + (1055/6)x^2$. For $I_1 = \langle x^4, y - (x^3 + 1) \rangle$, we have $\sigma_1 = -211/4$. Thus, $\text{Res}_{[0,1]}(\theta dz/f_1 f_2) = 4(-211/4) = -211$. For $I_2 = \langle (2x^2 - 1)^3, y - (x^3 + 1) \rangle$, we have $\sigma_2 = (935/128)x + (211/6)$. Then $G_2 = \langle -32768t^2 + 6914048t - 356848007, -2805x + 128t - 13504, -2805y + 64t - 3947 \rangle$. Thus we have $\varrho_2(t) = -32768t^2 + 6914048t - 356848007 = 0$ which $t = \text{Res}_{[V(I_2)]}(\theta dz/f_1 f_2)$ satisfies.

4.1 Localization

By using Corollary 3.6, we get an algorithm for computing the point residues of $\theta dz/f_1 \cdots f_n$. Let U_k , V_k and W_k be vector spaces given by

$$U_k := \{u(z_1) \in \mathbb{Q}[z_1] \mid \deg u(z_1) \leq d_k - 1\},$$

$$V_k := \{v(z_1) \in \mathbb{Q}[z_1] \mid \deg v(z_1) \leq d_k - 1, \text{Res}_\alpha \left(\frac{v(z_1)}{f_{1,k} \cdots f_{n,k}} dz \right) = 0, \alpha \in Y_k\},$$

and

$$W_k = \{w(z_1) \in \mathbb{Q}[z_1] \mid \deg w(z_1) \leq d_k - 1, \frac{w(z_1)}{f_{1,k} \cdots f_{n,k}} \text{ has at most simple poles}\}.$$

The dimensions of these spaces are $\dim U_k = d_k$, $\dim V_k = d_k - \nu_k$ and $\dim W_k = \nu_k$, respectively.

Denote by $v_{k,j}(z_1)$ the remainder of $P^* z_1^j$ by $f_{1,k}$, $j = 0, \dots, d_k - \nu_k - 1$. Let $w_{k,\ell}(z_1)$ the remainder of $\text{Jac} \cdot z_1^\ell$ by $f_{1,k}(z_1)$ for $\ell = 0, \dots, \nu_k - 1$. Then we have the next proposition.

Proposition 4.2

- (i) $U_k = V_k \oplus W_k$
- (ii) $V_k = \text{Span}\{v_{k,0}(z_1), \dots, v_{k,d_k-\nu_k-1}(z_1)\}$
- (iii) $W_k = \text{Span}\{w_{k,0}(z_1), \dots, w_{k,\nu_k-1}(z_1)\}$

Let $\underline{\theta}_k(z_1)$ be the remainder of $\underline{\theta}(z_1)$ by $f_{1,k}(z_1)$, where $\underline{\theta}(z_1)$ is the remainder of $\theta(z)$ by I . we can write $\underline{\theta}_k(z_1)$ into

$$\underline{\theta}_k(z_1) = \sum_{j=0}^{d_k-\nu_k-1} a_{k,j} v_{k,j}(z_1) + \sum_{\ell=0}^{\nu_k-1} b_{k,\ell} w_{k,\ell}(z_1)$$

and we have

$$\text{Res}_{\alpha \in Y_k} \left(\frac{\theta}{f_1 \cdots f_n} dz \right) = \text{Res}_{\alpha \in Y_k} \left(\left(\frac{\text{Jac}_k}{f_{1,k} \cdots f_{n,k}} \frac{\sum_{\ell=0}^{\nu_k-1} b_{k,\ell} z_1^\ell}{\prod_{j \neq k} f_{1,j}} \right) dz \right).$$

Thus we have that the residue of $\theta dz/f_1 \cdots f_n$ at $\alpha = (\alpha_1, \dots, \alpha_n) \in Y_k$ is equal to $\mu_k(\sum_{\ell=0}^{\nu_k-1} b_{k,\ell} \alpha_1^\ell / \prod_{j \neq k} f_{1,j}(\alpha_1))$. In other words, for computing residues, we can proceed as follows:

Let J_k be the ideal of $\mathbb{Q}[z, t]$ generated by $f_{1,k}, f_{2,k}, \dots, f_{n,k}$ and $\mu_k \sum_{\ell=0}^{\nu_k-1} b_{k,\ell} z_1^\ell - t \prod_{j \neq k} f_{1,j}$. We obtain an univariate polynomial $\varrho_k(t)$ of t as the generator of $J_k \cap \mathbb{Q}[t]$. Then $\varrho_k(t) = 0$ is the equation for residues of $\theta dz/f_1 \cdots f_n$ at Y_k .

Algorithm 2 (localized version)

Input $g_1(z_1), z_2 - g_2(z_1), \dots, z_n - g_n(z_1) : \text{the shape basis}, \theta(z) \in \mathbb{Q}[z]$
 $\underline{\theta}(z_1) \leftarrow \text{the remainder of } \theta(z) \text{ by } (g_1(z_1), z_2 - g_2(z_1), \dots, z_n - g_n(z_1))$
 $g_{1,1}^{\mu_1}(z_1) \cdots g_{1,N}^{\mu_N}(z_1) \leftarrow \text{the squarefree factorization of } g_1(z_1)$
for k **from** 1 **to** N
 $f_{1,k} \leftarrow g_{1,k}^{\mu_k}$
 $\nu_k \leftarrow \deg g_{1,k}$
 $d_k \leftarrow \mu_k \cdot \nu_k$
 $\underline{\theta}_k \leftarrow \text{the remainder of } \underline{\theta} \text{ by } f_{1,k}$
for i **from** 2 **to** n
 $g_{i,k} \leftarrow \text{the remainder of } g_i \text{ by } f_{1,k}$
 $f_{i,k} \leftarrow z_i - g_{i,k}$
for j **from** 0 **to** $d_k - \nu_k - 1$
 $v_{k,j} \leftarrow \text{the remainder of } -\frac{g_1}{\gcd(g_1, g'_1)} j z_1^{j-1} + \frac{g_1}{\gcd(g_1, g'_1)} \frac{\gcd(g_1, g'_1)'}{\gcd(g_1, g'_1)} z_1^j \text{ by } f_{1,k}$
for ℓ **from** 0 **to** $\nu_k - 1$
 $w_{k,\ell} \leftarrow \text{the remainder of } f_{1,k} z_1^\ell \text{ by } f_{1,k}$
 $\vartheta_k \leftarrow \underline{\theta}_k - \sum_{j=0}^{d_k - \nu_k - 1} a_{k,j} v_{k,j} - \sum_{\ell=0}^{\nu_k - 1} b_{k,\ell} w_{k,\ell}$
 $(a_{k,0}, \dots, a_{k,d_k - \nu_k - 1}, b_{k,0}, \dots, b_{k,\nu_k - 1}) \leftarrow \text{the coefficients s.t. } \vartheta_k = 0$
 $J_k \leftarrow \langle g_{1,k}, f_{2,k}, \dots, f_{n,k}, \gamma_k \sum_{\ell=0}^{\nu_k - 1} b_{k,\ell} z_1^\ell - t \prod_{j \neq k} f_{1,j} \rangle$
 $G_k \leftarrow \text{Gröbner basis of } J_k \text{ w.r.t. the lexicographic order } z \succ t$
Output $\{G_1, \dots, G_N\}$

Example 2 Let us consider the same f_1 and f_2 with Example 1. For $I_1 = \langle x^4, y - (x^3 + 1) \rangle$, $V_1 = \text{Span}\{14x^2 - 3, 12x^3 - 2x, -x^2\}$ and $W_1 = \text{Span}\{4x^3\}$. The remainder $\underline{\theta}_1$ of $\underline{\theta}$ by x^4 is $x^3 - x^2 + 35x$. It can be written as $\underline{\theta}_1 = (211/4)w_{1,0} - (35/2)v_{1,1} + v_{1,2}$. Thus we have $G_1 = \langle t + 211, x, y - 1 \rangle$. In the same way, we have $\underline{\theta}_2 = (211/24)w_{2,0} + (935/512)w_{2,1} + (375/256)v_{2,0} + (459/16)v_{2,1} + (1343/128)v_{2,2} - (513/16)v_{2,3}$. Thus we have $G_2 = \langle -32768t^2 + 6914048t - 356848007, -2805x + 128t - 13504, -2805y + 64t - 3947 \rangle$ for $I_2 = \langle (x^2 - 1)^3, y - (x^3 + 1) \rangle$.

Example 3 Put $z = (x, y)$. Let us consider $f_1 = (x^2 + 1)^{13}(2x^2 - 1)^9$ and $f_2 = y - (3x^6 + 3x^4 - 2x^3 + 2x^2 - 2x + 2)$. The annihilator P of the cohomology class $[1/f_1 f_2]$ given in (A) is $P = (x^2 + 1)(2x^2 - 1)\partial_x + (36x^9 + 42x^7 - 12x^6 + 2x^5 - 10x^4 - 8x^3 + 4x^2 - 4x + 2)\partial_y + 88x^3 + 10x$. Let us compute the residue of $\theta dz / f_1 f_2$, where $\theta = 35x^3 y^5 - 2y^4 + 2xy - 1$. Along the algorithm 2, for $I_1 = \langle (2x^2 - 1)^9, y - (3x^6 + 3x^4 - 2x^3 + 2x^2 - 2x + 2) \rangle$, we have that

$$\text{Res}_{[V(I_1)]} \left(\frac{\theta}{f_1 f_2} dz \right) = \text{Res}_{[V(I_1)]} \left(\frac{\sigma_1}{f_{1,1} f_{1,2}} dz \right),$$

where

$$\sigma_1 = \left(-\frac{747718501}{5036466357} - \frac{126787493190876461}{380240477766549504} x \right) \text{Jac}_1,$$

$\text{Jac}_1 = 9216x^{17} - 36864x^{15} + 64512x^{13} - 64512x^{11} + 40320x^9 - 16128x^7 + 4032x^5 - 576x^3 + 36x$. Thus we have

$$G_1 = \langle 417734204338866689619963936768t^2 - 1612447467044961048518379700224t - 1778835134830001896609499526073, \\ 1826154596005141x + 457019805007872t - 882044634267648, \\ 14609236768041128y - 10968475320188928t - 39094030445746101 \rangle.$$

On the other hand, for $I_2 = \langle (x^2 + 1)^{13}, y - (3x^6 + 3x^4 - 2x^3 + 2x^2 - 2x + 2) \rangle$, we have that $\text{Res}_{[V(I_2)]}(\theta dz / f_1 f_2)$ satisfies

$$\begin{cases} 855519650485998980341686142500864t^2 + 3302292412508080227365641626058752t \\ + 19261767639724614140497003918883305 = 0, \\ 126787493190876461x + 29249267520503808t + 56450856593129472 = 0, \\ y = 0. \end{cases}$$

We can apply our algorithms for any 0-dimensional ideal which has the shape basis even though the given generators are not the shape basis.

Example 4 Let I be the ideal in $\mathbb{Q}[x, y]$ generated by $(x^2 + y^2)^2 + 3x^2y - y^3$, $x^2 + y^2 - 1$. Then I has the shape basis

$$\{16x^6 - 24x^4 + 9x^2, y - (4x^4 - 5x^2 + 1)\}$$

with respect to the lexicographical order $y \succ x$. By the transformation law of the residue ([1]), we have

$$\text{Res}_{\alpha \in Y} \left(\left[\frac{h}{((x^2 + y^2)^2 + 3x^2y - y^3)(x^2 + y^2 - 1)} \right] \right) = \text{Res}_{\alpha \in Y} \left(\left[\frac{h\Delta}{f_1 f_2} \right] \right)$$

for some $h \in \mathbb{Q}[x, y]$, where Y is the set of common zeros of $(x^2 + y^2)^2 + 3x^2y - y^3$ and $x^2 + y^2 - 1$ and $\Delta = -4x^2 + 1$.

Let us compute residues

$$\text{Res}_{\alpha \in Y} \left(\left[\frac{h}{((x^2 + y^2)^2 + 3x^2y - y^3)(x^2 + y^2 - 1)} \right] \right)$$

for $h = 34x^5y + 2x^3y^4 - 3x^2 + 42$. Put $\theta = h\Delta = -136yx^7 + (-8y^4 + 34y)x^5 + 12x^4 + 2y^4x^3 - 171x^2 + 42$.

The annihilator P of the algebraic local cohomology class $[1/f_1 f_2]$ given in (A) is

$$P = x(4x^2 - 3)\partial_x + (-8x^4 + 6x^2)\partial_y + 24x^2 - 6.$$

Put $I_1 = \langle x^2, y - 1 \rangle$ and $I_2 = \langle 16x^4 - 24x^2 + 9, y - x^2 + 5/4 \rangle$. Then we have $\langle f_1, f_2 \rangle = I_1 \cap I_2$.

For I_1 , we have $v_{1,0} = -3$, $w_{1,0} = 2x$ and $\theta_1 = -14v_{1,0}$. Thus the residue

$$\text{Res}_{(0,0)} \left(\left[\frac{h}{((x^2 + y^2)^2 + 3x^2y - y^3)(x^2 + y^2 - 1)} \right] \right)$$

is equal to zero. On the other hand, for I_2 , we have $v_{2,0} = 12x^2 - 3$, $v_{2,1} = 8x^3$, $w_{2,0} = 64x^3 - 48x$, $w_{2,1} = 48x^2 - 36$ and

$$\theta_2 = \frac{213}{512}w_{2,0} + \frac{1}{8}w_{2,1} - \frac{53}{4}v_{2,0} + \frac{101}{32}v_{2,1}.$$

Thus we have $J_2 = \langle 2(213/512 + (1/8)x) - t, 4x^2 - 3, 2y + 1 \rangle$ and

$$G_2 = \langle -12288t^2 + 27264t - 14099, -64x + 192t - 213, 2y + 1 \rangle.$$

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