

UNIQUENESS OF PRODUCTS IN HIGHER ALGEBRAIC K-THEORY

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Let E be a higher algebraic K-theory defined on rings, that is to say, E assigns to each ring R a spectrum ER of algebraic K-theory of R . Fiedorowicz uniqueness theorem [2] says that if E has an external tensor product, then there is a natural map of spectra

$$f : ER \rightarrow GWR$$

which induces an equivalence between (-1) -connected covers of ER and the Gersten-Wagoner spectrum GWR ([3] and [13]). May [6] has given a similar uniqueness theorem for higher algebraic K-theories (or, infinite loop space machines) defined on permutative categories: given an infinite loop space machine E defined on permutative categories, there exists a natural equivalence of spectra between EU and the spectrum $SB\bar{U}$ constructed by Segal [9].

In this article we study the multiplicativity of such natural transformations between various higher algebraic K-theories defined on permutative categories, or exact categories, or rings. Here the term 'multiplicativity' is used in the following sense. Let E and E' be functors $C \rightarrow S$ from

permutative categories (or exact categories, or rings) to CW-spectra, and suppose that E (resp. E') functorially associates to each pairing $U \times V \rightarrow W$ in C a pairing $EU \wedge EV \rightarrow EW$ (resp. $E'U \wedge E'V \rightarrow E'W$) of CW-spectra. Then a natural transformation $f : E \rightarrow E'$ is called multiplicative if the following square commutes in the homotopy category HS ;

$$\begin{array}{ccc} EU \wedge EV & \longrightarrow & EW \\ f \wedge f \downarrow & & \downarrow f \\ E'U \wedge E'V & \longrightarrow & E'W. \end{array}$$

Notice that most of the constructions of products in higher algebraic K-theory, except for May's [7], provide only weak pairings, i.e., pairings in the sense of G. W. Whitehead. This notion of a weak pairing is inadequate for sophisticated spectrum level analysis. Hence we want to find a condition, as generous as possible, which ensures that a given machine functorially associates 'true' pairings. Thus we introduce a notion of a pairing of S_* -spectra which generalizes May's notion of a pairing of I_* -prespectra [7].

We now state the results.

A CW-spectrum $E = \{E_n \mid n \geq 0\}$ is called an S_* -spectrum if each E_n has an action by the symmetric group S_n which is compatible with the structure maps and restricts to a homotopically trivial A_n -action. There is a relevant notion of a pairing of S_* -spectra and we can show that pairings $(E, F) \rightarrow G$ of S_* -spectra functorially determine pairings $E \wedge F \rightarrow G$ in HS .

We use the term higher algebraic K-theory defined on permutative categories to denote a functor E which assigns to every permutative category U a connective CW-spectrum $EU = \{E_n U \mid n \geq 0\}$ together with a natural map $\lambda : BU \rightarrow E_0 U$ such that the composite $BU \rightarrow \Omega^\infty E_\infty U = \bigcup_n \Omega^n E_n U$ is a group completion.

Definition. E is called a multiplicative higher algebraic K-theory if (i) EU has a natural structure of an S_* -spectrum, and (ii) there associated, to every bipermutative functor $f : U \times V \rightarrow W$, a natural pairing $Ef = \{E_{m,n} f\} : (EU, EV) \rightarrow EW$ of S_* -spectra such that the following square commutes;

$$\begin{array}{ccc} BU \wedge BV & \xrightarrow{Bf} & BW \\ \lambda \wedge \lambda \downarrow & & \downarrow \lambda \\ E_0 U \wedge E_0 V & \xrightarrow{E_{0,0} f} & E_0 W. \end{array}$$

Thus a multiplicative higher algebraic K-theory E functorially associates a true pairing $Ef : EU \wedge EV \rightarrow EW$.

It will be shown that both May machine M [7] and Shimada-Shimakawa machine C [10] are multiplicative higher algebraic K-theories defined on permutative categories. (But Segal's machine [9] is not.)

Now our first theorem is

THEOREM A. Let E be a higher algebraic K-theory defined on permutative categories. Then there is a natural equivalence $\gamma : EU \rightarrow CU$ which is multiplicative when E is a multiplicative

higher algebraic K-theory.

Next let K denote the Waldhausen machine [14] which assigns to each exact category U a CW-spectrum $KU = \{BQ^n U^{[n]} \mid n \geq 0\}$ (cf. [11]). Then K associates to any biexact functor $f : U \times V \rightarrow W$ a pairing $Kf : (KU, KV) \rightarrow KW$ of S_* -spectra. (This is essentially the result of [11].) Let us denote by IsU the subcategory of all isomorphisms in a category U .

THEOREM B. There is a multiplicative natural transformation $\kappa : CIsU \rightarrow KU$ defined as the composite of a natural equivalence $\eta : \Omega CQU \cong KU$ with a natural map $\nu : CIsU \rightarrow \Omega CQU$ which deloops the familiar map $BIsU \rightarrow \Omega BQU$.

Note that by the "+ = Q" theorem [4], κ becomes an equivalence if every short exact sequence in U splits.

Finally we consider higher algebraic K-theories defined on rings. We do not know whether Loday's pairing $(GWR, GWR') \rightarrow GW(R \otimes R')$ induces a 'true' pairing $GWR \wedge GWR' \rightarrow GW(R \otimes R')$ or not. However we have

THEOREM C. There exists a functor A from rings to S_* -spectra which satisfies the followings:

(1) A assigns to every pair of rings R and R' a natural pairing $\mu : (AR, AR') \rightarrow A(R \otimes R')$ of S_* -spectra.

(2) For each $n \geq 1$, there exists a natural group completion $f_n : BIsP(S^n R) \rightarrow A_n R \approx K_0 S^n R \times BGLS^n R^+ = GW_n R$ such that

$$\begin{array}{ccc}
\text{BIsP}(S^m R) \wedge \text{BIsP}(S^n R') & \longrightarrow & \text{BIsP}(S^{m+n}(R \otimes R')) \\
\downarrow f_m \wedge f_n & & \downarrow f_{m+n} \\
A_m R \wedge A_n R' & \xrightarrow{\mu_{m,n}} & A_{m+n}(R \otimes R')
\end{array}$$

commutes. (Here $P(R)$ denotes the category of finitely generated projective modules over R .)

(3) The structure map $A_n R \wedge S^1 \rightarrow A_{n+1} R$ is given by the composite

$$A_n R \wedge S^1 \xrightarrow{1 \wedge \iota} A_n R \wedge A_1 Z \xrightarrow{\mu_{n,1}} A_{n+1}(R \otimes Z) = A_{n+1} R$$

where $\iota : S^1 \rightarrow A_1 Z$ represents the standard generator of $K_1 S Z = Z$ (cf. [5, Chapitre II]).

(4) There is a multiplicative natural transformation $\alpha : \text{CIsP}(R) \rightarrow AR$ such that the induced map $\Omega^\infty \text{CIsP}(R) \rightarrow \Omega^\infty A_\infty R$ is an equivalence.

Note that the condition (3) is similar to the description of the structure map of GWR given by Loday [5]. From (2) we see that $\mu_{m,n}$ is weakly homotopic to Loday's map $\text{GW}_m R \wedge \text{GW}_n R' \rightarrow \text{GW}_{m,n}(R \otimes R')$.

As a consequence we have

COROLLARY. The product structures in higher algebraic K-theory of rings constructed by Waldhausen [14], May [7], Shimada-Shimakawa [10], and Loday [5] (modified as in Theorem C) all agree with each other.

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