

The Weierstrass semigroup of a pair and moduli in $\mathcal{M}_3$

Jiryo Komeda (米田二良)
(Collaboration with Seon Jeong Kim)
Kanagawa Institute of Technology

0. Introduction

Let $\mathbb{N}$ be the additive semigroup of non-negative integers. A subsemigroup H of $\mathbb{N}$ is called a *numerical semigroup* if $\#(\mathbb{N} \setminus H) < \infty$. The number $g(H) := \#(\mathbb{N} \setminus H)$ is called the *genus* of H . A certain numerical semigroup of genus g is constructed from a pointed complete non-singular curve of genus g . In this paper we will define a subsemigroup of the additive semigroup $\mathbb{N} \times \mathbb{N}$ like a numerical semigroup. For such a subsemigroup of $\mathbb{N} \times \mathbb{N}$ we can also define its genus. Moreover, in the case where H is such a semigroup of genus 3 we will count the number of the moduli $\mathcal{M}_H$ of curves with a pair of points whose semigroup is H .

1. Numerical semigroups and Weierstrass semigroups

In this section we will review some facts on numerical semigroups which are useful for defining a subsemigroup of $\mathbb{N} \times \mathbb{N}$ like a numerical semigroup. First we give the examples of numerical semigroups of lower genus. For elements $a_1, \dots, a_n \in \mathbb{N}$ we denote by $\langle a_1, \dots, a_n \rangle$ the semigroup generated by $a_1, \dots, a_n$.

Example 1.1. The semigroup $\langle 2, 3 \rangle$ is only one numerical semigroup of genus 1.

Examples 1.2. The semigroups $\langle 3, 4, 5 \rangle$ and $\langle 2, 5 \rangle$ are the numerical semigroups of genus 2.

Examples 1.3. A numerical semigroup H of genus 3 is one of the following semigroups:

H	$\mathbb{N} \setminus H$
$\langle 4, 5, 6, 7 \rangle$	$\{1, 2, 3\}$
$\langle 3, 5, 7 \rangle$	$\{1, 2, 4\}$
$\langle 3, 4 \rangle$	$\{1, 2, 5\}$
$\langle 2, 7 \rangle$	$\{1, 3, 5\}$

The following invariant of a numerical semigroup is important to define a subsemigroup of $\mathbb{N} \times \mathbb{N}$ like a numerical semigroup.

Definition 1.4. Let H be a numerical semigroup. We set

$$c(H) = \text{Min}\{c \in \mathbb{N} \mid c + \mathbb{N} \subseteq H\},$$

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which is called the *conductor* of H .

The number $c(H)$ satisfies the following inequality:

Remark 1.5. $c(H) \leq 2g(H)$ (For example, see Lemma 2.1 (3) in Komeda [5]).

To describe a connection between numerical semigroups and pointed curves we introduce the following notations: Let C be a complete nonsingular irreducible algebraic curve of genus g over an algebraically closed field of characteristic 0, which is called a *curve* in this paper, and $\mathbf{K}(C)$ the field of rational functions on C . For any point P of C we define the set $H(P)$ by

$$H(P) := \{\alpha \in \mathbf{N} \mid \text{there exists } f \in \mathbf{K}(C) \text{ with } (f)_\infty = \alpha P\}.$$

We have the following well-known fact:

Fact 1.6. $H(P)$ is a numerical semigroup of genus g .

Hence, for a fixed numerical semigroup H we consider the moduli $\mathcal{M}_H$ of curves with a point whose semigroup is H .

Definition 1.7. Let $\mathcal{M}_g$ be the moduli variety of curves of genus g . For a numerical semigroup H of genus g we set

$$\mathcal{M}_H = \{[C] \in \mathcal{M}_g \mid \text{there exists } P \in C \text{ such that } H(P) = H\}.$$

If $\mathcal{M}_H \neq \emptyset$, H is called a *Weierstrass semigroup*.

We have the following facts on what numerical semigroups are Weierstrass or not.

Fact 1.8. (1) If $g(H) \leq 3$, then H is Weierstrass (Classical).

(2) If $g(H) = 4$, then H is Weierstrass (Lax [7]).

(3) If $5 \leq g(H) \leq 7$, then H is Weierstrass (Komeda [6]).

(4) For any $g \geq 16$ there exists a non-Weierstrass numerical semigroup of genus g (Buchweitz [2]).

2. Numerical semigroups of a pair

In this section we define a subsemigroup of $\mathbf{N} \times \mathbf{N}$ like a numerical semigroup.

Definition 2.1. A subsemigroup H of $\mathbf{N} \times \mathbf{N}$ is called a *numerical semigroup of a pair of genus g* if it satisfies the following three conditions:

(1) $\mathbf{N} \setminus \{\gamma \in \mathbf{N} \mid (\gamma, 0) \notin H\}$ and $\mathbf{N} \setminus \{\delta \in \mathbf{N} \mid (0, \delta) \notin H\}$ are numerical semigroups of genus g ,

(2) for any $(h_1, h_2) \in \mathbf{N} \times \mathbf{N}$ with $h_1 + h_2 \geq 2g$, we have $(h_1, h_2) \in H$, and

(3) we have a bijection

$$\sigma : \{\gamma \in \mathbf{N} \mid (\gamma, 0) \notin H\} \longrightarrow \{\delta \in \mathbf{N} \mid (0, \delta) \notin H\}$$

such that

$$\mathbf{N} \times \mathbf{N} \setminus H = \bigcup_{\alpha \in \{\gamma \in \mathbf{N} \mid (\gamma, 0) \notin H\}} \left(\{(\alpha, \beta) \mid \beta = 0, 1, \dots, \sigma(\alpha) - 1\} \cup \{(\mu, \sigma(\alpha)) \mid \mu = 0, 1, \dots, \alpha - 1\} \right).$$

In this case the set $\{(\alpha, \sigma(\alpha)) \mid \alpha \in \{\gamma \mid (\gamma, 0) \notin H\}\}$ is called the *generating set* of H , which is denoted by Γ_H . Thus, if $\pi_i : \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$ is the i -th projection for $i = 1, 2$, then $\mathbf{N} \setminus \pi_1(\Gamma_H)$ and $\mathbf{N} \setminus \pi_2(\Gamma_H)$ are numerical semigroups of genus g .

Example 2.2. (1) The semigroup H with generating set $\Gamma_H = \{(1, 1)\}$ is only one numerical semigroup of a pair of genus 1.

(2) A numerical semigroup H of a pair of genus 2 is one of the following types:

Type	Γ_H	$\mathbf{N} \setminus \pi_1(\Gamma_H)$	$\mathbf{N} \setminus \pi_2(\Gamma_H)$
I	$\{(1, 3), (3, 1)\}$	$\langle 2, 5 \rangle$	$\langle 2, 5 \rangle$
II	$\{(1, 2), (3, 1)\}$	$\langle 2, 5 \rangle$	$\langle 3, 4, 5 \rangle$
III	$\{(1, 3), (2, 1)\}$	$\langle 3, 4, 5 \rangle$	$\langle 2, 5 \rangle$
IVa	$\{(1, 1), (2, 2)\}$	$\langle 3, 4, 5 \rangle$	$\langle 3, 4, 5 \rangle$
IVb	$\{(1, 2), (2, 1)\}$	$\langle 3, 4, 5 \rangle$	$\langle 3, 4, 5 \rangle$

Fact 2.3. (Kim [3]) Let C be a curve of genus g and P, Q two distinct points of C . We define

$$H(P, Q) = \{(\alpha, \beta) \in \mathbf{N} \times \mathbf{N} \mid \text{there exists } f \in \mathbf{K}(C) \text{ with } (f)_\infty = \alpha P + \beta Q\}.$$

Then $H(P, Q)$ is a numerical semigroup of a pair of genus g .

Definition 2.4. Let H be a numerical semigroup of a pair of genus g . We set

$$\mathcal{M}_H = \{[C] \in \mathcal{M}_g \mid \text{there exist two distinct points } P \text{ and } Q \text{ of } C \text{ such that } H(P, Q) = H\}.$$

If $\mathcal{M}_H \neq \emptyset$, H is called a *Weierstrass semigroup of a pair*.

Fact 2.5. If H is a numerical semigroup of a pair of genus ≤ 2 , then it is Weierstrass (Kim [3]).

In the case of genus 3 the statement like Fact 2.8 does not hold.

Counterexample 2.6. The numerical semigroup H of a pair of genus 3 with $\Gamma_H = \{(1, 5), (3, 2), (5, 1)\}$ is not Weierstrass.

Proof. We note that $N \setminus \pi_1(\Gamma_H) = \langle 2, 7 \rangle$ and $N \setminus \pi_2(\Gamma_H) = \langle 3, 4 \rangle$. Suppose that H were Weierstrass. Then there exist a curve C and its two distinct points P, Q such that $H(P, Q) = H \supset (\langle 2, 7 \rangle \times \{0\}) \cup (\{0\} \times \langle 3, 4 \rangle)$. Thus, $H(P) = \langle 2, 7 \rangle$, hence C is hyperelliptic, and $H(Q) = \langle 3, 4 \rangle$, hence C is non-hyperelliptic. (It means that

$$\mathcal{M}_{N \setminus \pi_1(\Gamma_H)} \cap \mathcal{M}_{N \setminus \pi_2(\Gamma_H)} = \mathcal{M}_{\langle 2, 7 \rangle} \cap \mathcal{M}_{\langle 3, 4 \rangle} = \emptyset.)$$

This is a contradiction.

Q.E.D.

But we obtain the following result:

Theorem 2.7. *Let H be a numerical semigroup of a pair of genus 3 such that*

$$\mathcal{M}_{N \setminus \pi_1(\Gamma_H)} \cap \mathcal{M}_{N \setminus \pi_2(\Gamma_H)} \neq \emptyset.$$

Then the semigroup H is Weierstrass.

Proof. If $N \setminus \pi_i(\Gamma_H) = \langle 2, 7 \rangle$ for some i , this result is due to Kim [3]. Suppose that $N \setminus \pi_i(\Gamma_H) \neq \langle 2, 7 \rangle$ for $i = 1, 2$. Then we have the following table up to symmetries.

Type	Γ_H	$N \setminus \pi_1(\Gamma_H)$	$N \setminus \pi_2(\Gamma_H)$
I	$\{(1, 5), (2, 2), (5, 1)\}$	$\langle 3, 4 \rangle$	$\langle 3, 4 \rangle$
IIa	$\{(1, 2), (2, 4), (5, 1)\}$	$\langle 3, 4 \rangle$	$\langle 3, 5, 7 \rangle$
IIb	$\{(1, 4), (2, 2), (5, 1)\}$	$\langle 3, 4 \rangle$	$\langle 3, 5, 7 \rangle$
IIIa	$\{(1, 2), (2, 3), (5, 1)\}$	$\langle 3, 4 \rangle$	$\langle 4, 5, 6, 7 \rangle$
IIIb	$\{(1, 3), (2, 2), (5, 1)\}$	$\langle 3, 4 \rangle$	$\langle 4, 5, 6, 7 \rangle$
IVa	$\{(1, 2), (2, 4), (4, 1)\}$	$\langle 3, 5, 7 \rangle$	$\langle 3, 5, 7 \rangle$
IVb	$\{(1, 4), (2, 2), (4, 1)\}$	$\langle 3, 5, 7 \rangle$	$\langle 3, 5, 7 \rangle$
Va	$\{(1, 3), (2, 1), (4, 2)\}$	$\langle 3, 5, 7 \rangle$	$\langle 4, 5, 6, 7 \rangle$
Vb	$\{(1, 2), (2, 3), (4, 1)\}$	$\langle 3, 5, 7 \rangle$	$\langle 4, 5, 6, 7 \rangle$
Vc	$\{(1, 3), (2, 2), (4, 1)\}$	$\langle 3, 5, 7 \rangle$	$\langle 4, 5, 6, 7 \rangle$
VIa	$\{(1, 2), (2, 1), (3, 3)\}$	$\langle 4, 5, 6, 7 \rangle$	$\langle 4, 5, 6, 7 \rangle$
VIb	$\{(1, 3), (2, 1), (3, 2)\}$	$\langle 4, 5, 6, 7 \rangle$	$\langle 4, 5, 6, 7 \rangle$
VIc	$\{(1, 3), (2, 2), (3, 1)\}$	$\langle 4, 5, 6, 7 \rangle$	$\langle 4, 5, 6, 7 \rangle$
VId	$\{(1, 1), (2, 2), (3, 3)\}$	$\langle 4, 5, 6, 7 \rangle$	$\langle 4, 5, 6, 7 \rangle$

We note that every non-hyperelliptic curve of genus 3 can be expressed by a non-singular curve of degree 4 in the projective 2-space $\text{Proj } k[x, y, z]$ through a canonical embedding. For curves C with its points P and Q in the below table we have $H = H(P, Q)$. In fact, the case of Type VIc is trivial, for example, due to Arbarello, Cornalba, Griffiths and Harris [1, VIII Exercises B.7]. Using the Bertini's theorem and elementary calculation, we can easily prove that each curve

is nonsingular for general constants a and b , and that the given points P and Q satisfy $H = H(P, Q)$. Note that the canonical series on each curve in the table are cut out by lines on the plane.

Type	C	P	Q
I	$y^3z - yz^3 - x^4 = 0$	$(0 : 0 : 1)$	$(0 : 1 : 0)$
IIa	$-x^4 + xy^3 + 2yz^3 = 0$	$(0 : 0 : 1)$	$(0 : 1 : 0)$
IIb	$-(x - z)^4 + xy^3 + 2yz^3 = 0$	$(1 : 0 : 1)$	$(0 : 1 : 0)$
IIIa	$yz^3 - x^4 + xy^3 - 2y^2z^2 = 0$	$(0 : 0 : 1)$	$(0 : 1 : 0)$
IIIb	$a(yz^3 - (x - z)^4) + b(xy^3 + y^2z^2) = 0$	$(1 : 0 : 1)$	$(0 : 1 : 0)$
IVa	$-x^3z + xy^3 + 2yz^3 = 0$	$(0 : 0 : 1)$	$(0 : 1 : 0)$
IVb	$-(x - z)^3z + xy^3 + 2yz^3 = 0$	$(1 : 0 : 1)$	$(0 : 1 : 0)$
Va	$a(yz^3 - x^3(x - z)) + by^4 = 0$	$(0 : 0 : 1)$	$(1 : 0 : 1)$
Vb	$a(yz^3 - x^3z) + b(xy^3 + y^2z^2) = 0$	$(0 : 0 : 1)$	$(0 : 1 : 0)$
Vc	$a(yz^3 - (x - z)^3z) + b(xy^3 + y^2z^2) = 0$	$(1 : 0 : 1)$	$(0 : 1 : 0)$
VIa	$a(yz^3 - x^2(x - z)^2) + by^4 = 0$	$(0 : 0 : 1)$	$(1 : 0 : 1)$
VIb	$a(yz^3 - x^2(x - z)(x - 2z)) + by^4 = 0$	$(0 : 0 : 1)$	$(1 : 0 : 1)$
VIc	any curve	general	general

Q.E.D.

Theorem 2.8. *We can count the dimension of the moduli $\mathcal{M}_H$ of curves of genus 3 with a fixed Weierstrass semigroup H of a pair as follows:*

Type	Γ_H	$\dim \mathcal{M}_H$
I	$\{(1, 5), (2, 2), (5, 1)\}$	4
IIa	$\{(1, 2), (2, 4), (5, 1)\}$	4
IIb	$\{(1, 4), (2, 2), (5, 1)\}$	5
IIIa	$\{(1, 2), (2, 3), (5, 1)\}$	5
IIIb	$\{(1, 3), (2, 2), (5, 1)\}$	5
IVa	$\{(1, 2), (2, 4), (4, 1)\}$	5
IVb	$\{(1, 4), (2, 2), (4, 1)\}$	6
Va	$\{(1, 3), (2, 1), (4, 2)\}$	6
Vb	$\{(1, 2), (2, 3), (4, 1)\}$	6
Vc	$\{(1, 3), (2, 2), (4, 1)\}$	6
VIa	$\{(1, 2), (2, 1), (3, 3)\}$	6
VIb	$\{(1, 3), (2, 1), (3, 2)\}$	6
VIc	$\{(1, 3), (2, 2), (3, 1)\}$	6
VIId	$\{(1, 1), (2, 2), (3, 3)\}$	5

Proof. See Kim-Komeda[4].

Q.E.D.

References

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