

Refined Young inequality with Kantorovich constant

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1 Introduction

Throughout this note, A, B are positive operators on a Hilbert space, we use the following notations: $A\nabla_\mu B = (1 - \mu)A + \mu B$, $A\sharp_\mu B = A^{1/2}(A^{-1/2}BA^{-1/2})^\mu A^{1/2}$, and $A!_\mu B = ((1 - \mu)A^{-1} + \mu B^{-1})^{-1}$, see F. Kubo and T. Ando [6]. When $\mu = 1/2$ we write $A\nabla B$, $A\sharp B$ and $A!B$ for brevity, respectively. The Kantorovich constant is defined as $K(t, 2) = \frac{(t+1)^2}{4t}$ for $t > 0$, while the Specht ratio [9] is denoted by

$$S(t) = \frac{t^{\frac{1}{t-1}}}{e \log t^{\frac{1}{t-1}}} \quad \text{for } t > 0, t \neq 1; \quad \text{and} \quad S(1) = \lim_{t \rightarrow 1} S(t) = 1.$$

We start from the famous Young inequality:

$$a\nabla_\mu b \geq a^{1-\mu}b^\mu \quad (1)$$

for positive numbers a, b and $\mu \in [0, 1]$. The inequality (1) is also called a weighted arithmetic-geometric mean inequality and its reverse inequality was given in [10] with the Specht ratio as follows:

$$a\nabla_\mu b \leq S(h)a^{1-\mu}b^\mu \quad (2)$$

for all $\mu \in [0, 1]$, where $0 < m \leq a, b \leq M$ and $h = \frac{M}{m}$.

Recently, an improvement of the inequality (1) was given in [2] as follows:

Theorem F For $a, b > 0$, if $\mu \in [0, 1]$, $r = \min\{\mu, 1 - \mu\}$ and $h = \frac{b}{a}$, then

$$a\nabla_\mu b \geq S(h^r)a^{1-\mu}b^\mu. \quad (3)$$

Based on this, the refined weighted arithmetic-geometric operator mean inequality is given by

$$A\nabla_\mu B \geq S(h^r)A\sharp_\mu B. \quad (4)$$

See [3, 4] for recent developments of the improved Young inequality. See also [5] for another type of improvement for the classical Young inequality.

In this short paper, we improve the inequality (3) via the Kantorovich constant as follows:

$$a\nabla_\mu b \geq K(h, 2)^r a^{1-\mu} b^\mu$$

for all $\mu \in [0, 1]$, where $r = \min\{\mu, 1 - \mu\}$ and $h = \frac{b}{a}$. It admits an operator extension

$$A\nabla_\mu B \geq K(h, 2)^r A\sharp_\mu B$$

for positive operators A, B on a Hilbert space. While we provide a new viewpoint and method which is different from that of the refinement given in [2].

2 Refinement of Young Inequalities

First of all, we cite a refinement of the weighted arithmetic-geometric mean inequality for n positive numbers, which was shown by Pečarić et.al., see [7; Theorem 1, P.717] and also [1, 8].

Lemma 1. Let $x_1, \dots, x_n$ belong to a fixed closed interval $I = [a, b]$ with $a < b$, $p_1, \dots, p_n \geq 0$ with $\sum_{i=1}^n p_i = 1$ and $\lambda = \min\{p_1, \dots, p_n\}$. If f is a convex function on I , then

$$\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \geq n\lambda \left[\sum_{i=1}^n \frac{1}{n} f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \right]. \quad (5)$$

We will use lemma 1 as the following form by applying $f(x) = -\log x$:

Corollary 2. If $x_i \in [a, b]$, $0 < a < b$, $p_1, \dots, p_n \geq 0$ with $\sum_{i=1}^n p_i = 1$ and $\lambda = \min\{p_1, \dots, p_n\}$, then

$$\frac{\sum_{i=1}^n p_i x_i}{\prod_{i=1}^n x_i^{p_i}} \geq \left(\frac{\frac{1}{n} \sum_{i=1}^n x_i}{\prod_{i=1}^n x_i^{\frac{1}{n}}} \right)^{n\lambda}. \quad (6)$$

The case $n = 2$ in (6) is simplified to the following one, which is a loose extension of [2].

Corollary 3. If $a, b > 0$, $\mu \in [0, 1]$, then

$$a\nabla_\mu b \geq K(h, 2)^r a^{1-\mu} b^\mu, \quad (7)$$

where $r = \min\{\mu, 1 - \mu\}$ and $h = \frac{b}{a}$.

Replacing a, b by a^{-1}, b^{-1} , respectively, we have the counterpart of (7) itself.

Corollary 4. If $a, b > 0$ and $\mu \in [0, 1]$, then

$$a^{1-\mu}b^\mu \geq K(h, 2)^r a!_\mu b. \quad (8)$$

Furthermore Corollary 3 implies Theorem F because of the following fact.

Lemma 5. If $t > 0$ and $0 \leq r \leq \frac{1}{2}$, then

$$K(t, 2)^r \geq S(t^r). \quad (9)$$

To prove Lemma 5, we need the following lemma.

Lemma 6. ([2] Lemma 2.3) If $t > 0$ and $t \neq 1$, then

$$\frac{t^{\frac{t}{t-1}}}{e} \leq \frac{t^2 + 1}{t + 1}. \quad (10)$$

Proof. We give it a proof for convenience. By taking logarithms in (10), it is enough to prove that $f(t) = \log(t^2 + 1) - \log(t + 1) - \frac{t}{t-1} \log t + 1 \geq 0$ for $t > 0$ and $t \neq 1$.

Since $f'(t) = \frac{2t}{t^2+1} - \frac{1}{t+1} - \frac{1}{t-1} + \frac{\log t}{(t-1)^2} = \frac{4t}{t^4-1} + \frac{\log t}{(t-1)^2}$, it follows that $f'(t) \leq 0$ for $0 < t < 1$ and $f'(t) \geq 0$ for $t > 1$. Thus we have $f(t) \geq \lim_{t \rightarrow 1} f(t) = 0$ for all $t > 0$ with $t \neq 1$.

□

Proof of Lemma 5. If $t = 1$, then it is easy to get $S(1) = 1 = K(1, 2)$.

If $t > 0$ and $t \neq 1$, then, logarithmic-arithmetic mean inequality implies

$$\frac{t^r - 1}{\log t^r} \leq \frac{t^r + 1}{2} \quad \text{for } 0 \leq r \leq \frac{1}{2}.$$

Combining with (10) we have

$$S(t^r) = \frac{t^{\frac{1}{t^r-1}}}{e} \frac{t^r - 1}{\log t^r} = \frac{1}{t^r} \frac{t^{\frac{t^r}{t^r-1}}}{e} \frac{t^r - 1}{\log t^r} \leq \frac{1}{t^r} \frac{t^{2r} + 1}{t^r + 1} \frac{t^r + 1}{2} = \frac{t^{2r} + 1}{2t^r}.$$

Since $f(x) = x^{2r} (x \geq 0)$ is concave for $0 \leq r \leq \frac{1}{2}$, it follows that

$$\frac{t^{2r} + 1}{2} \leq \left(\frac{t + 1}{2} \right)^{2r} = \left[\frac{(t + 1)^2}{4} \right]^r.$$

Hence we have

$$S(t^r) \leq \frac{t^{2r} + 1}{2} \frac{1}{t^r} \leq \left[\frac{(t + 1)^2}{4t} \right]^r = K(t, 2)^r. \quad \square$$

3 Applications to Operator Young Inequality

Theorem 7. Suppose that two operators A, B and positive real numbers m, m', M, M' satisfy either of the following conditions:

- (i) $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$
- (ii) $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$.

Then

$$A\nabla_\mu B \geq K(h, 2)^r A\sharp_\mu B \quad (11)$$

for all $\mu \in [0, 1]$, where $r = \min\{\mu, 1 - \mu\}$, $h \equiv \frac{M}{m}$ and $h' \equiv \frac{M'}{m'}$.

Proof. From Corollary 3, we have

$$(1 - \mu) + \mu x \geq K(x, 2)^r x^\mu$$

for any $x > 0$. And hence

$$(1 - \mu)I + \mu X \geq \min_{h \leq x \leq h'} K(x, 2)^r X^\mu$$

for the positive operator X such that $0 < hI \leq X \leq h'I$.

Substituting $A^{-1/2}BA^{-1/2}$ for X in the above inequality we have:

In the case of (i), $1 < h = \frac{M}{m} \leq A^{-1/2}BA^{-1/2} \leq \frac{M'}{m'} = h'$, we have

$$(1 - \mu)I + \mu A^{-1/2}BA^{-1/2} \geq \min_{h \leq x \leq h'} K(x, 2)^r (A^{-1/2}BA^{-1/2})^\mu.$$

It is easy to check that $K(x, 2)$ is an increasing function for $x > 1$, then

$$(1 - \mu)I + \mu A^{-1/2}BA^{-1/2} \geq K(h, 2)^r (A^{-1/2}BA^{-1/2})^\mu. \quad (12)$$

In the case of (ii), we have $0 < 1/h' \leq A^{-1/2}BA^{-1/2} \leq 1/h < 1$, then

$$(1 - \mu)I + \mu A^{-1/2}BA^{-1/2} \geq \min_{1/h' \leq x \leq 1/h} K(x, 2)^r (A^{-1/2}BA^{-1/2})^\mu.$$

Since $K(x, 2)$ is a decreasing function for $0 < x < 1$, we have

$$(1 - \mu)I + \mu A^{-1/2}BA^{-1/2} \geq K(1/h, 2)^r (A^{-1/2}BA^{-1/2})^\mu. \quad (13)$$

Multiplying both sides by $A^{1/2}$ to inequality (12) and (13) and using $K(1/h, 2) = K(h, 2)$ for $h > 0$, we obtain the refined arithmetic-geometric operator mean inequality. $\square$

By replacing A, B by A^{-1}, B^{-1} , respectively, then the noncommutative geometric-harmonic mean inequality can be obtained as follows:

Theorem 8. Assume the conditions as in Theorem 7. Then

$$A\sharp_{\mu}B \geq K(h, 2)^r A!_{\mu}B. \quad (14)$$

From Lemma 5, it's easy to get the following

Corollary 9. [2] Assume the conditions as in Theorem 7. Then

$$A\nabla_{\mu}B \geq S(h^r)A\sharp_{\mu}B. \quad (15)$$

In the remainder, we focus on extending the refined weighted arithmetic-harmonic mean inequality to an operator version for another type of improvement.

Lemma 10. If $x_1, \dots, x_n > 0$ and $p_1, \dots, p_n \geq 0$ with $\sum_{i=1}^n p_i = 1$, then

$$\sum_{i=1}^n p_i x_i^{-1} - \left(\sum_{i=1}^n p_i x_i \right)^{-1} \geq n\lambda \left[\sum_{i=1}^n \frac{1}{n} x_i^{-1} - \left(\sum_{i=1}^n \frac{1}{n} x_i \right)^{-1} \right], \quad (16)$$

where $\lambda = \min\{p_1, p_2, \dots, p_n\}$.

Proof. Let $f(x) = x^{-1}$ in lemma 1, then the desired inequality is obtained. $\square$

Theorem 11. If $\mu \in [0, 1]$, A and B are positive operators, then

$$A\nabla_{\mu}B \geq A!_{\mu}B + 2r(A\nabla B - A!B), \quad (17)$$

where $r = \min\{\mu, 1 - \mu\}$.

Proof. From the case $n = 2$ in Lemma 10, we have, for $x > 0$ and $\mu \in [0, 1]$,

$$(1 - \mu) + \mu x^{-1} - ((1 - \mu) + \mu x)^{-1} \geq 2r \left[\frac{1 + x^{-1}}{2} - \left(\frac{1 + x}{2} \right)^{-1} \right].$$

Thus it follows that

$$(1 - \mu)I + \mu T^{-1} \geq ((1 - \mu)I + \mu T)^{-1} + 2r \left[\frac{I + T^{-1}}{2} - \left(\frac{I + T}{2} \right)^{-1} \right] \quad (18)$$

for a strictly positive operator T and $\mu \in [0, 1]$.

We may assume that A, B are invertible. Put $T = A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ in (18), then

$$(1 - \mu)I + \mu(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-1} \geq ((1 - \mu)I + \mu A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-1} \\ + 2r \left[\frac{I + (A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-1}}{2} - \left(\frac{I + A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}}{2} \right)^{-1} \right].$$

Multiplying both sides by $A^{\frac{1}{2}}$ we have

$$(1 - \mu)A + \mu B \geq ((1 - \mu)A^{-1} + \mu B^{-1})^{-1} + 2r \left[\frac{A + B}{2} - \left(\frac{A^{-1} + B^{-1}}{2} \right)^{-1} \right],$$

so that

$$A \nabla_{\mu} B \geq A!_{\mu} B + 2r(A \nabla B - A!B).$$

□

References

- [1] J. Barić, M. Matic, and J. Pečarić, On the bounds for the normalized Jensen functional and Jensen-Steffensen inequality. *Math. Inequal. Appl.*, 12(2009), 413-432.
- [2] S. Furuichi, Refined Young inequalities with Specht's ratio, *ArXiv:1004. 0581v2*.
- [3] S. Furuichi, On refined Young inequalities and reverse inequalities, *ArXiv:1001. 0535v2*.
- [4] T. Furuta, The Hölder-McCarthy and the Young inequalities are equivalent for Hilbert space operators, *Amer. Math. Monthly*, 108(2001), 68-69.
- [5] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrices, *J. Math. Anal. Appl.*, 36(2010), 262-269.
- [6] F. Kubo and T. Ando, Means of positive operators, *Math. Ann.*, 264(1980), 205-224.
- [7] D.S. Mitrinović, J. Pečarić and A.M. Fink, *Classical and new inequalities in analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.
- [8] S. Simić, On a new converse of Jensen inequality. *Publ. de L'institut Math.*, 85(2009), 107-110.
- [9] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.*, 74(1960), 91-98.
- [10] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, 55(2002), 583-588.