

THREEFOLDS OF GENERAL TYPE WITH SMALL INVARIANTS

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1. INTRODUCTION

Varieties of general type, together with Fano varieties and varieties with $\kappa = 0$, are considered to be the building blocks for higher dimensional varieties. In order to have better understanding of varieties of general type, usually people are interested in geography problems, i.e. distribution of invariants, and study some further detailed geometry for those varieties with extremal invariants.

Take surface of general type for example. Noether Inequality asserts that $\text{vol}(S) \geq 2p_g(S) - 4$. Moreover, equality holds if and only if S admits a double covering $\rho : S \rightarrow Y$ over a polarized (possibly singular) surface $(Y, \mathcal{L})$ so that $\mathcal{O}_S(K_S) = \rho^*\mathcal{L}$ and $c_1(\mathcal{L})^2 = h^0(S, \mathcal{L}) - 2$. Indeed, in Horikawa's work (cf. [Hor75]), surfaces satisfying $\text{vol}(S) = 2p_g(S) - 4$, which we call *surfaces on the Noether line*, was intensively studied.

Theorem 1.1. *Let S be a minimal surface on the Noether line. Then S is of the minimal resolution of one of the following surfaces:*

- (1) *double covering of $\mathbb{P}^2$ with branch locus of degree 8;*
- (2) *double covering of $\mathbb{P}^2$ with branch locus of degree 10;*
- (3) *double covering of $\mathbb{F}_e$ with branch locus in $|6\sigma + (2k + 4e + 4)l|$ for some $k \geq 1$ with $p_g = 2k + e + 2$;*
- (4) *double covering of $\mathbb{F}_e$ with branch locus of the form $\sigma + B_0$ in $|6\sigma + (4e + 4)l|$ and $p_g = e + 2$. Moreover $e = 2, 3, 4$.*

In the first three cases, the canonical image is $\mathbb{P}^2$ or $\mathbb{F}_e$ respectively. In the last case, the canonical image is $\mathbb{P}(1, 1, e)$. Moreover, S is simply connected in any of above 4 cases.

Consequently, one has the following interesting corollary:

Corollary 1.2. *Let S be a surface on the Noether line. If $p_g(S) \geq 7$, then the canonical image is non-singular.*

It is thus natural to seek for the three-dimensional or even higher dimensional analogue. In [Kob92], Kobayashi studied higher dimensional varieties of general type.

Theorem 1.3. *Let X be a n -dimensional variety of general type with canonical dimension n . Then $\text{vol}(X) \geq 2(p_g(X) - n)$.*

On the other hand, if the canonical dimension is not maximal, he constructed examples in dimension three so that $p_g(X) = 3e - 2$ and $\text{vol}(X) = 4e - 6$ for some $e \geq 3$ (cf. [Kob92, Proposition 3.2] as follows.

Example 1. Let $W := \mathbb{F}_e$ be a Hirzebruch surface. We take a divisor $D = 2\sigma + 2el$ and consider $Y = \mathbb{P}_W(\mathcal{O} \oplus \mathcal{O}(-D))$.

On Y , let Σ be the section corresponds to $\mathcal{O}(1)$. We take a general divisor $B_0 \in |5\Sigma + 5p^*D|$, where $p : Y \rightarrow W$ is the projection. Let X be the double cover branching along $B_0 + \Sigma$. Computation shows that $p_g(X) = 3e - 2$ and $\text{vol}(X) = 4e - 6$. Notice that X is a threefold with canonical dimension $d_X = 2$.

Inspired by the work of Kobayashi and the Example 1, the best one can hope for threefold of general type is

$$(1) \quad \text{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3}$$

There were various attempts and partial results during the past three decades. The recent work [CHJ25] shows that this inequality holds for any threefold of general type. In the work of [HZ25], more refined description of threefolds near the Noether line are demonstrated. Combining with the recent work of [CHPZ25], one has the following:

Theorem 1.4. *Let X be a minimal threefold of general type with $p_g(X) \geq 3$, then the following holds.*

- (1) *If $p_g(X) \equiv 1 \pmod{3}$, then $\text{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3}$. If the equality holds, then X is Gorenstein. We call such a threefold X on the Noether line.*
- (2) *If $p_g(X) \equiv 2 \pmod{3}$, then $\text{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{19}{6}$. If the equality holds, then X has a non-Gorenstein of type $\frac{1}{2}(1, 1, 1)$. We call such a threefold X on the second Noether line.*
- (3) *If $p_g(X) \equiv 0 \pmod{3}$, then $\text{vol}(X) \geq \frac{4}{3}p_g(X) - 3$. If the equality holds, then X has non-Gorenstein singularity with basket $2 \times \frac{1}{2}(1, 1, 1)$. We call such a threefold X on the third Noether line.*

In the literature, the study of threefolds near the Noether line stemmed from the study of canonical map. Let $d_K(X)$ be the dimension of the canonical map.

Suppose that X is a n -dimensional variety of general type and with canonical dimension n . The joint work of the author and Bangere and Gallego (cf. [BCG23]) investigates varieties of general type with $\text{vol}(X) = 2(p_g(X) - n)$ and $d_K(X) = n$. It is shown that X doubly cover over a polarized variety $(Y, \cdot)$ with minimal degree. Moreover, X is simply connected. Also, if $p_g(X) \geq n + 5$, then the canonical image is non-singular.

Let us consider threefolds of general type from now on. If $d_K(X) = 2$, then one focuses on genus 2 fibration over surfaces since higher genus fibration usually implies that the threefold in question is far away from the Noether line. If $d_K(X) = 1$, then one needs to work on threefolds fibered by surface of general type. Notice that $p_g(X) > 0$ implies that $p_g(F) > 0$. It turns out that most efforts were made on fibration such that general fiber F has $(\text{vol}(F), p_g(F)) = (1, 2)$. It is well-known that the canonical model for a surface of type $(1, 2)$ can be realized as a hypersurface of degree 10 in $\mathbb{P}(1, 1, 2, 5)$. Therefore, it is illuminating to consider a hypersurface of relative degree 10 in a $\mathbb{P}(1, 1, 2, 5)$ -bundle over a curve. We refer the reader to [CP23] for this.

Instead of giving a complete description of the whole story, the purpose of this note is to give a brief review of the developments, focusing on the explicit description of the cases with $p_g = 5$ and 6. We hope that the readers will find this note accessible and maybe instructive instead of working on various technical details.

2. FEASIBLE RESOLUTION OF LINEAR SYSTEM

Let X be a threefold of general type such that $d_K(X) = 2$. We recall the notion of *feasible resolution* introduced in [CC15]. With such explicit resolution, we are able to distill some geometric properties when invariants are small and extreme.

The feasible resolution of $|K_X|$ consists of two parts: first, we have a sequence of w -morphisms (cf. [CH11])

$$X_k \rightarrow \dots \rightarrow X_0 = X$$

resolving $\text{Bs}|K_{X_i}| \cap \text{Sing}(X_i)$, and second we have a sequence of smooth blowup

$$X' = X_t \rightarrow \dots \rightarrow X_k$$

resolving $\text{Bs}|K_{X_i}|$. Then finally we have $\phi : X' \rightarrow Z$ that $\dim Z = 2$.

In each step, $\tau_i : X_i \rightarrow X_{i-1}$ is a divisorial contraction with exceptional divisor F_i . We set an integer r_i to be the Cartier index of $\tau_i(F_i)$ if $\tau_i(F_i)$ is a singular point, otherwise we set $r_i = 1$. Let $D \in |K_X|$ be a general member and D_{X_i} be its proper transform on X_i . We have

$$\begin{aligned} K_{X_i} &= \tau_i^* K_{X_{i-1}} + \frac{a_i}{r_i} F_i; \\ D_{X_i} &= \tau_i^* D_{X_{i-1}} - \frac{m_i}{r_i} F_i, \end{aligned}$$

where a_i , r_i and m_i are positive numbers satisfying one of the following:

- (1) If $i \leq k$, then τ_i is a w -morphism to a point, $a_i = 1$, $m_i \in \mathbb{Z}_{\geq 1}$. The integer r_i is the Cartier index of the point $\tau_i(F_i)$.
- (2) If $k < i \leq t$, then one of the following holds
 - (a) τ_i is a blowup over a smooth point. Then $r_i = 1$, $a_i = 2$ and $m_i \in \mathbb{Z}_{\geq 1}$.
 - (b) τ_i is a blowup along a smooth curve in smooth locus. Then $r_i = 1$, $a_i = 1$ and $m_i \in \mathbb{Z}_{\geq 1}$.

Let E_i be the proper transform of F_i on X' . We can write $\mu_i^* F_i = E_i + \sum_{l>i} \beta_{il} E_l$, where β_{il} are non-negative rational numbers. Then we organize the difference $K_{X'/X} := K_{X'} - \mu^* K_X$ and $D_{X'/X} := \mu^* D - D_{X'}$ as the following:

$$(2) \quad \begin{cases} K_{X'/X} = \sum_{j=1}^t c_j E_j, & \text{where } c_j = \frac{a_j}{r_j} + \sum_{i<j} \frac{a_i}{r_i} \beta_{ij}; \\ D_{X'/X} = \sum_{j=1}^t b_j E_j, & \text{where } b_j = \frac{m_j}{r_j} + \sum_{i<j} \frac{m_i}{r_i} \beta_{ij}. \end{cases}$$

Observation 2.1. We will need the following easy but very useful observation:

$$\begin{cases} b_j \geq c_j & \text{if } j \leq k; \\ 2b_j \geq c_j & \text{for all } j. \end{cases} \quad \dagger$$

We introduce some notations. Let $J_H := \{j \mid \dim \phi_1(E_j) = 2\}$ and $J_V := \{j \mid \dim \phi(E_j) < 2\}$. Let D_H, K_H (resp. D_V, K_V) denote the ϕ -horizontal part (reps. ϕ -vertical part) of $D_{X'/X}$ and $K_{X'/X}$.

Let $t_j := E_j \cdot C$ if $j \in J_H$. Then we have

$$\begin{aligned}\xi &= \sum_{j \in J_H} c_j t_j; \\ 2 &= \sum_{j \in J_H} (b_j + c_j) t_j.\end{aligned}$$

Also, as $|K_{X'}| = |M| + \sum_j (b_j + c_j) E_j$, we have $b_j + c_j \in \mathbb{Z}$, where $|M|$ is the free system defining ϕ . We thus have the following possibilities if $g(C) = 2$:

- i. $|J_H| = 2$. Then $t_j = 1$ and $b_j + c_j = 1$ for all $j \in J_H$. Hence $j \leq k$ and $b_j \geq \frac{1}{2} \geq c_j$.
- ii. $|J_H| = 1$ and $t_j = 1$. Then $b_j + c_j = 2$. If $j > k$, then $b_j, c_j \geq 1$ and hence we have $b_j = c_j = 1$. If $j \leq k$, then $b_j \geq 1 \geq c_j$.
- iii. $|J_H| = 1$ and $t_j = 2$. Then we have $b_j + c_j = 1$ and hence $j \leq k$ and $b_j \geq \frac{1}{2} \geq c_j$.

In particular, we have the following easy estimation.

Lemma 2.2. [Che22, Lemma 3.3, 3.4] *Let C be a general connected fiber of ϕ and $\xi := \mu^* K_X \cdot C$. If $g(C) \geq 3$, then $\xi \geq \frac{4}{3}$. If $g(C) = 2$, then $\xi \geq 1$.*

We will also need the following Lemma.

Lemma 2.3. [Che22, Lemma 4.3] *Suppose that $d_K(X) = 2$ and $g(C) = 2$. Suppose furthermore that $2b_j - c_j \leq 1$ for some $j \in J_H$. Then*

$$\mu^* K_X|_S \cdot E_j|_S \geq \frac{1 + c_j - b_j}{b_j + c_j + 1} t_j (p_g - 2) - \frac{2b_j}{b_j + c_j + 1}.$$

In particular, we have

$$\mu^* K_X|_S \cdot E_j|_S \geq \frac{1}{3} t_j (p_g - 2) - \frac{2}{3}.$$

Remark 2.4. Let $\mu' : X' \rightarrow X_t$ be a higher model extracting no horizontal divisors. Then the general fiber $C' \subset X'$ maps to general fiber $C \subset X_t$. The horizontal divisors E'_j on X' are exactly the proper transforms of horizontal divisors E_j on X_t . Thus the classification of horizontal divisors (as well as the coefficients) are the same. Moreover, over the general member S' of $\mu^*[M]$, we have

$$(\mu \circ \mu')^* K_X|_{S'} \cdot E'_j|_{S'} = (\mu \circ \mu')^* K_X \cdot \mu'^* S \cdot E'_j = \mu^* K_X \cdot S \cdot \mu'_* E'_j = \mu^* K_X|_S \cdot E_j|_S.$$

In short, the classification of horizontal divisors as well as the formula in Lemma 2.3 works on higher model extracting on horizontal divisors.

3. CANONICAL IMAGE

Let $\varphi_K : X \dashrightarrow \mathbb{P}^{p_g-1}$ be the canonical map $d_K(X) := \dim \varphi_K(X)$ which we call the canonical dimension. The purpose of this section is to study the canonical image for threefold with $p_g(X) = 5, 6$ and $d_K(X) = 2$.

As we are interested in threefolds near the Noether line, we will focus on threefolds under the following assumptions.

Assumption $\mathfrak{b}_5$. X is a minimal threefold of general type with $p_g(X) = 5$, $d_K(X) = 2$ and $\text{vol}(X) < 4$.

Assumption $\mathfrak{b}_6$. X is a minimal threefold of general type with $p_g(X) = 6$, $d_K(X) = 2$ and $\text{vol}(X) \leq 5$.

Proposition 3.1. Under the Assumption $\mathfrak{b}_5$, the canonical image is either $\mathbb{F}_1$ or $\mathbb{P}(1, 1, 3)$. Moreover, ϕ is a fibration with general fiber C of $g(C) = 2$.

Proof. Let $\mu : X' \rightarrow X$ be the resolution of indeterminacy of the canonical map so that we have a morphism $\phi : X' \rightarrow Z \subset \mathbb{P}^4$, where Z denotes the canonical image. Let $M := \phi^*\mathcal{O}(1)$ and $S \in |M|$ a general member. We have $\mu^*[K_X] \geq |M|$. Let $\phi : X' \xrightarrow{\Phi} Z' \xrightarrow{\Psi} Z$ be the Stein factorization. Let C be the general fiber of Φ . We define $\xi := \mu^*K_X \cdot C$. Then we have

$$(3) \quad (\mu^*K_X)^3 \geq \mu^*K_X|_S \cdot M|_S \geq (\mu^*K_X \cdot C) \deg(Z) \cdot \deg(\Psi) = \xi \cdot \deg(Z) \cdot \deg(\Psi).$$

By [Che22, Lemma 3.4, 3.5], we know that $\xi \geq 1$. and and by [EH87], a non-degenerate surface of in $\mathbb{P}^4$ has degree ≥ 3 . So under the assumption that $4 > (\mu^*K_X)^3$, Z is a surface of degree 3 and Ψ is an isomorphism. Therefore, ϕ is a fibration and $(Z, \mathcal{O}(1)) \cong (\mathbb{F}_1, \sigma + 2l)$ where $\sigma^2 = -1$ and l is the ruling, or $(Z, \mathcal{O}(1)) \cong (\mathbb{P}(1, 1, 3), \mathcal{O}(1))$ by the classification of varieties with minimal degree (cf. [EH87]).

Moreover, if $g(C) \geq 3$, then $\xi \geq \frac{4}{3}$ and hence $\text{vol}(X) \geq 4$, a contradiction. Thus $g(C) = 2$. $\square$

Lemma 3.2. Under the Assumption $\mathfrak{b}_6$, the canonical image Z is either $\mathbb{P}^2, \mathbb{F}_0, \mathbb{F}_2$ or $\mathbb{P}(1, 1, 4)$. Moreover, ϕ is fibered by $g = 2$ curves.

Proof. By (3), we have now $5 \geq \xi \cdot \deg(Z) \cdot \deg(\Psi)$. Again a surface in $\mathbb{P}^5$ has degree ≥ 4 and $\xi \geq 1$, so $\deg(\Psi) = 1$ and $\deg(Z) \leq 5$. Therefore, ϕ is a fibration. If $g(C) \geq 3$, then $\xi \geq \frac{4}{3}$, a contradiction. Thus, $g(C) = 2$.

It remains to exclude the case that $d := \deg(Z) = 5$. Suppose that $d = 5$ and $g(C) = 2$, then $\xi = 1$. Recall that the ϕ -horizontal divisors can be classified into three cases. Under the condition that $\xi = \sum_{j \in J_H} b_j t_j = 1$, one has $b_j = c_j$ for any $j \in J_H$. Hence, $2b_j - c_j \leq 1$ in each case. Therefore, by Lemma 2.3, we have

$$\mu^*K_X \cdot M \cdot E_j \geq \frac{1}{3}(p_g - 2)t_j - \frac{2}{3} \geq \frac{2}{3} > 0.$$

In particular, $\mu^*K_X \cdot M \cdot E_H > 0$. However, we have

$$5 \geq \mu^*K_X^3 = \mu^*K_X^2 \cdot M = \mu^*K_X \cdot M^2 + \mu^*K_X \cdot M \cdot E_H \geq d\xi + \mu^*K_X \cdot M \cdot E_H > 5,$$

a contradiction.

Now Z is a surface of degree 4 in $\mathbb{P}^5$. Therefore, $(Z, \mathcal{O}(1)) \cong (\mathbb{F}_0, \sigma + 2l)$, or $(\mathbb{F}_2, \sigma + 3l)$, or $(\mathbb{P}(1, 1, 4), \mathcal{O}(4))$, or $(\mathbb{P}^2, \mathcal{O}(2))$ by the classification of varieties with minimal degree (cf. [EH87]). $\square$

Following Horikawa's notation, we say that a threefold X under the Assumption b_5, b_6 is of type (n) (resp. (n') , (∞)) if the canonical image is $\mathbb{F}_n$ (resp. $\mathbb{P}(1, 1, n)$, $\mathbb{P}^2$).

Indeed, if $Z \cong \mathbb{P}(1, 1, n)$ (resp. $\mathbb{P}^2$) then we consider $\tau : \mathbb{F}_n \rightarrow \mathbb{P}(1, 1, n)$ its minimal resolution (resp. $\tau : \mathbb{F}_1 \rightarrow \mathbb{P}^1$ a blowup). There is a higher model $\mu' : X' \rightarrow X_t$ resolving the indeterminacy of $\tau^{-1} \circ \phi : X_t \dashrightarrow \mathbb{F}_n$. Let $p : \mathbb{F}_n \rightarrow \mathbb{P}^1$ be the bundle map and $f : X' \rightarrow \mathbb{P}^1$ be the composition. If $Z \cong \mathbb{F}_n$ then τ and μ' are identity. Therefore, we always have the following diagram for some n .

$$(4) \quad \begin{array}{ccccc} & & \pi & & \\ & & \curvearrowright & & \\ & X' & \xrightarrow{\mu'} & X_t & \xrightarrow{\mu_t} & X \\ & \downarrow \psi & & \downarrow \phi & & \\ f \swarrow & \mathbb{F}_n & \xrightarrow{\tau} & Z & & \\ & \downarrow p & & & & \\ & \mathbb{P}^1 & & & & \end{array}$$

Notice that μ' is a birational morphism extracting no horizontal divisors. Also, the general fiber of ψ maps to general fiber of ϕ . Therefore, we can compute horizontal divisors and their intersections on X' as in the feasible resolution X_t (see also Remark 2.4).

Lemma 3.3. *Assume b_5 or b_6 . There is an induce morphism $f : X' \rightarrow \mathbb{P}^1$ as above. Let F be the general fiber of this induced morphism. Then F is a surface of type $(\text{vol}, p_g) = (1, 2)$.*

Proof. Notice first that $h^0(F, K_F) \geq h^0(X', K_{X'}|_F) \geq 2$, otherwise the canonical map has 0-dimensional image along general fiber F , which is a contradiction. Then we need to prove that $\text{vol}(F) = 1$. To this end, we use the argument in [HZ25, Prop. 2.1]. We reproduce the arguemnt for the reader's convenience.

Notice that $\pi^*K_X \geq M = \mu'^*\phi^*\mathcal{O}(1) = \psi^*\tau^*\mathcal{O}(1) = \psi^*(a\sigma + ml)$ for some positive integer $a \geq 1$ and $m \geq 2$.¹ Thus we have $\psi^*ml \leq M \leq \pi^*K_X$, and hence $\pi^*K_X - mF \geq 0$. Let $\sigma : F \rightarrow F_0$ be the contraction to its minimal model. By [CCJ20, Corollary 2,3], $(1 + \frac{1}{m})\pi^*K_X - \sigma^*K_{F_0}$ is $\mathbb{Q}$ -effective. In particular,

$$C \cdot \sigma^*K_{F_0} \leq C \cdot \frac{m+1}{m}\pi^*K_X = \frac{m+1}{m}\xi < 2$$

as $\frac{m+1}{m} \leq \frac{3}{2}$ and $\xi < \frac{4}{3}$ by (3). Hence

$$C \cdot \sigma^*K_{F_0} = \sigma_*(C) \cdot K_{F_0} \leq 1.$$

Notice also that

$$K_F \geq \pi^*K_X|_F \geq M|_F \geq C.$$

Thus

$$K_{F_0}^2 = \sigma^*K_{F_0} \cdot K_F \geq \sigma^*K_{F_0} \cdot C = \sigma_*(C) \cdot K_{F_0}.$$

¹In fact, $(a, m) = (1, 2)$ (resp. $(1, 3)$) if $p_g(X) = 5$ and $Z = \mathbb{F}_1$ (resp. $\mathbb{P}(1, 1, 3)$). Also, if $p_g = 6$ then $(a, m) = (1, 2)$ (resp. $(1, 3), (1, 4), (2, 2)$) if $Z = \mathbb{F}_0$ (resp. $\mathbb{F}_2, \mathbb{P}(1, 1, 4), \mathbb{P}^2$).

By Hodge Index Theorem and parity, we have $K_{F_0}^2 = (\sigma_* C)^2 = 1$ and $K_{F_0} = \sigma_* C$. Hence $\text{vol}(F) = 1$ and therefore $p_g(F) \leq 2$ by the Noether Inequality for surfaces of general type. Hence $p_g(F) = 2$ and we are done. $\square$

Proposition 3.4. Under the Assumption b_6 , the canonical image can not be $\mathbb{P}^2$.

Proof. Suppose that $Z \cong \mathbb{P}^2$, then $K_{X'} \geq M = \psi^*(2\sigma + 2l)$. Pushing forward the inclusion $\psi^*\mathcal{O}_{\mathbb{P}^1}(2\sigma + 2l) \hookrightarrow \omega_{X'}$, we have an inclusion

$$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \hookrightarrow f_*\omega_{X'}.$$

However, $f_*\omega_{X'}$ has rank 2 by Lemma 3.3. Thus we have the desired contradiction. $\square$

By the classification of canonical image, we have $\psi^*\mathcal{O}_{\mathbb{F}_n}(\sigma + ml) \hookrightarrow \omega_{X'}$ for some $4 \geq m \geq n$. Therefore, we have inclusion $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(m - n) \hookrightarrow f_*\omega_{X'}$. Hence $f_*\omega_{X'} = \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)$ for some $d_1 \geq m$ and $d_2 \geq m - n$. However, $d_1 + d_2 + 2 = h^0(X', K_{X'}) = h^0(\mathbb{F}_1, \sigma + ml) = 2m - n + 2$. One sees that $f_*\omega_{X'} = \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(m - n)$. To sum up, we have the following explicit decomposition.

Corollary 3.5. Assume b_5 or b_6 . Then $f_*\omega_{X'} = \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(m - n)$ for some $4 \geq m \geq n$.

Proposition 3.6. Under the assumption b_5 or b_6 , we have $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$. Hence $\chi(\mathcal{O}_X) = 1 - p_g(X)$.

Proof. Since these cohomologies are birational invariants, we may compute them on X' instead.

By Kollár's work ([Kol86]), we know that $R^i f_*\omega_X$ are torsion free for $i = 1, 2$ and hence $R^1 f_*\omega_X = 0$ as its has rank $h^1(F, \omega_F) = 0$. We also have $R^2 f_*\omega_X = \omega_{\mathbb{P}^1}$. By Lemma 3.3, we have $f_*\omega_{X'} = \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)$ for some $d_1, d_2 \geq 0$.

Then easy Leray spectral sequence computation shows that

$$\begin{aligned} h^1(X, \mathcal{O}_X) &= h^2(X, \omega_{X'}) = h^0(\mathbb{P}^1, R^2 f_*\omega_{X'}) + h^1(\mathbb{P}^1, R^1 f_*\omega_{X'}) = 0, \\ h^2(X, \mathcal{O}_X) &= h^1(X, \omega_{X'}) = h^0(\mathbb{P}^1, R^1 f_*\omega_{X'}) + h^1(\mathbb{P}^1, f_*\omega_{X'}) = 0. \end{aligned}$$

It follows immediately that $\chi(\mathcal{O}_X) = 1 - p_g(X)$. $\square$

In order to find the lower bound of canonical volume, we need to analyze the ϕ -horizontal divisors in greater detail.

Lemma 3.7. Under the Assumption b_5 or b_6 , there is a unique ϕ -horizontal divisor E_0 in X' such that $\psi|_{E_0} : E_0 \rightarrow \mathbb{F}_n$ is birational.

Proof. By the classification of horizontal divisor at the end of section 2, we aim to prove that only Case ii. is possible.

Recall that [Che22, Lemma 4.3] now reads,

$$\pi^* K_X|_S \cdot E_j|_S \geq \frac{1 + c_j - b_j}{b_j + c_j + 1} 3t_j - \frac{2b_j}{b_j + c_j + 1} > 0.$$

In Case iii, we have

$$\begin{aligned} \text{vol}(X) &\geq (p_g - 2)\xi + b_j \pi^* K_X|_S \cdot E_j|_S \geq 2(p_g - 2)b_j + b_j \left(\frac{1+c_j-b_j}{2} 2(p_g - 2) - \frac{2b_j}{2} \right) \\ &\geq 2(p_g - 2)b_j + \frac{1}{2}((p_g - 2)(1 + c_j) - (p_g - 1)b_j) = \frac{1}{2}(2p_g - 4 + (2p_g - 5)b_j) \\ &\geq \frac{1}{4}(6p_g - 13), \end{aligned}$$

where we used $b_j \geq \frac{1}{2}$ in the third and fourth inequality. One sees that $\text{vol}(X) \geq \frac{17}{4}$ (resp. $\frac{23}{4}$) if $p_g(X) = 5$ (resp. $p_g(X) = 6$). Thus we can exclude this case.

In Case i, we have similarly

$$\begin{aligned} \text{vol}(X) &\geq \sum_{i=1}^2 (p_g - 2)b_{j_i} + b_{j_i} \left(\frac{1+c_{j_i}-b_{j_i}}{2} (p_g - 2) - \frac{2b_{j_i}}{2} \right) \\ &\geq \sum_{i=1}^2 (p_g - 2)b_{j_i} + \frac{1}{4}(p_g - 2)(1 + c_{j_i}) - p_g b_{j_i} \\ &= \sum_{i=1}^2 \frac{1}{4}(2(p_g - 2) + b_{j_i}(2p_g - 6)) \geq \frac{1}{2}(3p_g - 7), \end{aligned}$$

where we used $b_{j_1}, b_{j_2} \geq \frac{1}{2}$ in the second and third inequality. One sees that $\text{vol}(X) \geq 4$ (resp. $\frac{11}{2}$) if $p_g(X) = 5$ (resp. $p_g(X) = 6$). Thus we can also exclude this case.

Therefore, only Case ii is possible. In Case ii, there is a unique ϕ -horizontal divisor, denoted E_0 , so that $C \cdot E_0 = 1$. Hence $\psi|_{E_0} : E_0 \rightarrow \mathbb{F}_n$ is birational. $\square$

4. CANONICAL VOLUME

We study the lower bound of canonical volume in this section. Let us recall some known results.

Proposition 4.1. Let X be a variety of general type with $d_K(X) = \dim X$, then $\text{vol}(X) \geq 2(p_g(X) - \dim X)$. In particular, $\text{vol}(X) \geq 4$ (resp. $p_g(X) = 6$) for threefolds with $p_g(X) = 5$ (resp. $p_g(X) = 6$) and $d_K(X) = 3$.

Also, the recent work of [CHJ25] shows that

Proposition 4.2. Let X be a threefold of general type with $p_g(X) = 5$ (resp. 6) and $d_K(X) = 1$, then $\text{vol}(X) \geq \frac{109}{30}$ (resp. $\frac{61}{12}$).

Theorem 4.3. Assume b_5 or b_6 , then we have

$$\text{vol}(X) \geq (p_g(X) - 2) + \frac{1}{2} \left\lceil \frac{2}{3}(p_g - 4) \right\rceil.$$

Proof. Let S be the general member of $|M|$. One sees that $\phi_S := \psi|_S : S \rightarrow \gamma \cong \mathbb{P}^1$ (resp. $\phi_S := \phi|_S : S \rightarrow \gamma \cong \mathbb{P}^1$) is a surface fibered by genus 2 curves with general fiber C , where $\gamma \in |\sigma + 3l|$ in $\mathbb{F}_3$ (resp. $|\sigma + 2l|$ in $\mathbb{F}_1$) if the canonical image is $\mathbb{P}(1, 1, 3)$ (resp. $\mathbb{F}_1$). Since $\Gamma_S := S \cap E_0$ is a smooth irreducible curve birational to γ , Γ_S is a section and hence $g(\Gamma_S) = 0$.

We have $\pi^* K_X = M + \xi E_0 + D_V$ and $K_{X'/X} = (2 - \xi)E_0 + K_V$, where D_V and K_V are ψ -vertical. By adjunction, we have

$$K_S = (K_{X'} + S)|_S \equiv 2dC + 2\Gamma_S + \Delta_S,$$

where $\Delta_S := (D_V + K_V)|_S$ is ψ -vertical.

Let $\sigma : S \rightarrow S_0$ be the contraction to its minimal model and Γ_{S_0} (resp. C_0) be the image of Γ_S (resp. C). By [CCJ20, Corollary 3.7], we have

$$\pi^* K_X|_S \sim_{\mathbb{Q}} \frac{1}{2} \sigma^* K_{S_0} + \Delta'_S,$$

for some effective Δ'_S . It follows that

$$2\xi = 2\pi^* K_X \cdot C \geq \sigma^* K_{S_0} \cdot C = K_{S_0} \cdot C_0.$$

Pushing forward K_S via σ , we get

$$(5) \quad K_{S_0} \equiv 2dC_0 + 2\Gamma_{S_0} + \sigma_*(\Delta_S).$$

Therefore, $\frac{8}{3} > 2\xi \geq K_{S_0} \cdot C_0 \geq 2dC_0^2$. It follows that $C_0^2 = 0$. This implies that $\phi_S : S \rightarrow \mathbb{P}^1$ descends to a fibration $\phi_{S_0} : S_0 \rightarrow \mathbb{P}^1$ so that $\phi_S = \phi_{S_0} \circ \sigma$. In particular, Γ_{S_0} is a section of ϕ_{S_0} .

Then we have

$$-2 = K_{S_0} \cdot \Gamma_{S_0} + \Gamma_{S_0}^2 = \frac{3}{2} K_{S_0} \cdot \Gamma_{S_0} - d - \frac{1}{2} \sigma_* \Delta_S \cdot \Gamma_{S_0},$$

where the last equality follows from (5). In particular, $K_{S_0} \cdot \Gamma_{S_0} \geq \frac{2}{3}(d-2)$ and hence

$$\sigma^* K_{S_0} \cdot \Gamma_S = K_{S_0} \cdot \Gamma_{S_0} \geq \lceil \frac{2}{3}(d-2) \rceil$$

We thus have

$$\pi^* K_X \cdot M \cdot E_H = \xi \pi^* K_X \cdot \Gamma_S \geq \frac{\xi}{2} \sigma^* K_{S_0} \cdot \Gamma_S \geq \frac{\xi}{2} \lceil \frac{2}{3}(d-2) \rceil \geq \frac{1}{2} \lceil \frac{2}{3}(d-2) \rceil.$$

Therefore,

$$(6) \quad \text{vol}(X) = \pi^* K_X^3 \geq \pi^* K_X \cdot M^2 + \pi^* K_X \cdot M \cdot E_H \geq \xi + \frac{1}{2} \lceil \frac{2}{3}(d-2) \rceil \geq p_g(X) - 2 + \frac{1}{2} \lceil \frac{2}{3}(p_g(X) - 4) \rceil.$$

This completes the proof. $\square$

Corollary 4.4. *Let X be a threefold of general type with $p_g(X) = 5$, then $\text{vol}(X) \geq \frac{7}{2}$. If X is a threefold of general type with $p_g(X) = 6$, then $\text{vol}(X) \geq 5$.*

Proof. This follows from Proposition 4.1, 4.2 and Theorem 4.3. $\square$

Theorem 4.5. *Let X be a threefold of general type with $p_g(X) = 5$ and $\text{vol}(X) = \frac{7}{2}$. Then $P_2(X) = 14$ and X has a unique non-Gorenstein singularity of type $\frac{1}{2}(1, 1, 1)$.*

Proof. Suppose that $p_g(X) = 5$ and $\text{vol}(X) = \frac{7}{2}$. Then $d_K(X) = 2$ by Proposition 4.1 and 4.2. Hence $d_K(X) = 2$ and Assumption b_5 holds. In particular, $\chi(\mathcal{O}_X) = -4$ (cf. Proposition 3.6). The singular Riemann-Roch yields

$$P_2(X) = -3\chi(\mathcal{O}_X) + \frac{1}{2} \text{vol}(X) + l_2 = 12 + \frac{7}{4} + l_2,$$

where $l_2 := \sum_{(r,b) \in \mathcal{B}(X)} \frac{b(r-b)}{2r}$. Clearly, $P_2(X) \geq 14$ and $P_2(X) = 14$ if and only if $l_2 = \frac{1}{4}$ if and only if the basket of X consists of one $\frac{1}{2}(1, 1, 1)$. By classification of terminal

singularities, the only singularities with such basket is of type $\frac{1}{2}(1, 1, 1)$. Thus, it is sufficient to prove that $P_2(X) \leq 14$.

Next, we follow the line of [HZ25, Proposition 3.7]. For the reader's convenience, we outline their proof.

Set $|M_0| := \text{Mov}|2\pi^*K_X|$, $|M_1| := \text{Mov}|2\pi^*K_X - M|$, and $|M_2| := \text{Mov}|2\pi^*K_X - 2M|$. Replacing X' by higher model, we may assume that M_i are free for all i . Let $u_i := \dim \text{Im}(H^0(X', M_i) \rightarrow H^0(S, M_i|_S))$, for $i = 0, 1$.

1. $u_0 \leq \lfloor 2K_X^3 \rfloor + 2$.

Let ϕ_0 be the morphism defined by $|M_0|_S|$. Depending on the dimension of image of ϕ_0 , we have

$$\begin{cases} 4(\pi^*K_X|_S) \geq (|M_0|_S)^2 \geq 2(h^0(S, M_0|_S) - 4), & \text{if } \dim \phi_0(S) = 2; \\ 2(\pi^*K_X|_S)^2 \geq M_0|_S \cdot \pi^*K_X|_S \geq (h^0(S, M_0|_S) - 1)\pi^*K_X \cdot C, & \text{if } \dim \phi_0(S) = 1. \end{cases}$$

Since $K_X^3 \geq (\pi^*K_X|_S)^2$ and $u_0 \leq h^0(S, M_0|_S)$ is an integer, we have the required estimate.

2. $u_1 \leq \lfloor 2K_X^3 - \frac{5(p_g(X)-1)}{3} \rfloor + 4$.

Now we consider ϕ_1 to be the morphism defined by $|M_1|_S|$.

If $\dim \phi_1(S) = 1$, then ϕ_1 defines the same pencil as ϕ_S since $M_1|_S \geq M|_S$. Thus we have $M_1|_S \equiv bC$ for some $b \geq h^0(S, M_1|_S) - 1$ and $M|_S \equiv dC$ where $d = p_g(X) - 2$.

Then one consider

$$D := K_{S_0} - \frac{2(d-2)}{3}C_0 \equiv \frac{2}{3}((2d+2)C_0 + 3\Gamma_{S_0} + \frac{3}{2}\sigma_*(\Delta_S)).$$

By checking its intersection with C_0, Γ_{S_0} and $\sigma_*(\Delta)$ (note that Δ is ϕ_S -vertical), one sees that D is nef. Then $D \cdot (2\pi_X^K - (b+d)C) \geq 0$, which implies that

$$2\mu^*K_X^3 \geq \pi^*K_X|_S \cdot \sigma^*K_{S_0} \geq b + \frac{5}{3}d - \frac{4}{3},$$

since $\mu^*K_X \cdot C = 1$ and $K_{S_0} \cdot C_0 = \sigma^*K_{S_0} \cdot C = 2\mu^*K_X|_S \cdot C = 2$. This leads to the desired upper bound since $u_1 \leq h^0(S, M_1|_S) \leq b + 1$ is an integer.

If $\dim \phi_1(S) = 2$, then $(M_1|_S) \geq 2h^0(S, M_1|_S) - 4$ and hence

$$4(\pi^*K_X|_S)^2 \geq (M_1|_S + M|_S)^2 \geq 2h^0(S, M_1|_S) - 4 + 2(M_1|_S \cdot M|_S).$$

As $(M_1|_S)|_C$ is free on the genus 2 curve C , one sees that $M_1|_S \cdot C \geq 2$. Therefore, $M_1|_S \cdot M|_S \geq 2(p_g(X) - 2)$. It follows that $u_1 \leq h^0(S, M_1|_S) \leq 2K_X^3 + 6 - 2p_g(X)$.

3. $h^0(X', 2\pi^*K_X - 2M) = 1$.

Suppose that $h^0(X', 2\pi^*K_X - 2M) \geq 2$. Then one proves that $2\pi^*K_X \geq 2M + F$. Then

$$K_X^3 \geq \pi^*K_X^2 \cdot M + \frac{1}{2}\pi^*K_X^2 \cdot F.$$

By (6), $K_X^3 = \pi^*K_X^2 \cdot M$. However, $\pi^*K_X^2 \cdot F \geq \delta\sigma_F^*K_{F_0}^2 > 0$ for some $\delta > 0$, where $\sigma_F : F \rightarrow F_0$ is the contraction to its minimal model. This is a contradiction. $\square$

Theorem 4.6. *Let X be a threefold of general type with $p_g(X) = 6$ and $\text{vol}(X) = 5$. Then $P_2(X) = 18$ and X has basket of singularities of type $2 \times \frac{1}{2}(1, 1, 1)$.*

Proof. Suppose that $p_g(X) = 6$ and $\text{vol}(X) = 5$. Then similarly Assumption $\mathfrak{b}_6$ holds and the singular Riemann-Roch yields

$$P_2(X) = -3\chi(\mathcal{O}_X) + \frac{1}{2}\text{vol}(X) + l_2 = 15 + \frac{5}{2} + l_2,$$

where $l_2 := \sum_{(r,b) \in \mathcal{B}(X)} \frac{b(r-b)}{2r}$. Clearly, $P_2(X) \geq 18$. Easy computation of l_2 shows that $l_2 = \frac{1}{2}$ if and only if the basket of X consists of $2 \times \frac{1}{2}(1, 1, 1)$.

Similarly, one sees that $u_0 \leq 12$, $u_1 \leq 5$ and $h^0(X', 2\pi^*K_X - 2M) = 1$. Therefore, $P_2(X) \leq 18$ as desired. $\square$

Remark 4.7. By the classification of terminal singularities, the singularities of a threefold with basket of type $2 \times \frac{1}{2}(1, 1, 1)$ belongs to one of the following:

- 2 cyclic quotient singularities of type $\frac{1}{2}(1, 1, 1)$.
- one singularity of type $cA/2$ analytically locally given by $xy + f(z^2, u) = 0 \subset \mathbb{A}^4/\frac{1}{2}(1, 1, 1, 0)$ with $\deg(f) = 2$ and $u^2 \in f$.
- one singularity of type $cAx/2$ analytically locally given by $x^2 + y^2 + f(z, u) = 0 \subset \mathbb{A}^4/\frac{1}{2}(0, 1, 1, 1)$ with f purely of even degree ≥ 4 .

5. EXAMPLES

The purpose of this section is to provide some examples with $(\text{vol}(X), p_g(X)) = (\frac{7}{2}, 5)$ or $(5, 6)$.

Example 2. We start with $W = \mathbb{F}_1$. Over W , we take a divisor $D = 2\sigma + 3l$ and a rank 2 vector bundle $\mathcal{E} = \mathcal{O}_W \oplus \mathcal{O}_W(-D)$. Let $p : Y = \mathbb{P}_W(\mathcal{E}) \rightarrow W$ be the associated $\mathbb{P}^1$ -bundle.

On Y , let Σ_2 be the distinguished section and $B_0 \in |5\Sigma_2 + 10p^*\sigma + 16p^*l|$ be a general member. The linear system is free, hence B_0 is non-singular and $\gamma := B_0 \cap \Sigma_2$ is the ruling of $\Sigma_2 \cong \mathbb{F}_1$.

Next, we consider $Y_1 \rightarrow Y$ the blowup along γ with exceptional divisor F . Let $\tilde{B}_0$ and $\tilde{\Sigma}_2$ be the proper transform of B_0 and Σ_2 respectively. Then We construct a double covering $\rho : X_1 \rightarrow Y_1$ with branch divisor $\tilde{B}_0 + \tilde{\Sigma}_2$. Easy computation on ramification formula shows that

$$K_{X_1} \sim_{\mathbb{Q}} \rho^*\tau_1^*(K_Y + \frac{1}{2}(B_0 + \Sigma_2)) = \rho^*\tau_1^*(\Sigma_2 + p^*(\sigma + 2l)).$$

One sees that $p_g(X_1) = 5$. Moreover, one can compute $h^0(X_1, mK_{X_1})$ for all m to get $\text{vol}(X_1) = \frac{7}{2}$.

Indeed, let T and E be the pre-image of $\tilde{\Sigma}_2$ and F respectively. Then we have decomposition $K_{X_1} = T + \frac{1}{2}E + P$, where $P \equiv \rho^*\tau_1^*(\frac{1}{2}\Sigma_2 + p^*(\sigma + 2l))$ is nef and big with $P^3 = \frac{7}{2}$.

The minimal model program contracts T to a curve Γ as the ruling of T is the only K -negative curve. We get a divisorial contraction $\pi_1 : X_1 \rightarrow X_0$. The image of E in X_0 , denoted $\bar{E}$, is isomorphic to $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-2)$. The next step of the minimal model program $\pi_0 : X_0 \rightarrow X$ contracting $\bar{E}$ to a point $p \in X$ of type $\frac{1}{2}(1, 1, 1)$. One can

verify that $K_{X_1} = \pi_1^* \pi_0^*(K_X) + T + \frac{1}{2}E$. Therefore, $\pi_1^* \pi_0^*(K_X) = P$ is nef and big. This implies in particular that X is indeed a minimal model of general type.

Alternatively, let X'_0 be the Stein factorization of $\tau_1 \circ : X_1 \rightarrow Y$. One sees that $\pi'_1 : X_1 \rightarrow X'_0$ contracts E to a canonical singularity along a ruling of $\pi'_1(T)$. The map $\rho' : X'_0 \rightarrow Y$ is the branch cover along $B_0 + \Sigma_2$. As a summary, we have the following diagram.

$$(7) \quad \begin{array}{ccccc} & X_1 & \xrightarrow{\rho} & Y_1 & \\ \pi_1 \swarrow & & \searrow \pi'_1 & & \downarrow \tau_1 \\ X_0 & & X'_0 & \xrightarrow{\rho'} & Y \\ \pi_0 \searrow & & \swarrow \pi'_0 & & \downarrow p \\ & X & & & W \end{array}$$

Example 3. Let $W = \mathbb{F}_3$. We consider $D \sim 2\sigma + 5l$, where σ denotes the negative section and l is the ruling. Let $\mathcal{E} = \mathcal{O}_W \oplus \mathcal{O}_W(-D)$ and $p : Y = \mathbb{P}_W(\mathcal{E}) \rightarrow W$ be a $\mathbb{P}^1$ -bundle.

On Y , we pick a general member $B_0 \subset |5\Sigma_2 + p^*(10\sigma + 26l)|$ and then we construct a double covering $X'_0 \rightarrow Y$ with branch divisor $B := B_0 + \Sigma_2$. Let T'_0 be the preimage of Σ_2 in X'_0 . Contract the ruling of T'_0 , we get a minimal model X . Similar computation shows that $p_g(X) = 5$ and $\text{vol}(X) = \frac{7}{2}$.

One can see the canonical image is $\mathbb{P}(1, 1, 3)$ factoring through $W = \mathbb{F}_3$.

Remark 5.1. In [HZ25, Proposition 5.8], they consider minimal models $X_{e,k}$ with $e \geq 3$ and $k \geq 0$. The conditions on e, k are not sharp in lower genus case. In fact, our first example here corresponds to $e = 1$ and $k = 0$. Our second example here corresponds to $e = 3$ and $k = -1$.

Example 4. The examples with $p_g(X) = 6$ and $\text{vol}(X) = 5$ can be constructed similarly. Take $(e, k, a) = (0, 1, 2), (2, 0, 4), (4, -1, 6)$ respectively. We consider Y , which is a $\mathbb{P}^1$ -bundle over $W = \mathbb{F}_e$. Take general member $B_0 \in |5\Sigma_2 + p^*(10\sigma + 2bl)|$ with $b = 6, 11, 16$ respectively. Then we construct a double cover $\rho' : X'_0 \rightarrow Y$ branching along $B_0 + \Sigma_2$. Now $B_0 \cap \Sigma = \gamma_1 \cup \gamma_2$, where γ_i s are distinct rulings of Σ_2 . Let $Y_1 \rightarrow Y$ be the blowups of γ_1 and γ_2 with exceptional divisors F_1, F_2 . Let E_i be the preimage of F_i in X_1 . One sees similarly that the minimal model program contracts T first and then contracts E_1 and E_2 . Thus, X has two singularities of type $\frac{1}{2}(1, 1, 1)$.

Notice that the canonical image is $\mathbb{F}_e$ if $e = 0$ or 2 . If $e = 4$, the canonical image is $\mathbb{P}(1, 1, 4)$ factoring through $W = \mathbb{F}_4$.

We would like to remark that if one chooses B_0 particularly so that $B_0 \cdot \Sigma_2 = 2\gamma$. With more involved computation, one gets X with singularity of type $cA/2$ or $cAx/2$ (with basket $2 \times \frac{1}{2}(1, 1, 1)$) instead.

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