

ON MEROMORPHIC CONVEX AND STARLIKE FUNCTIONS

By

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Abstract

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad \text{in } |z| < 1.$$

Then it is well known that [1, 3]

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{2} \quad \text{in } |z| < 1.$$

Corresponding the above theorem, if $f(z) = 1/z + \sum_{n=0}^{\infty} a_n z^n$ is analytic in the punctured disk $0 < |z| < 1$ and

$$\operatorname{Re} \left[-\left(1 + \frac{zf''(z)}{f'(z)}\right) \right] > 0 \quad \text{in } |z| < 1,$$

then there exists no positive constant $\alpha > 0$ for which

$$\operatorname{Re} \left[-\frac{zf'(z)}{f(z)} \right] > \alpha \quad \text{in } |z| < 1.$$

1. Introduction.

Let Σ denote the class of function of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$$

which are regular in the punctured disk $E = \{z : 0 < |z| < 1\}$.

A function $f(z) \in \Sigma$ is called meromorphic starlike of order α ($0 \leq \alpha < 1$) if

$$\operatorname{Re} \left[-\frac{zf'(z)}{f(z)} \right] > \alpha$$

for all $z \in U = \{z : |z| < 1\}$.

We denote by $\Sigma^*(\alpha)$ the subclass of Σ consisting of functions which are meromorphic starlike of order α in U .

Further, a function $f \in \Sigma$ is called meromorphic convex of order α ($0 \leq \alpha < 1$) if

$$\operatorname{Re} \left[-\left(1 + \frac{zf''(z)}{f'(z)}\right) \right] > \alpha$$

for all $z \in U$.

We denote by $\Sigma_c(\alpha)$ the subclass of Σ consisting of functions which are meromorphic convex of order α in U .

2. Preliminaries.

Lemma 1. [1, Theorem 1] Let $p(z)$ be regular in U , $p(0) = 1$ and suppose that there exists a point $z_0 \in U$ such that

$$\operatorname{Re} p(z) > 0 \quad \text{for } |z| < |z_0|,$$

$$\operatorname{Re} p(z_0) = 0 \quad \text{and } p(z_0) \neq 0.$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is real and $|k| \geq 1$.

Lemma 2. Let $p(z)$ be regular in U , $p(0) = 1$ and

$$(1) \quad \operatorname{Re} \left(p(z) - \frac{zp'(z)}{p(z)} \right) > 0 \quad (z \in U),$$

then

$$\operatorname{Re} p(z) > 0 \quad (z \in U).$$

Then this result is sharp for the function $p(z) = \frac{1+z}{1-z}$.

Proof. First, we want to prove $p(z) \neq 0$ ($z \in U$).

If $p(z)$ has a zero of order n ($n \geq 1$) at a point z_0 ($z_0 \neq 0$), then $p(z)$ can be written as $p(z) = (z - z_0)^n g(z)$ ($g(z_0) \neq 0$, $g(z)$ is regular in U), and it follows that

$$p(z) - \frac{zp'(z)}{p(z)} = (z - z_0)^n g(z) - \frac{nz}{z - z_0} - \frac{zg'(z)}{g(z)}.$$

When z approaches to z_0 on the line segment satisfying the conditions $\arg z = \arg z_0 = \theta$ and $|z_0| < |z| < 1$, we have

$$\begin{aligned} & \lim_{\substack{z \rightarrow z_0 \\ \arg z = \arg z_0, |z_0| < |z| < 1}} \operatorname{Re} \left(p(z) - \frac{zp'(z)}{p(z)} \right) \\ &= \lim_{\substack{z \rightarrow z_0 \\ \arg z = \arg z_0, |z_0| < |z| < 1}} \operatorname{Re} \left((z - z_0)^n g(z) - \frac{nz}{z - z_0} - \frac{zg'(z)}{g(z)} \right) \\ &= \text{negative infinite real value,} \end{aligned}$$

because we have

$$\begin{aligned} & \lim_{\substack{z \rightarrow z_0 \\ \arg z = \arg z_0, |z_0| < |z| < 1}} \left(\arg \left(-\frac{nz}{z - z_0} \right) \right) \\ &= \lim_{\substack{z \rightarrow z_0 \\ \arg z = \arg z_0, |z_0| < |z| < 1}} \left(\arg(-1) + \arg nz - \arg(z - z_0) \right) \\ &= \pi + \theta - \theta = \pi. \end{aligned}$$

This result contradicts (1).

Therefore we have

$$p(z) \neq 0 \quad (z \in U).$$

If there exists a point $z_0 \in U$ such that

$$\operatorname{Re} p(z) > 0 \quad \text{for } |z| < |z_0|,$$

$$\operatorname{Re} p(z_0) = 0 \quad \text{and } p(z_0) \neq 0,$$

then from Lemma 1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is real and $|k| \geq 1$.

For the case $p(z_0) = ia$ ($a > 0$), we have

$$\operatorname{Re} \left(\frac{z_0 p'(z_0)}{p(z_0)} - p(z_0) \right) = \operatorname{Re}(ik - ia) = 0.$$

This contradicts our assumption.

For the case $p(z_0) = -ia$ ($a > 0$), applying the same method as the above, we have

$$\operatorname{Re} \left(\frac{z_0 p'(z_0)}{p(z_0)} - p(z_0) \right) = 0.$$

This contradicts our assumption.

Therefore we complete our proof.

The result is sharp for the function $p(z) = \frac{1+z}{1-z}$.

3. Main result.

Theorem. If $f(z) \in \Sigma_c(0)$, then $f(z) \in \Sigma^*(0)$, and there exists no positive constant $\alpha > 0$ such that $\Sigma_c(0) \subset \Sigma^*(\alpha)$.

Proof. Setting

$$p(z) = -\frac{zf'(z)}{f(z)},$$

then we have $p(0) = 1$ and

$$-\left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z) - \frac{zp'(z)}{p(z)}.$$

From the assumption of theorem, we have

$$\operatorname{Re}\left[-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] = \operatorname{Re}\left[p(z) - \frac{zp'(z)}{p(z)}\right] > 0 \quad \text{in } U,$$

then from Lemma 2, we have

$$\operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}p(z) > 0 \quad \text{in } U.$$

Next, we prove that there exists no positive constant $\alpha > 0$ such that $\Sigma_c(0) \subset \Sigma^*(\alpha)$. Because the extremal function of Lemma 2 is

$$p(z) = \frac{1+z}{1-z},$$

so we put

$$-\frac{zf'(z)}{f(z)} = \frac{1+z}{1-z}.$$

Then by a brief calculation, we have

$$\frac{f'(z)}{f(z)} = -\frac{1}{z} - \frac{2}{1-z}.$$

Adding $1/z$ to both sides and integrating from zero to z ($0 < |z| < 1$), we have

$$\int_0^z \left(\frac{1}{z} + \frac{f'(z)}{f(z)}\right) dz = -\int_0^z \frac{2}{1-z} dz,$$

and it follows that

$$f(z) = \frac{(1-z)^2}{z}.$$

This function belong to $\Sigma_c(0)$ and $\Sigma^*(0)$ but there exists no positive constant $\alpha > 0$ for which $f(z) \in \Sigma^*(\alpha)$.

References

- [1] A. Marx, Untersuchungen über schlichte Funktionen, Math. Ann., 107(1932), 40-67.
- [2] M. Nunokawa, On properties of Non-Carathéodory functions, Proc. Japan Acad. 68 (1992), 152-153.
- [3] E. Strohäcker, Beiträge zur Theorie der schlichten Funktionen, Math. Z., 37(1933), 356-380.

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