

A random search for discrete robust design optimization of linear-elastic steel frames under interval parametric uncertainty

Bach Do^{1*} and Makoto Ohsaki¹

¹Department of Architecture and Architectural Engineering, Graduate School of Engineering, Kyoto University, Kyoto-Daigaku Katsura, Nishikyo, Kyoto 615-8540, Japan.

(*Corresponding author email: do.bach.35a@st.kyoto-u.ac.jp)

Abstract

This study presents a new random search method for solving discrete robust design optimization (RDO) problem of planar linear-elastic steel frames. The optimization problem is formulated with an explicit objective function of discrete design variables, unknown-but-bounded uncertainty in material properties and external loads, and black-box constraint functions of the structural responses. Radial basis function (RBF) models serve as approximations of structural responses. The anti-optimization problem is approximated by an RBF-based optimization problem that is solved using an adaptive strategy coupled with a difference-of-convex functions algorithm. The adaptive strategy is embedded in an iterative process for solving the upper-level optimization problem. This process starts with a set of candidate solutions and iterates through selecting the best candidate among the available candidates and generating new promising candidates by performing a small random perturbation around the best solution found so far to refine the RBF approximations. It terminates when the number of iterations reaches an upper bound, and outputs the optimal solution that is the best solution obtained through the optimization process. Two test problems and two design examples demonstrate that the exact optimal or a good approximate solution can be found by the proposed method with a few trials of the algorithm.

Keywords: Discrete robust design optimization; Random search; Interval uncertainty; Radial basis function; Adaptive optimization strategy; Frame structure.

1 Introduction

The interaction between buildings with their surrounding environment and with human activities leads structural engineering design to be a place full of uncertainty that is associated with material properties, external loads, boundary conditions, and structural geometry as well as human errors involved in the design, construction, and operation of structures [1]. This problem has been addressed in structural design by traditional use of partial factors since the late 1970s, when structural design philosophy changed from the allowable stress design to the load and resistance factor design [2]. These factors simultaneously scale up the effects of external loads on structural members and scale down the resistance capacity of these members to produce a conservative design, which guarantees that the scaled resistance capacity should exceed the sum of the scaled relevant load effects [3]. This design approach, however, may suffer from lack of invariance of safety measure, that is, the set of the partial factors may not be unique for a given failure mode [4]. Furthermore, the partial factors do not demonstrate their ability to well address the economic aspect of a structure. This arises from the fact that the derivations of these factors only consider the tradeoff between the structural safety and economy in a general context, based on the properties of similar structures rather than rigorous calculations for a particular structure. Structural optimization considering uncertainty, therefore, has become a powerful tool in recent years for addressing the tradeoff between structural safety and economy. It consists of three primary branches, namely, reliability-based design optimization (RBDO), risk-based optimization (RBO), and robust design optimization (RDO).

The RBDO explicitly takes into account uncertainty to minimize a structural cost function under constraints on allowable probabilities of structural failure, while the expected cost of failure is ignored [5–9]. By contrast, the RBO considers the structural failure cost by moving the constraints of the RBDO problem to its objective function and formulates an unconstrained optimization problem [10]. Meanwhile, the RDO aims at finding an optimal design of the structure that is robust against the effects of uncertainty [11–13], which is the focus of this study.

Depending on the description of uncertainty, the RDO can be formulated using either probabilistic or non-probabilistic approaches. In the probabilistic approaches, uncertainty information is described by probability density functions, or equivalently, by cumulative distribution functions of the random parameters. Then, known input distributions of the random parameters are propagated through the mechanical model for obtaining quantities associated with the output distribution, such as the mean and variance of the objective and constraint functions. Mathematical methods to the uncertainty propagation include the Monte-Carlo simulation (MCS), Taylor series expansion [14], polynomial chaos expansion [15], and Bayes-Hermite quadrature [16]. When statistical information to identify the distributions of the uncertain parameters is insufficient, it is natural to capture all uncertainty scenarios by an uncertainty set rather than assumptions on specific probability distributions, which forms the foundation of the non-probabilistic approaches [17]. Two classical non-probabilistic approaches involve the objective discrepancy minimization [18] and the worst-case scenario optimization [19–21]. The former approach formulates a bi-objective optimization problem that attempts to simultaneously minimize the nominal objective function and the discrepancy of the objective values in a case with uncertainty and a nominal case. Meanwhile, the latter approach, which is of interest in this study, is to find a minimum worst value of the objective function under its feasibility against all realizations of the random parameters within the uncertainty set. This approach is associated with a mathematical program that first defines a reasonable uncertainty set, e.g., box, ellipsoidal, or polyhedral set, and then reformulates the original problem to a computationally tractable problem based upon the defined set [22]. However, since the exact worst value can be found only for convex optimization problems, the mathematical program is not suitable for many structural design problems, where the structural responses are complex nonlinear functions of the uncertain parameters.

To further enlarge the application of the non-probabilistic approaches to real-world engineering problems, one can use surrogate models to approximate the objective and/or constraint functions and solve the original problem based on these approximations. For example, Marzat et al. [23] incorporated Kriging-based modeling and an expected improvement (EI) criterion into the scenario approach for solving a worst-case global optimization problem. The problem was formulated for a finite set of

uncertain parameters, utilizing the Kriging objective function and the EI criterion for selecting the next sampling points in the design variable space where the objective function and accuracy of the Kriging objective function are expected to improve [24]. Similarly, ur Rehman et al. [25] and ur Rehman and Langelaar [26] used Kriging and the EI metric for solving an unconstrained surrogate-based RDO problem with a maximum of ten design variables. A novel EI criterion was devised to determine the worst value of the Kriging objective function and sequentially locate the next sampling points in the design and random variable spaces for updating the approximate objective function. However, as the design optimization problem of a steel frame is discrete in nature [27,28], the aforementioned studies are not applicable to the optimization problem considered in this study because the EI criterion has been considered as a function of continuous design variables.

Due to the above limitations of the mathematical program, it is not surprising that most of the RDO problems formulated for steel frames belong to the probabilistic approaches, while only few studies concerning the non-probabilistic approaches are available. In the probabilistic perspective, when the input variables are endowed with specified probability distributions, the quest is pursued to efficiently propagate the uncertainty and obtain the output distribution. Doltsinis and Kang [29] evaluated the mean and variance values of the structural response in static and dynamic problems using the second-order perturbation based stochastic finite element analysis. Papadrakakis et al. [30] combined a neural network and an evolutionary algorithm for minimizing the total mass and standard deviation of the maximum inter-story drift of a steel frame against seismic motions. Lagaros and Papadrakakis [31] used the MCS to estimate the standard deviation of the structural responses required for solving the RDO problem of a steel frame. Csébfalvi and Lógó [32] formulated a volume-constrained expected-compliance minimization problem for a truss structure and solved it under uncertainty in the loading directions using a symbolic computation software for calculating the expected value of the truss compliance. Recently, Do and Ohsaki [33] used Gaussian mixture model coupled with the Taylor series expansion for propagating aleatory uncertainty in external loads and material properties to the responses of steel frame structures under static loads. In the non-probabilistic perspective, solving the anti-optimization problem for finding the worst response is still a challenge [34], especially for a

combinatorial problem with many black-box constraint functions. Ohsaki and Katsura [35] introduced a concept for the anti-optimization problem where the accuracy of the solution is defined as the order rather than the global worst value among possible combinations of the uncertain-but-bounded parameters; therefore, only approximate solutions have been found. Ohsaki et al. [36] used the order statistics of quantile responses to approximate the worst value. Other approaches based on sensitivity region information [37,38] or conventional interval analysis [39] do not demonstrate their applicability to discrete RDO problems. Still, to the best of the authors' knowledge, there is a knowledge gap on an approach that directly addresses the global optimal and anti-optimal solutions to the discrete RDO problem of structures, which is a common problem in practice. This study attempts to fill this gap.

In particular, we formulate the following RDO problem for a steel frame:

$$\begin{aligned}
& \min_{\mathbf{s}} C(\mathbf{s}) \\
& \text{subject to} \\
& \quad g_i(\mathbf{s}, \mathbf{r}) \leq 0, i = 1, \dots, n \\
& \quad h_j(\mathbf{s}) \leq 0, j = 1, \dots, m \\
& \quad \mathbf{s}_L \leq \mathbf{s} \leq \mathbf{s}_U, \forall \mathbf{r} \in \mathcal{R}
\end{aligned} \tag{1}$$

where $\mathbf{s} \in \mathbb{N}^{k_1}$ is the vector of the discrete design variables; $\mathbf{s}_L$ and $\mathbf{s}_U$ indicate the lower and upper bounds of $\mathbf{s}$, respectively; $C(\mathbf{s}): \mathbb{N}^{k_1} \rightarrow \mathbb{R}$ represents the total mass of the structure; $\mathbf{r} \in \mathcal{R} \subseteq \mathbb{R}^{k_2}$ is the vector of the random parameters, for example, the magnitude of external loads and mechanical properties of the steel material; $\mathcal{R} = [\mathbf{r}_0 - \alpha_L \Delta \mathbf{r}, \mathbf{r}_0 + \alpha_U \Delta \mathbf{r}]$ denotes an interval uncertainty set characterized by a nominal vector $\mathbf{r}_0$ and two positive scalars α_L and α_U ; $g_i(\mathbf{s}, \mathbf{r}): \mathbb{N}^{k_1} \times \mathbb{R}^{k_2} \rightarrow \mathbb{R}$ and $h_j(\mathbf{s}): \mathbb{N}^{k_1} \rightarrow \mathbb{R}$ are respectively the semi-infinite and deterministic constraint functions associated with the strength, serviceability, and geometric requirements for designing the structure.

A necessary condition for a given candidate solution $\mathbf{s}$ being a feasible solution to problem (1) is that the worst values of $g_i(\mathbf{s}, \mathbf{r})$ for all $\mathbf{r} \in \mathcal{R}$ are non-positive. In other words, the global minimum of the following anti-optimization problem should be non-negative before the feasibility of $\mathbf{s}$ is verified:

$$\begin{aligned}
& \min_{\mathbf{r}}. -g(\mathbf{s}, \mathbf{r}) \\
& \text{subject to } \mathbf{r} \in \mathcal{R}
\end{aligned} \tag{2}$$

As all the constraint functions $g_i(\mathbf{s}, \mathbf{r})$ are treated equally, we drop the subscript to simplify the exposition.

Since checking the inequality $h_j(\mathbf{s}) \leq 0$ is a trivial task when the deterministic constraint functions are explicitly given, the confirmation for the feasibility of $\mathbf{s}$ mainly depends on the accuracy of finding the global minimum of problem (2). This problem formulated for a steel frame has a complex nonlinear objective function $g(\mathbf{s}, \mathbf{r})$, which is implicit within a finite element analysis program. Therefore, most of the existing methods are not applicable for solving the problem because they require the function $g(\mathbf{s}, \mathbf{r})$ should be convex with respect to the random parameters $\mathbf{r}$. Although one can use metaheuristic techniques, e.g., genetic algorithm [40] or particle swarm optimization [41], for solving problem (2), these techniques are expensive in terms of function evaluations, especially when many random parameters are considered. Thus, to solve problem (2) using an intelligent way ignoring unpromising region in random parameter space, we first approximate the black-box constraint functions $g_i(\mathbf{s}, \mathbf{r})$ by the corresponding radial basis function (RBF) models $\hat{g}_i(\mathbf{s}, \mathbf{r})$, derived as shown in Appendix A, and formulate an approximate optimization problem for a given $\mathbf{s}$. Here, to enable the construction of $\hat{g}_i(\mathbf{s}, \mathbf{r})$, we assume that the constraint functions $g_i(\mathbf{s}, \mathbf{r})$ are continuous with respect to both $\mathbf{s}$ and $\mathbf{r}$, even though the design variables $\mathbf{s}$ are discrete. Then, we devise an adaptive optimization strategy to sequentially reduce the difference in the solutions to the approximate problem and problem (2). At each iteration of the adaptive process, the approximate problem is solved using a difference-of-convex functions algorithm (DCA) since the RBF model can be decomposed as the difference of two convex functions. The resulting solution serves as the next sampling point to refine the approximate problem. Finally, to solve problem (1), we extend the work by Müller et al. [42] to discrete design variables and develop an iterative optimization process. This process starts with a set of integer candidate solutions and iterates through selecting the best solution among the available candidates and locally generating new promising candidates around the best solution found so far for improving the accuracy of the RBF

models until the limit for number of iterations is reached. The global optimal solution is approximated by the best solution found once the optimization process has terminated. Furthermore, the solution by the proposed method is compared with that by other methods in the literature or verified by the MCS.

The remainder of this paper is organized as follows. Section 2 presents the adaptive optimization strategy for solving problem (2). Section 3 describes the optimization procedure for solving problem (1). In Section 4, numerical tests for a two-bar truss and a welded beam are carried out to verify the performance of the proposed method. Section 5 applies the proposed method to two planar steel frames. Section 6 summarizes and concludes the paper. Theoretical aspects of the RBF model are illustrated in Appendix A. The semi-infinite constraint functions $g_i(\mathbf{s}, \mathbf{r})$ associated with the serviceability and strength requirements for frame designs are briefly described in Appendix B.

2 DCA using RBF models

As mentioned in Introduction, the candidate solution $\mathbf{s}$ is feasible for problem (1) if $h_j(\mathbf{s}) \leq 0$ and the global minima of the constraint functions $-g_i(\mathbf{s}, \mathbf{r})$ are non-negative over the parameter set $\mathcal{R}$. Since the confirmation of the former requirement is a trivial task, checking the latter requirement is the central focus of this section, where problem (2) is solved based on the RBF model $\hat{g}(\mathbf{s}, \mathbf{r})$ of the black-box constraint function $g(\mathbf{s}, \mathbf{r})$.

An interesting property of $\hat{g}(\mathbf{s}, \mathbf{r})$ is that it can be defined as the difference of two convex functions, i.e., $\hat{g}(\mathbf{s}, \mathbf{r}) = l(\mathbf{s}, \mathbf{r}) - h(\mathbf{s}, \mathbf{r})$, where $l(\mathbf{s}, \mathbf{r})$ and $h(\mathbf{s}, \mathbf{r})$ at a given $\mathbf{s}$ are both convex with respect to $\mathbf{r}$; see Eqs. (A.4)-(A.6) in Appendix A. Graphically, Fig. 1 shows an additive decomposition of a specified RBF function $\hat{g}(\mathbf{s}, \mathbf{r})$ into a convex function $l(\mathbf{s}, \mathbf{r})$ and a concave function $-h(\mathbf{s}, \mathbf{r})$. Based on this, problem (2) can be reformulated as

$$\begin{aligned} & \min_{\mathbf{r}} h(\mathbf{s}, \mathbf{r}) - l(\mathbf{s}, \mathbf{r}) \\ & \text{subject to } \mathbf{r} \in \mathcal{R} \end{aligned} \tag{3}$$

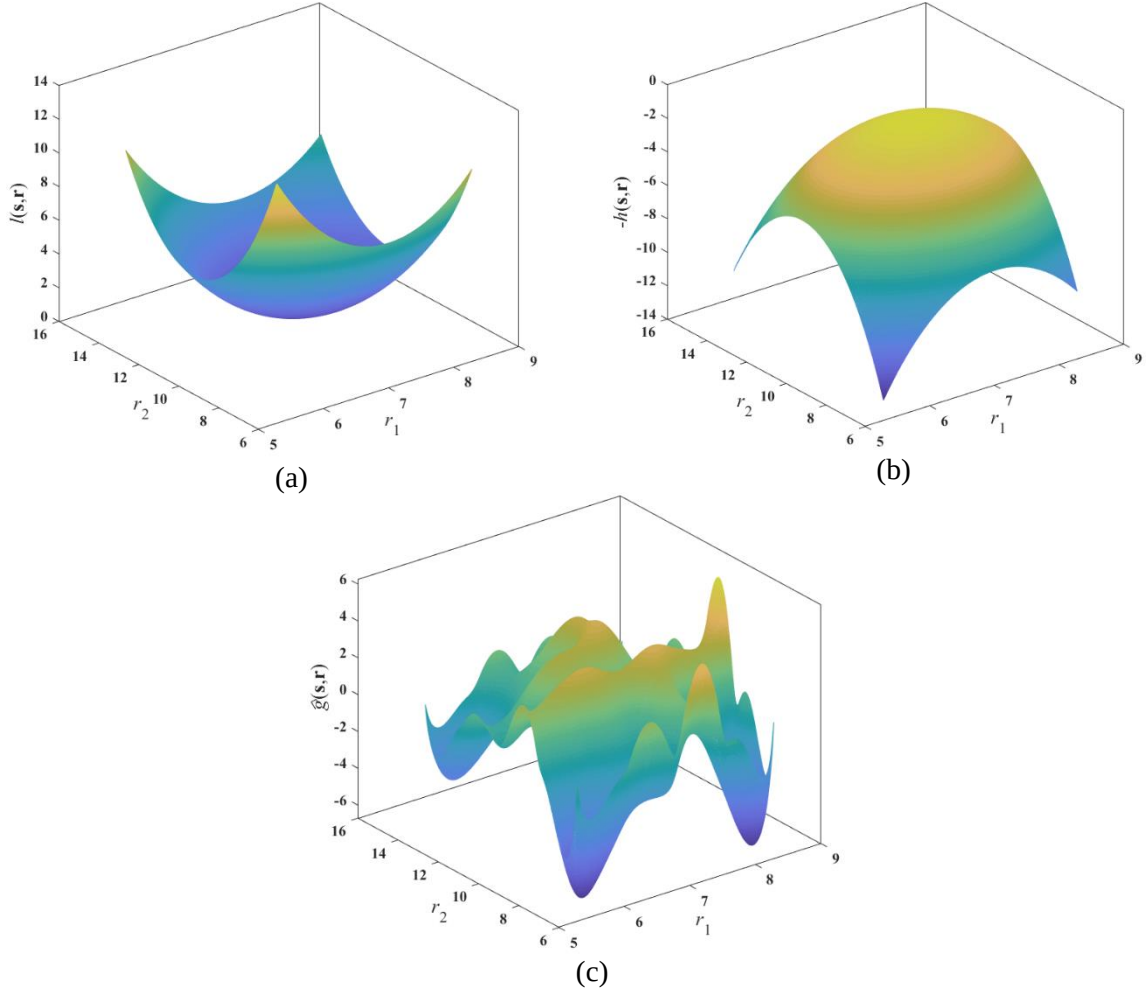


Fig. 1 Decomposition of an RBF model for a given value of $\mathbf{s}$: (a) convex function $l(\mathbf{s}, \mathbf{r})$; (b) concave function $-h(\mathbf{s}, \mathbf{r})$; (c) RBF model $\hat{g}(\mathbf{s}, \mathbf{r}) = l(\mathbf{s}, \mathbf{r}) - h(\mathbf{s}, \mathbf{r})$.

Problem (3) is solved using the DCA, which has been proved to be a robust and efficient algorithm for dealing with non-convex and especially large-scale optimization problems [43]. The DCA is an iterative method that guarantees to offer a local optimal solution. However, it usually converges to the global optimal solution with an appropriate initial starting point [44]. At its core, the DCA approximates problem (3) by a sequence of convex problems in which the concave term $-l(\mathbf{s}, \mathbf{r})$ at the t th iteration is replaced by its affine majorization using its gradient $\mathbf{y}^{(t)} = \nabla_{\mathbf{r}} l(\mathbf{s}, \mathbf{r}^{(t)})$. Then, the global minimum of the resulting convex problem serves as the starting point, as follows, for the next iteration of the algorithm [43]:

$$\mathbf{r}^{(t+1)} = \underset{\mathbf{r}}{\operatorname{argmin}} \left\{ h(\mathbf{s}, \mathbf{r}) - l(\mathbf{s}, \mathbf{r}^{(t)}) - \begin{bmatrix} \mathbf{0} \\ \mathbf{r} - \mathbf{r}^{(t)} \end{bmatrix}^T \mathbf{y}^{(t)} \right\}$$

(4)

subject to $\mathbf{r} \in \mathcal{R}$

The DCA for solving problem (3) is described in **Algorithm 1**. Since the Jacobian and Hessian of the RBF model are easy to derive as shown in Appendix A, the implementation of **Algorithm 1** is quite simple.

Although problem (3) can be efficiently solved using the DCA, its solution may be far different from that of problem (2) due to the use of RBF approximations. Therefore, we devise an adaptive optimization strategy, as described in Fig. 2, to harmonize the solutions to both problems. In particular, we first solve problem (3) using the DCA for obtaining its optimal solution, denoted by $\mathbf{r}_{\min}$. Then, this solution is assigned as the next sampling point $\mathbf{r}_{\text{new}}$, i.e., $\mathbf{r}_{\text{new}} = \mathbf{r}_{\min}$, for updating problem (3). We also assign $\mathbf{r}_{\text{new}}$ as the starting point for the DCA of the next iteration of the adaptive process. We carry out the iterative process of finding $\mathbf{r}_{\text{new}}$ and updating the approximate optimization problem until one of the following criteria is satisfied: (1) the norm of the difference of the optimal solutions at two consecutive iterations is less than an allowable threshold ε , and (2) the limit for number of additional sampling points $\mathbf{r}_{\text{new}}$, denoted by $k_{\max}$, has been reached, e.g., $k_{\max} = 100$. Here, the adaptive optimization strategy represents our interest in refining problem (3) in the promising region that may contain the global optimal solution to problem (2) rather than finding the best model of problem (3), which exactly fits all the points in the random parameter space.

Algorithm 1: DCA

Inputs $\mathbf{r}^{(0)}$: initial guess

$\mathcal{R}$: uncertainty set of $\mathbf{r}$

$l(\mathbf{s}, \mathbf{r})$ and $h(\mathbf{s}, \mathbf{r})$: convex functions for decomposition of the RBF model

ε : small positive value, e.g., $\varepsilon = 0.01$

$t_{\max}$: limit for number of additional sampling points, e.g., $t_{\max} = 200$

Outputs $\mathbf{r}_{\min}$ and $\hat{g}(\mathbf{s}, \mathbf{r}_{\min})$: minimizer and minimum of the RBF model with $\mathbf{r} \in \mathcal{R}$

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1:  $0 \leftarrow t; +\infty \leftarrow \delta$ 
2: while  $\delta > \varepsilon$  and  $t < t_{\max}$  do
3:    $\mathbf{y}^{(t)} \leftarrow \nabla_{\mathbf{r}} l(\mathbf{s}, \mathbf{r}^{(t)})$ 
4:   Solve the convex problem (4) to obtain  $\mathbf{r}^{(t+1)}$ 
5:    $\delta \leftarrow \|\hat{g}(\mathbf{s}, \mathbf{r}^{(t+1)}) - \hat{g}(\mathbf{s}, \mathbf{r}^{(t)})\|$ 
6:    $t \leftarrow t + 1$ 
7: end
8: return  $\mathbf{r}_{\min}$  and  $\hat{g}(\mathbf{s}, \mathbf{r}_{\min})$ 
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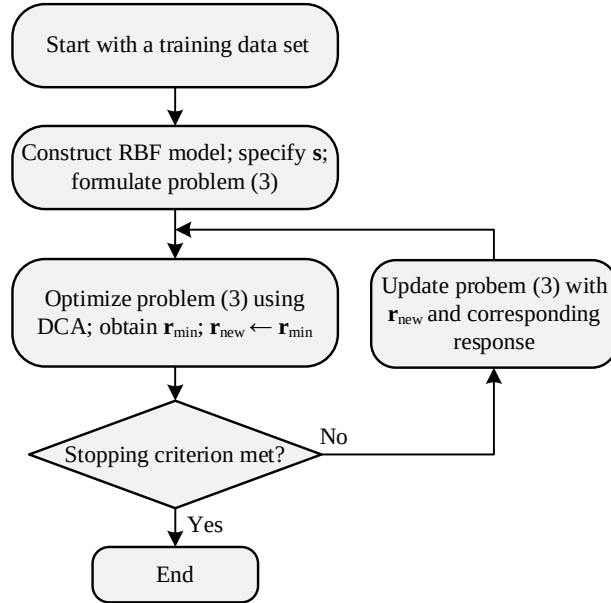


Fig. 2 Proposed adaptive optimization strategy for solving problem (2).

3 Robust design optimization procedure

3.1 Initial sampling design

We consider an initial training data set $\mathcal{D} = \{\mathbf{X}, \mathbf{y}\} = \{\mathbf{x}_i, y_i\}_{i=1}^N$, where $\mathbf{x}_i = [\mathbf{s}_i^T, \mathbf{r}_i^T]^T \in \mathbb{R}^k$ are k -dimensional vectors of input variables ($k = k_1 + k_2$) and $y_i \in \mathbb{R}$ are the corresponding output variables (or structural responses). $\mathcal{D}$ is randomly generated by the following two steps. First, a total of N_s samples for the discrete design variables $\mathbf{s}$ are randomly generated over the domain defined by the bounds $\mathbf{s}_L$ and $\mathbf{s}_U$ using Latin-hypercube sampling. The integer values $\mathbf{s}$ are then determined by rounding the corresponding values to the nearest integers. Second, for each sample of $\mathbf{s}$, a total of N_r samples of the random parameters $\mathbf{r}$ are generated over the uncertainty set $\mathcal{R}$. The rationale behind this step is to reduce the computation time for executing the adaptive optimization strategy. As a result, $\mathcal{D}$ initially has a total of $N = N_s N_r$ samples once the two steps have been done. The structural responses (or constraint function values) corresponding to each sample are obtained via the finite element analysis. Then, each sample of $\mathbf{s}$ is checked whether it violates the constraints, and the sample that yields any positive constraint function value in problem (1) is deleted from $\mathcal{D}$; therefore, the number of samples considerably decreases from N .

3.2 Optimization procedure

Besides checking the feasibility of a discrete candidate solution, another important task of the optimization process is to explore the design variable space. A good search strategy can lead the optimization algorithm to quickly converge to a good approximation of the global solution to problem (1). Branch-and-bound method or heuristic techniques, such as simulated annealing, genetic algorithm, swarm algorithms, and scatter search, can be used for discrete problems [45]. These techniques, however, may not be suitable for our RDO problem since they require a large number of function calls during the implementation of the adaptive strategy to find the global minimum of problem (2). Hence, to explore the design variable space, we employ a random search strategy, which has been successfully applied to continuous and mixed-integer surrogate-based global optimization problems [42,46].

Once $\mathcal{D}$ has been created, initial RBF models are developed for describing our initial belief about the underlying models of the constraint functions. Based on these models, we start searching the best solution, denoted by $\mathbf{s}_{\text{best}}$, among the available candidate solutions in $\mathcal{D}$ with support from the adaptive strategy in Section 2. If $\mathcal{D}$ contains only infeasible candidate solutions, we regenerate it until at least one feasible solution has been found.

With $\mathbf{s}_{\text{best}}$, we create a new set of M integer candidate solutions, denoted by $\Omega_s = \{\mathbf{s}_j\}_{j=1}^M$, to carry out the local search. Here, Ω_s is a set of neighborhood solutions of $\mathbf{s}_{\text{best}}$ generated by randomly increasing or decreasing each integer element of the vector by an integer value that is randomly selected from a given integer set specifying the neighborhood of $\mathbf{s}_{\text{best}}$, e.g., $\{1, 2, \dots, 5\}$. The purpose of generating Ω_s is to examine the variation of the objective function when the discrete design variables vary in the promising region that is deemed likely to contain the global solution, which may increase the chance of finding the global solution with a small number of trials of the optimization algorithm. The generation of Ω_s is analogous with the generation of the so-called “N” candidate points used for searching global solution to a deterministic surrogate-based optimization problem proposed by Regis and Shoemaker [46], where “N” stands for new candidate points that are normally generated around the best minimizer found so far. After Ω_s is created, we find the best candidate among its members and assign the candidate as the next sampling point, denoted by $\mathbf{s}_n$, based on the following objective-function and distance criteria [42,46].

Let $\Omega_{fs} = \{\mathbf{s}_l\}_{l=1}^L \subseteq \Omega_s$ be the set of feasible solutions among the total of M solutions in Ω_s . The best candidate selected from Ω_{fs} by the objective-function criterion should have the smallest value of the objective function $C(\mathbf{s})$. The score $V_f(\mathbf{s}_l)$ for the objective-function criterion of a candidate $\mathbf{s}_l$ in Ω_{fs} is defined as

$$V_f(\mathbf{s}_l) = \frac{C(\mathbf{s}_l) - C_{\min}}{C_{\max} - C_{\min}} \quad (5)$$

where $C_{\min} = \min(C(\mathbf{s}_1), \dots, C(\mathbf{s}_L))$ and $C_{\max} = \max(C(\mathbf{s}_1), \dots, C(\mathbf{s}_L))$. If $C_{\max} = C_{\min}$, then $V_f(\mathbf{s}_l) = 1$.

Let $d(\mathbf{s}_l, \mathcal{D}) = \min(d(\mathbf{s}_l, \mathbf{s}_1), \dots, d(\mathbf{s}_l, \mathbf{s}_N))$ denote the distance from the candidate $\mathbf{s}_l$ to the initial training data set $\mathcal{D}$, here, $d(\mathbf{s}_l, \mathbf{s}_i)$ is the Euclidean distance between $\mathbf{s}_l$ and $\mathbf{s}_i$. A good candidate in Ω_{fs} selected by the distance criterion should be far away from the sampled points in $\mathcal{D}$ since we expect that the selected candidate can explore unobserved regions of the design variable space. For the candidate $\mathbf{s}_l$, the score for the distance criterion is defined as follows:

$$V_d(\mathbf{s}_l) = \frac{d_{\max} - d(\mathbf{s}_l, \mathcal{D})}{d_{\max} - d_{\min}} \quad (6)$$

where $d_{\min} = \min(d(\mathbf{s}_1, \mathcal{D}), \dots, d(\mathbf{s}_L, \mathcal{D}))$ and $d_{\max} = \max(d(\mathbf{s}_1, \mathcal{D}), \dots, d(\mathbf{s}_L, \mathcal{D}))$. If $d_{\max} = d_{\min}$, then $V_d(\mathbf{s}_l) = 1$.

The criteria in Eqs. (5) and (6) conflict with each other since they can be regarded as the criteria for local and global search algorithms, respectively. The following weighted score, therefore, is defined for the candidate $\mathbf{s}_l$ in Ω_{fs} to mitigate the conflict based on the scalarization approach in multi-objective programming problem:

$$V(\mathbf{s}_l) = w_f V_f(\mathbf{s}_l) + w_d V_d(\mathbf{s}_l) \quad (7)$$

where w_f and w_d are the weight factors with $w_f + w_d = 1$. Müller et al. [47] examined different settings of w_f and w_d and found that $w_f = 0.9$ and $w_d = 0.1$ can generate a good solution. The best candidate solution in Ω_{fs} minimizing $V(\mathbf{s}_l)$ is assigned as the next sampling point $\mathbf{s}_n$ in the design variable space for updating the RBF models. If $\mathbf{s}_n$ coincides with any already sampled candidate in $\mathcal{D}$, it will be discarded and a new set of candidate solutions Ω_s are regenerated until at least one unsampled point $\mathbf{s}_n$ has been found.

For $\mathbf{s}_n$, a total of N_r samples of the random parameters are generated over $\mathcal{R}$ to produce new N_r samples in the input variable space, which is identical to the second step of generation of the initial training data set $\mathcal{D}$ in Section 3.1. These new samples and the corresponding structural responses are

then added to $\mathcal{D}$ for updating the RBF models. The optimization process iterates through updating the best solution $\mathbf{s}_{\text{best}}$ found so far, generating the set of new candidate solutions Ω_s , and selecting a new sampling point $\mathbf{s}_n$ until the limit for number of additional new sampling points of the design variables, denoted by p_{max} , has been reached. Fig. 3 summarizes the proposed optimization procedure for solving problem (1).

Müller et al. [42] showed that the random search process can be regarded as an asymptotically complete method, which means that it is able to converge to the global minimum with probability one, although we should allow an indefinitely long run-time computation and the exact moment at which the global minimum has been found cannot be identified. However, for solving complex optimization problems of various types, the asymptotically complete method is usually successful and is the only feasible choice [48], because we can terminate the computation within a moderately long time to converge to the approximate global minimum with probability arbitrarily close to one. This is justified since finding an exact global optimal solution to a non-convex nonlinear optimization problem as problem (1) is challenging and may be impossible [49].

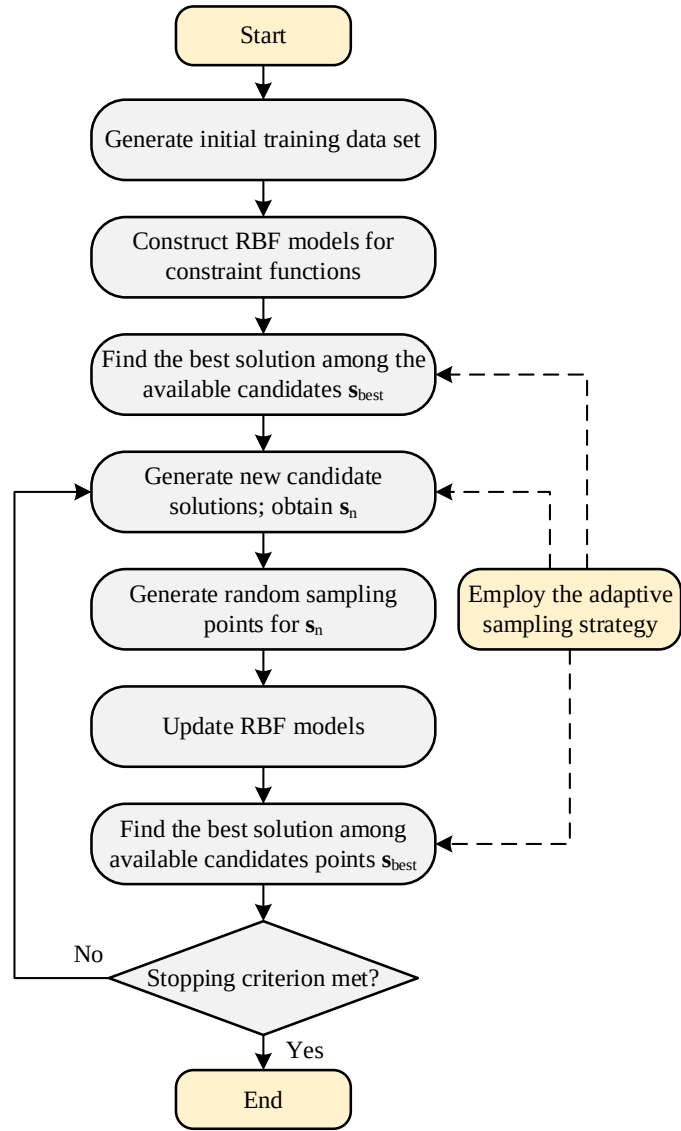


Fig. 3 Proposed RDO procedure.

4 Test problems

4.1 Two-bar truss

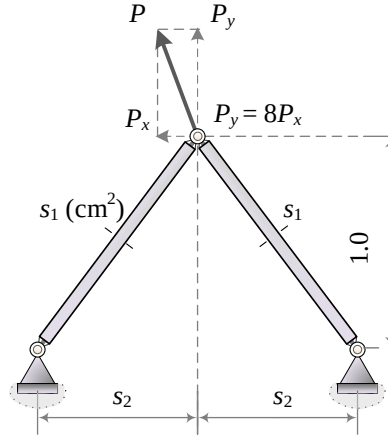


Fig. 4 Two-bar truss.

Consider a two-bar truss as shown in Fig. 4 [33]. The cross-sectional area of two truss members s_1 and the horizontal span of the truss s_2 are the discrete design variables that are selected from the sets $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, as shown in Table 1. The two random parameters are the magnitude of the external load P and the truss material's tensile strength σ_y , i.e., $\mathbf{r} = [P, \sigma_y]^T$. Their associated uncertainty sets are also given in Table 1. Density of the truss material ρ is 10^4 kg/m^3 . The axial load due to the self-weight of the truss elements is not considered to make the problem considered here identical to the original problem [33]. Therefore, problem (1) for minimizing the total material volume of the truss under constraints on the axial strength of its members can be written as

$$\min_{\mathbf{s}} C(\mathbf{s}) = 10^{-4} \rho s_1 \sqrt{1 + s_2^2}$$

subject to

$$g_1(\mathbf{s}, \mathbf{r}) = \frac{5P}{\sqrt{65}s_1\sigma_y} \sqrt{1 + s_2^2} \left(8 + \frac{1}{s_2}\right) - 1 \leq 0$$

$$g_2(\mathbf{s}, \mathbf{r}) = \frac{5P}{\sqrt{65}s_1\sigma_y} \sqrt{1 + s_2^2} \left(8 - \frac{1}{s_2}\right) - 1 \leq 0$$

(8)

$$s_1 \in \mathcal{S}_1; s_2 \in \mathcal{S}_2$$

$$\forall P \in \mathcal{R}_1; \forall \sigma_y \in \mathcal{R}_2$$

Table 1 Design variables and random parameters for the two-bar truss.

Variable	Description	Design variable/Uncertainty set
s_1	Cross-sectional area [cm ²]	$\mathcal{S}_1 = \{1.0, 1.5, \dots, 20.0\}$
s_2	Horizontal span [m]	$\mathcal{S}_2 = \{0.1, 0.2, \dots, 2.0\}$
P	External load [kN]	$\mathcal{R}_1 = [500, 1100]$
σ_y	Tensile strength [MPa]	$\mathcal{R}_2 = [672, 1428]$

To create the initial training data set $\mathcal{D}$, we randomly generate a total of $N_s = 200$ samples for the discrete design variables $\mathbf{s}$ over the joint space of $\mathcal{S}_1$ and $\mathcal{S}_2$. Only different samples among them are kept in $\mathcal{D}$. For each available sample of $\mathbf{s}$, we randomly generate a total of $N_r = 15$ samples of the parameters $\mathbf{r}$ over the uncertainty sets $\mathcal{R}_1$ and $\mathcal{R}_2$. Then, we calculate the values of the constraint functions g_1 and g_2 for each sample of the input variables $\mathbf{s}$ and $\mathbf{r}$. A design variable sample $\mathbf{s}$ is removed from $\mathcal{D}$ if it and any corresponding sampled value of $\mathbf{r}$ yield a positive value of either of g_1 or g_2 . As a result, a total of 405 samples of the input variables, which correspond to 27 samples of $\mathbf{s}$ (indicated as initial candidates in Fig. 5), are kept in $\mathcal{D}$ and used to train the RBF models for both g_1 and g_2 . However, it is worth noting that the number of input variables to train the RBF models is a random value since it depends on the samples of $\mathbf{s}$ and $\mathbf{r}$, which are randomly generated.

To carry out the optimization process, we set $M = 100$ and $p_{\max} = 10$. The obtained optimal solution is compared with a solution to a continuous RDO problem taken from Do and Ohsaki [33] and with the exact solution found by solving both the optimization and anti-optimization problems using a genetic algorithm. Here, the genetic algorithm is performed two times and the resulting solutions are identical. Note that we can also obtain the exact solution by enumeration because at a specified value of the design variables $\mathbf{s}$, g_1 and g_2 are monotonic functions of the random parameters and the solution to the anti-optimization problem is found at upper bounds of P and σ_y .

Fig. 5 shows the history of adding the design variables in the optimization process for the two-bar truss, where the discrete values, e.g., 1.0, 1.5, ..., 20.0 for s_1 are indexed as 1, 2, ..., 39. Even though the initial samples of $\mathcal{D}$ are generated using Latin-hypercube sampling, the initial design candidates are not space-filling samples as those violating the constraints are discarded. Table 2 shows that the RDO solution is found at $(s_1, s_2) = (11.5, 0.4)$, which agrees with that found in the previous study by Do and Ohsaki [33] and is identical to the exact solution by the genetic algorithm. The obtained optimal solution is also used to examine the variation in the constraint functions g_1 and g_2 when the random parameters are varied. Fig. 6 shows that the worst values of both g_1 and g_2 are negative over the uncertainty set, which confirms the feasibility of the obtained optimal solution.

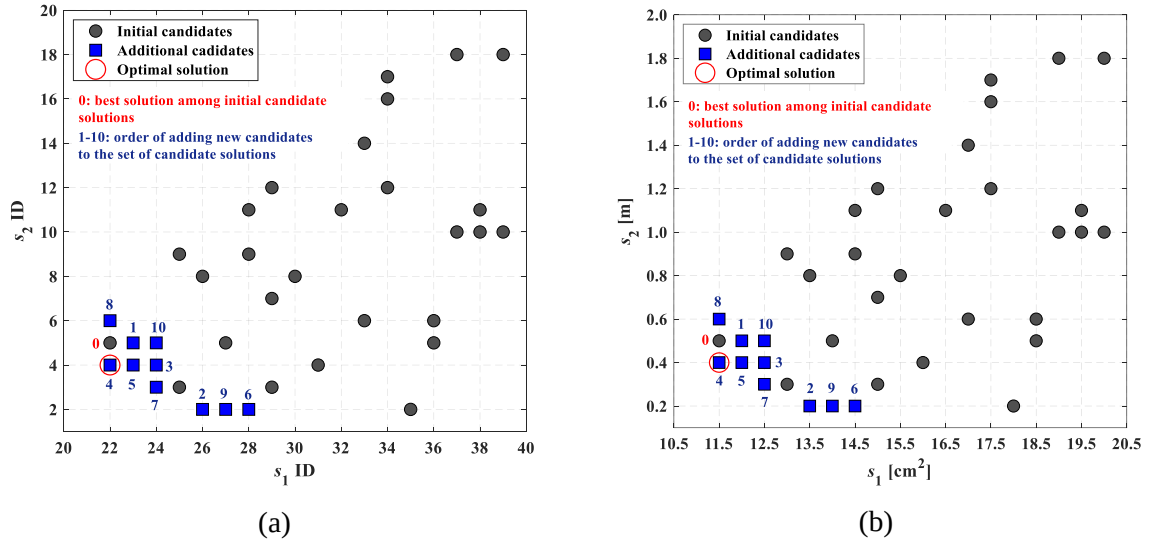


Fig. 5 History of adding design variables for the two-bar truss: (a) discrete values; (b) real values.

Table 2 Comparison of robust optimization results for the two-bar truss.

Variable and objective function	Continuous RDO [33]	Discrete RDO (This study)	Discrete RDO (Exact)
s_1 [cm ²]	11.53	11.50	11.50
s_2 [m]	0.36	0.40	0.40
$C(\mathbf{s})$ [kg]	12.25	12.39	12.39

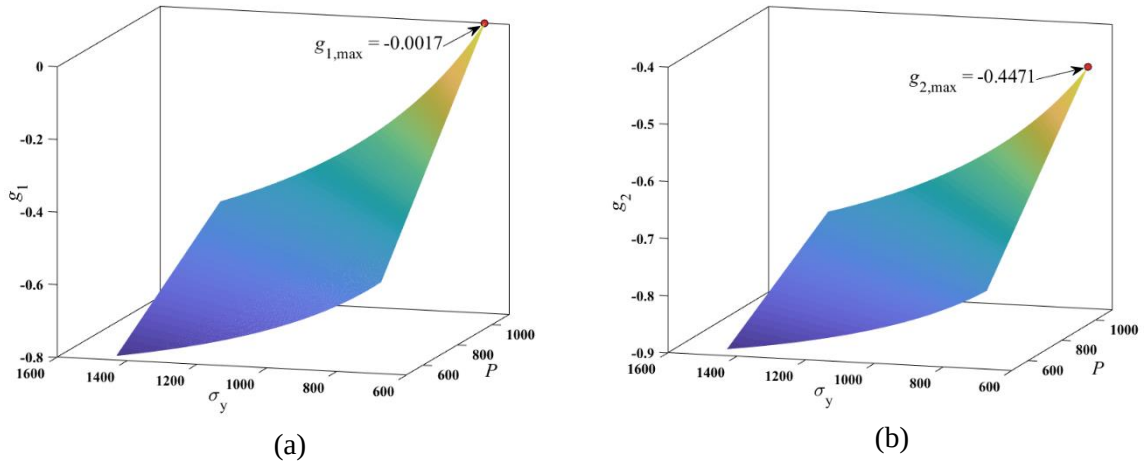


Fig. 6 Surfaces of the constraint functions for the two-bar truss at the optimal design: (a) g_1 ; (b) g_2 .

4.2 Welded beam

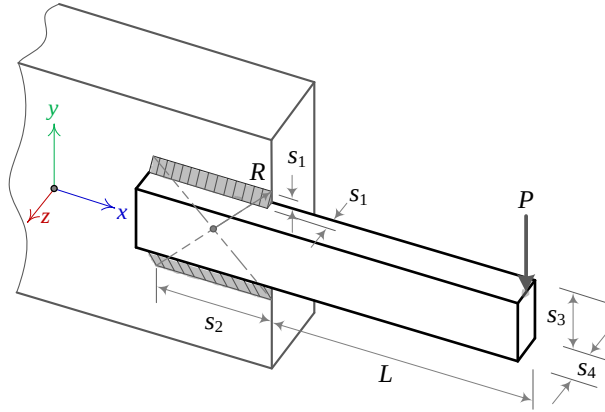


Fig. 7 Welded beam.

The welded beam as shown in Fig. 7, adapted from Tran et al. [50], is optimized for minimum total cost of the beam and welding materials under uncertainty in the external load and material properties. The beam has the constant length $L = 14$ in. Note that the English units are used in this example to be consistent with the original problem. Four discrete design variables, namely, the weld thickness s_1 , weld length s_2 , beam height s_3 , and beam width s_4 are selected from the design variable sets $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, and $\mathcal{S}_4$, respectively, as provided in Table 3. The beam is subject to the constraints on the weld shear stress τ , the maximum normal stress in the beam σ_x , the tip deflection Δ , the lateral torsional buckling load P_c , and the ratio of weld thickness to the beam width. The uncertain parameters are the

external load P , Young's modulus E , and shear modulus G of beam material, i.e., $\mathbf{r} = [P, E, G]^T$, and their intervals are described in Table 3.

Table 3 Design variables and random parameters for the welded beam.

Variable	Description	Design variable/Uncertainty set
s_1	Thickness of the welds [in]	$\mathcal{S}_1 = \{0.2, 0.4, \dots, 3.2\}$,
s_2	Length of the welds [in]	$\mathcal{S}_2 = \{0.5, 1.0, \dots, 8.0\}$
s_3	Height of the beam [in]	$\mathcal{S}_3 = \{0.5, 1.0, \dots, 8.0\}$
s_4	Width of the beam [in]	$\mathcal{S}_4 = \{0.2, 0.4, \dots, 3.2\}$
P	External load [lb]	$\mathcal{R}_1 = [5400, 6600]$
E	Young's modulus [psi]	$\mathcal{R}_2 = [27, 33] \times 10^6$
G	Shear modulus [psi]	$\mathcal{R}_3 = [10.8, 13.2] \times 10^6$

The weld shear stress τ , including the primary stress component τ_1 acting over the weld throat area and the secondary torsional stress component τ_2 , is calculated as

$$\tau = \sqrt{\tau_1^2 + \frac{\tau_1 \tau_2 s_2}{R} + \tau_2^2} \quad (9)$$

where

$$R = \sqrt{[s_2^2 + (s_1 + s_3)^2]/4} \quad (10)$$

$$\tau_1 = \frac{P}{\sqrt{2}s_1s_2} \quad (11)$$

$$\tau_2 = \frac{P(L + s_2/2)R}{2\sqrt{2}s_1s_2[s_2^2/12 + (s_1 + s_3)^2/4]} \quad (12)$$

The maximum normal stress σ_x of the beam is estimated by

$$\sigma_x = \frac{6PL}{s_4s_3^2} \quad (13)$$

Assuming that the beam acts as a cantilever of length L , its tip deflection Δ is

$$\Delta = \frac{4PL^3}{Es_4s_3^3} \quad (14)$$

For a narrow rectangular cantilever beam, its lateral torsional buckling load P_c is approximated by [51]

$$P_c = \frac{4.013\sqrt{EG}}{6L^2} s_3 s_4^3 \left(1 - \frac{s_3}{2L} \sqrt{\frac{E}{4G}} \right) \quad (15)$$

The RDO problem of the beam, therefore, is formulated as

$$\min_{\mathbf{s}} C(\mathbf{s}) = 1.10471s_1^2 s_2 + 0.04811s_3 s_4 (L + s_2)$$

subject to

$$g_1(\mathbf{s}, \mathbf{r}) = \frac{\tau}{13600} - 1 \leq 0$$

$$g_2(\mathbf{s}, \mathbf{r}) = \frac{\sigma_x}{30000} - 1 \leq 0$$

$$g_3(\mathbf{s}, \mathbf{r}) = \frac{\Delta}{0.25} - 1 \leq 0$$

$$g_4(\mathbf{s}, \mathbf{r}) = \frac{P}{P_c} - 1 \leq 0$$

$$h(\mathbf{s}) = \frac{s_1}{s_4} - 1 \leq 0$$

$$s_i \in \mathcal{S}_i, i = 1, 2, 3, 4$$

$$\forall P \in \mathcal{R}_1; \forall E \in \mathcal{R}_2; \forall G \in \mathcal{R}_3 \quad (16)$$

The initial training data set $\mathcal{D}$ is generated with $N_s = 100$ samples for the discrete design variables $\mathbf{s}$ and $N_r = 15$ random parameter samples for each sample of $\mathbf{s}$. The optimization process is performed two times with $M = 100$ and $p_{\max} = 10$. The computational time required for each optimization process is 1.77 hours using an Intel i7-7700HQ 2.80 GHz CPU and 8.0 GB memory. Fig. 8 shows the histories of adding design variables for two optimization attempts for the welded beam. The corresponding optimal solutions are listed in Table 4. It is observed that there is a slight difference between the solutions obtained from the two attempts of the optimization process. This is due to the fact that the solution to the second attempt does not appear in the set of candidate solutions of the first attempt with $p_{\max} = 10$. The solutions to the two attempts should be identical when $p_{\max}$ is increased

to a sufficiently large value, e.g., $p_{\max} = 20$. For comparison, an optimal solution to problem (16) is found by solving the upper-level and lower-level problems using a genetic algorithm with a population size of 2000 candidates for 10 generations and a multi-start algorithm with 50 starting points, respectively. These algorithms are available in the MATLAB 2018a Global Optimization Toolbox. Two attempts to perform the genetic algorithm are made and the computational time required for each attempt is 16.63 hours. The resulting solutions are identical and can be regarded as the exact solution, which is given in Table 4. It shows that the RBF approximations made in the proposed method do not hinder convergence to the exact global solution even though the required computational time is considerably reduced. The feasibility of the obtained optimal solutions is also confirmed by using the MCS with 10^5 random samples uniformly generated over the bounded intervals of the random parameters. The results in Table 4 demonstrate the feasibility of the obtained optimal solutions as the maximum value of each constraint function among the uncertain parameter samples obtained by the MCS is negative.

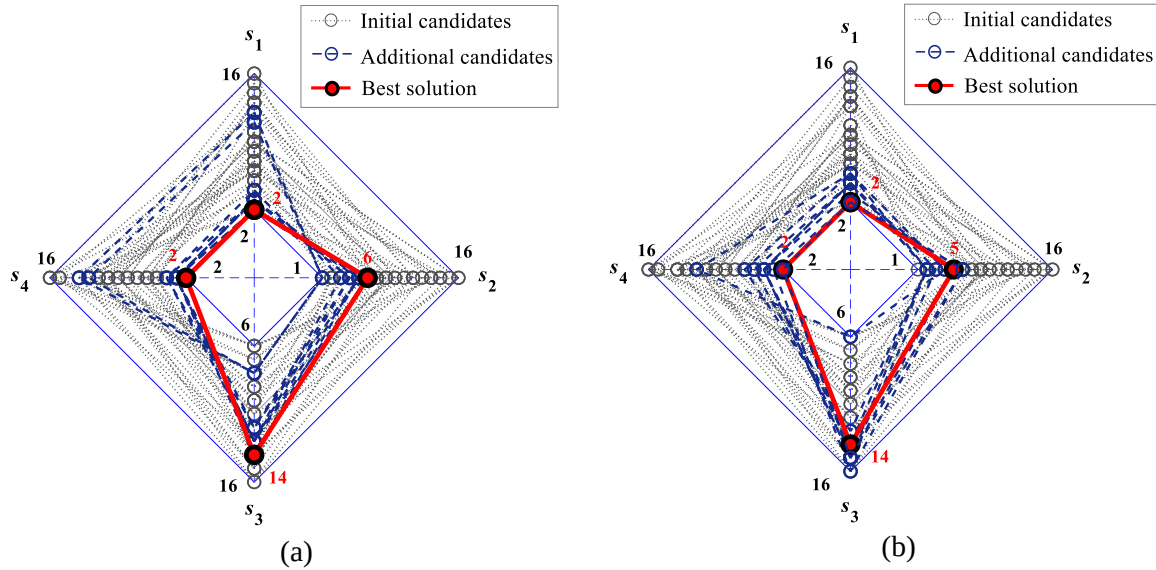


Fig. 8 Histories of adding design variables for two optimization attempts for the welded beam: (a) 1st attempt; (b) 2nd attempt.

Table 4 Comparison of robust optimization results for the welded beam.

Variable, constraint and objective functions	Discrete RDO with DCA (1st attempt)	Discrete RDO with DCA (2nd attempt)	Discrete RDO (Exact)
s_1 [in]	0.40	0.40	0.40
s_2 [in]	3.00	2.50	2.50
s_3 [in]	7.00	7.00	7.00
s_4 [in]	0.40	0.40	0.40
max. g_1	-0.2326	-0.1096	-0.1096
max. g_2	-0.0571	-0.0571	-0.0571
max. g_3	-0.9218	-0.9219	-0.9219
max. g_4	-0.6862	-0.6857	-0.6857
$C(\mathbf{s})$	2.8203	2.6646	2.6646

5 Design examples

5.1 Three-story two-bay steel frame

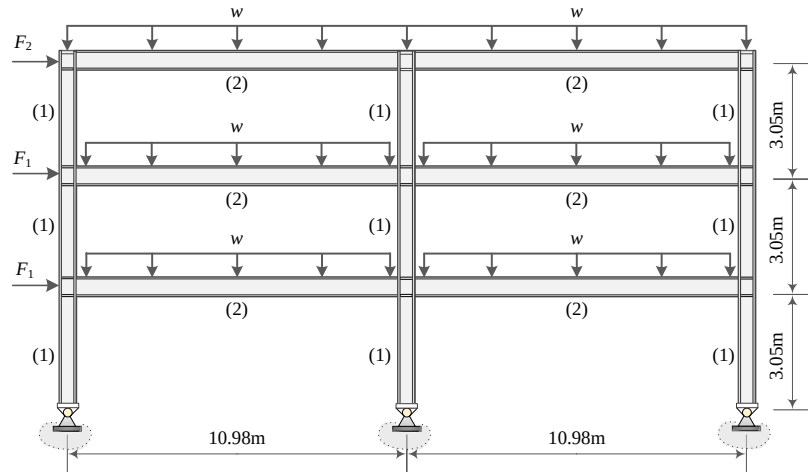


Fig. 9 Three-story two-bay steel frame.

A three-story two-bay steel frame as shown in Fig. 9 is optimized for minimizing its total mass C . The discrete design variables $\mathbf{s}$ of the frame are the steel sections of its members, which are classified into group (1) for columns and group (2) for beams. The members in each group are selected from the list of American wide-flange steel sections in Table 5. The random parameters $\mathbf{r}$ are independent, including

the uniformly distributed load w , lateral loads F_1 and F_2 , and material properties, i.e., Young's modulus E , yield stress σ_y , and ultimate tensile strength σ_u , and their associated intervals are specified in Table 6.

The frame is subjected to five constraints on the normalized values of the maximum inter-story drift g_1 , the total drift of the structure g_2 , the maximum long-term displacement of the beam g_3 , the maximum stress in the column g_4 , and the maximum stress in the beam g_5 . These constraints are in accordance with ANSI/AISC 360-16 [52] and their detailed descriptions can be found in Appendix B and in Do and Ohsaki [33]. Note that, to calculate g_4 and g_5 , the maximum values of the required axial (either tension or compression) and flexural strengths of the structural members are determined by a linear elastic frame analysis under static loads. The available axial and flexural strengths of each structural member is a function of the cross-sectional properties of the steel section, Young's modulus, the yield stress, the ultimate tensile strength, and the effective length of the structural member. A deterministic geometric constraint $h(\mathbf{s}) \leq 0$ is also considered so that the flange width of the beam should not exceed the flange width of the column. The RDO problem of the frame, therefore, is formulated as follows:

$$\min_{\mathbf{s}} C(\mathbf{s}) = 27.45\rho_1 + 65.88\rho_2$$

subject to

$$g_i(\mathbf{s}, \mathbf{r}) - 1 \leq 0, i = 1, \dots, 5$$

$$h(\mathbf{s}) \leq 0$$

$$s_1 \in \mathcal{S}_1; s_2 \in \mathcal{S}_2$$

$$\forall w \in \mathcal{R}_1; \forall F_1 \in \mathcal{R}_2; \forall F_2 \in \mathcal{R}_3; \forall E \in \mathcal{R}_4; \forall \sigma_y \in \mathcal{R}_5; \forall \sigma_u \in \mathcal{R}_6 \quad (17)$$

where ρ_l ($l = 1, 2$) are the nominal mass of the steel section for the l th group; see Table 5.

The initial training data set $\mathcal{D}$ is generated with $N_s = 100$ and $N_r = 15$. Only the design variable samples that have non-positive values of all constraint functions are retained as the initial candidate solutions. The optimal design is approximated by the result obtained after the optimization process is performed with $M = 100$ and $p_{\max} = 10$.

Table 5 List of sections of columns and beams for the three-story two-bay steel frame.

ID	Column S_1	ρ_1 [kg/m]	Beam S_2	ρ_2 [kg/m]
1	W14×90	134.0	W24×94	140.0
2	W14×82	122.0	W24×84	125.0
3	W14×74	110.0	W24×76	113.0
4	W14×68	101.0	W24×68	101.0
5	W14×61	91.0	W24×62	92.0
6	W12×72	107.0	W21×93	138.0
7	W12×65	97.0	W21×83	123.0
8	W12×58	86.0	W21×73	109.0
9	W12×53	79.0	W21×68	101.0
10	W12×50	74.0	W21×62	92.0
11	W10×68	101.0	W18×71	106.0
12	W10×60	89.0	W18×65	97.0
13	W10×54	80.0	W18×60	89.0
14	W10×49	73.0	W16×77	114.0
15	W10×45	67.0	W16×67	100.0

Table 6 Random parameters for the three-story two-bay steel frame.

Parameter	Description	Uncertainty set
w	Vertical load [kN/m]	$\mathcal{R}_1 = [37, 45]$
F_1	Lateral load [kN]	$\mathcal{R}_2 = [14, 30]$
F_2	Lateral load [kN]	$\mathcal{R}_3 = [7, 15]$
E	Young's modulus [GPa]	$\mathcal{R}_4 = [192, 210]$
σ_y	Yield stress [MPa]	$\mathcal{R}_5 = [259, 292]$
σ_u	Tensile strength [MPa]	$\mathcal{R}_6 = [395, 462]$

Fig. 10 shows the history of adding the design variables in the optimization process for the three-story two-bay steel frame. The optimal discrete design variables are found at $(s_1, s_2) = (5, 5)$ that correspond to the column and beam sections listed in Table 7. The minimum total mass of the frame is 8559 kg that is one member of the Pareto front to a multi-objective RDO problem of the frame presented by Do and Ohsaki [33]. The minimum mass design and another compromise design (Pareto optimal solution), are also given in Table 7 for comparison. Note that, to make this comparison, the uncertainty

sets of the random parameters in Table 6 are regarded as the confidence intervals of the corresponding probabilistic design parameters given in Do and Ohsaki [33]. To confirm the feasibility of the obtained optimal design, a total of 10^5 random samples are uniformly generated over the intervals of the random parameters $\mathbf{r}$. The constraint function values for each sample are determined by the finite element analysis. As a result, the maximum value of each constraint functions among all samples are negative, i.e., $\max. g_1 - 1 = -0.1880$, $\max. g_2 - 1 = -0.5085$, $\max. g_3 - 1 = -0.2538$, $\max. g_4 - 1 = -0.1623$, and $\max. g_5 - 1 = -0.0083$, which indicates the feasibility of the obtained optimal solution.

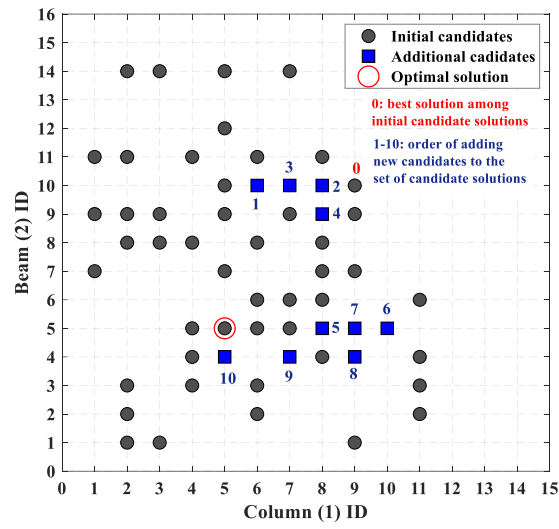


Fig. 10 History of adding design variables for the three-story two-bay steel frame.

Table 7 Comparison of robust optimization results for the three-story two-bay steel frame.

Variable and objective function	Min. mass design [33]	Compromise design [33]	Discrete RDO (This study)
Column (1)	W14×53	W18×65	W14×61
Beam (2)	W24×62	W24×62	W24×62
$C(\mathbf{s})$ [kg]	8229	8724	8559

5.2 Four-story three-bay steel frame

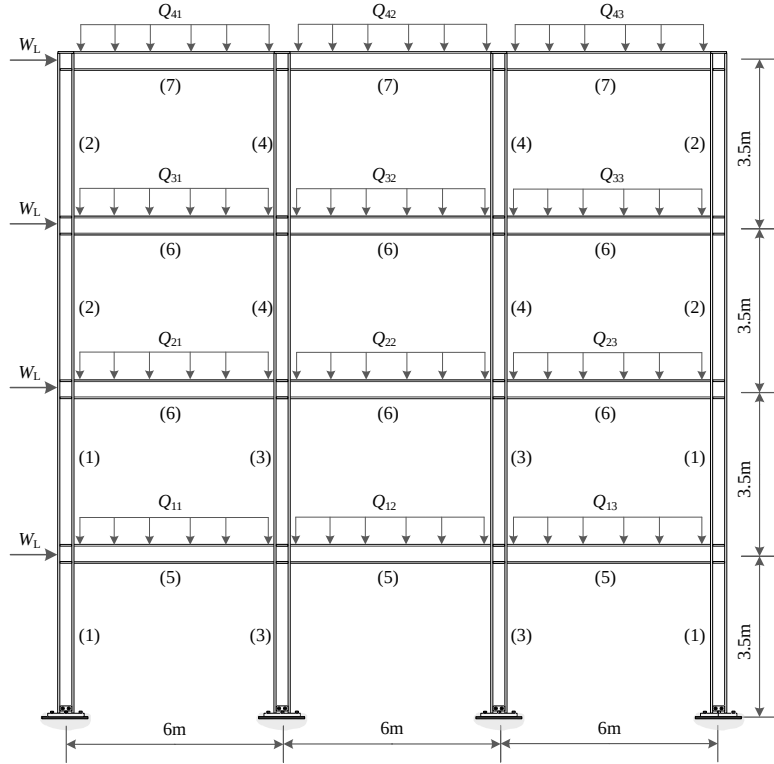


Fig. 11 Four-story three-bay steel frame.

Problem (1) is formulated for a four-story three-bay steel frame in Fig. 11. The frame consists of 28 members classified into seven groups of column and beam members. The steel sections in each group are considered as the discrete design variables $\mathbf{s}$ and chosen from the list of standard steel sections in Table 8. Table 9 shows the intervals of the random parameters $\mathbf{r}$, which include the external loads and mechanical properties of the steel material, i.e., Young's modulus, yield stress, and ultimate tensile strength. The combinations of vertical loads applied to beam members, i.e., $Q_{11}, \dots, Q_{43}$, depicted in Fig. 11, are described in Table 10.

Table 8 List of sections of columns and beams for the four-story three-bay steel frame.

ID	Column $\mathcal{S}_1$	$\rho_1, \dots, \rho_4$ [kg/m]	Beam $\mathcal{S}_2$	ρ_5, ρ_6, ρ_7 [kg/m]
1	W14×82	122.0	W24×68	101.0
2	W14×74	110.0	W24×62	92.0
3	W14×68	101.0	W24×55	82.0
4	W14×61	91.0	W21×62	92.0
5	W14×53	79.0	W21×55	82.0
6	W12×65	97.0	W21×57	85.0
7	W12×58	86.0	W16×77	114.0
8	W12×53	79.0	W16×67	100.0
9	W12×50	74.0	W16×57	85.0
10	W12×45	67.0	W16×50	75.0
11	W10×77	115.0	W16×45	67.0
12	W10×68	101.0	W16×40	60.0
13	W10×60	89.0	W14×61	91.0
14	W10×54	80.0	W14×53	79.0
15	W10×49	73.0	W14×48	72.0
16	W8×40	59.0	W14×38	57.8
17	W8×35	52.0	W14×34	51.0
18	W8×31	46.1	W14×30	44.0

Table 9 Random parameters for the four-story three-bay steel frame.

Parameter	Description	Uncertainty set
D_L	Vertical load [kN/m]	$\mathcal{R}_1 = [18, 22]$
SL_1	Short-term live load 1[kN/m]	$\mathcal{R}_2 = [9, 11]$
SL_2	Short-term live load 2[kN/m]	$\mathcal{R}_3 = [3.5, 6.5]$
LL_1	Long-term live load 1 [kN/m]	$\mathcal{R}_4 = [7, 13]$
LL_2	Long-term live load 2 [kN/m]	$\mathcal{R}_5 = [3.5, 6.5]$
S_L	Snow load [kN/m]	$\mathcal{R}_6 = [3.7, 6.3]$
W_L	Wind load [kN]	$\mathcal{R}_7 = [5, 11]$
E	Young's modulus [GPa]	$\mathcal{R}_8 = [192, 210]$
σ_y	Yield stress [MPa]	$\mathcal{R}_9 = [259, 292]$
σ_u	Tensile strength [MPa]	$\mathcal{R}_{10} = [395, 462]$

Table 10 Vertical loads applied to beam members of the four-story three-bay steel frame.

Load	Combination
$Q_{11}, Q_{13}, Q_{21}, Q_{23}, Q_{31}, Q_{33}$	$D_L + SL_1 + LL_1$
Q_{12}, Q_{22}, Q_{32}	$D_L + SL_2 + LL_2$
Q_{41}, Q_{42}, Q_{43}	$D_L + S_L$

Design constraints include serviceability, strength, and geometric requirements. The serviceability constraints are related to the normalized values of the maximum inter-story drift g_1 , the total drift g_2 , and the maximum long-term displacements g_3 , g_4 , and g_5 of beams (5), (6), and (7), respectively. Here, the long-term displacement of a beam member is evaluated as 1.5 times its elastic displacement, which is determined by a linear elastic frame analysis. Meanwhile, the constraints on strength of column and beam members are associated with the normalized values of the maximum stresses g_6 , g_7 , g_8 , and g_9 in columns (1), (2), (3), and (4), respectively, and the maximum stresses g_{10} , g_{11} , and g_{12} in beams (5), (6), and (7), respectively. The required resistances for these members are calculated from the structural analysis of the frame with factors 1.2, 1.6, 1.6, and 1.0 for the dead load, live loads, snow load, and wind load, respectively. These factors are based on the requirement in ASCE/SEI 7-16 [3]. Also, eight geometric constraints, i.e., $h_j(\mathbf{s}) \leq 0, (j = 1, \dots, 8)$ are imposed at beam-column connections and column-column joints of the frame to ensure the following two geometric requirements: (1) the flange width of a beam connected to a column should be less than or equal to the flange width of the column and (2) the depth of the column section on the upper story should not exceed the depth of the column section in the lower story. The formulation of problem (1) for the frame is given as follows:

$$\min_{\mathbf{s}} C(\mathbf{s}) = 14\rho_1 + 14\rho_2 + 14\rho_3 + 14\rho_4 + 18\rho_5 + 36\rho_6 + 18\rho_7$$

subject to

$$g_i(\mathbf{s}, \mathbf{r}) - 1 \leq 0, i = 1, \dots, 12$$

$$h_j(\mathbf{s}) \leq 0, j = 1, \dots, 8$$

$$s_1, s_2, s_3, s_4 \in \mathcal{S}_1; s_5, s_6, s_7 \in \mathcal{S}_2 \quad (18)$$

$$\forall D_L \in \mathcal{R}_1; \forall SL_1 \in \mathcal{R}_2; \forall SL_2 \in \mathcal{R}_3; \forall LL_1 \in \mathcal{R}_4; \forall LL_2 \in \mathcal{R}_5;$$

$$\forall S_L \in \mathcal{R}_6; \forall W_L \in \mathcal{R}_7; \forall E \in \mathcal{R}_8; \forall \sigma_y \in \mathcal{R}_9; \forall \sigma_u \in \mathcal{R}_{10}$$

where ρ_l ($l = 1, \dots, 7$) are the nominal mass of the steel section for the l th group; see Table 8.

The initial training data set $\mathcal{D}$ is generated with $N_s = 100$ and $N_r = 15$. The optimal design is found by performing the optimization process three times with $M = 100$ and $p_{\max} = 15$ to examine the variation in the obtained solutions. The histories of adding design variables in the optimization processes are shown in Fig. 12 and the corresponding column and beam sections for seven groups are given in Table 11. It is observed that the obtained optimal designs of the frame are not unique. However, the difference in the minimum mass between those designs is not much higher than 4% of these values. Table 11 demonstrates that this difference is due to the variation in size of the columns, while the beam sections are the same. For each design, a total of 10^5 random samples are uniformly generated over the intervals of the random parameters to confirm its feasibility. Results in Table 11 show that the maximum values of the constraint functions among the samples for each design are all negative, which indicates the feasibility of the obtained solutions. The maximum values of functions $g_4 - 1$ and $g_5 - 1$ corresponding to the displacement constraints for beams (6) and (7) give values close to 0, which leads to the unique set of beam sections for three solutions.

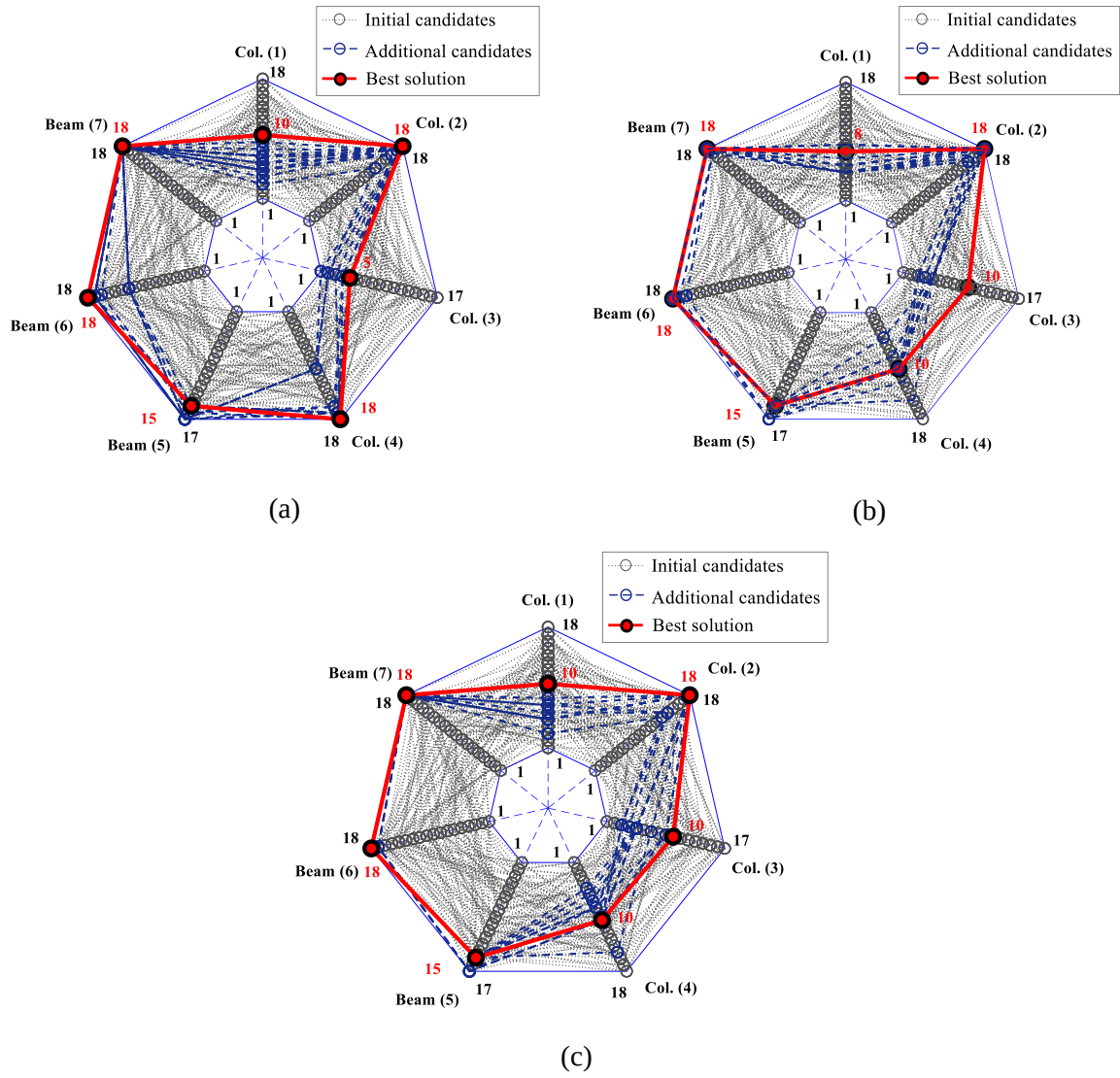


Fig. 12 Histories of adding design variables for three attempts of the optimization process for the four-story three-bay steel frame: (a) 1st attempt; (b) 2nd attempt; (c) 3rd attempt.

Table 11 Comparison of robust optimization results for the four-story three-bay steel frame.

Variable, constraint and objective functions	Discrete RDO (1st attempt)	Discrete RDO (2nd attempt)	Discrete RDO (3rd attempt)
Column (1)	W12×45	W12×53	W12×45
Column (2)	W8×31	W8×31	W8×31
Column (3)	W14×53	W12×45	W12×45
Column (4)	W8×31	W12×45	W12×45
Beam (5)	W14×48	W14×48	W14×48
Beam (6)	W14×30	W14×30	W14×30
Beam (7)	W14×30	W14×30	W14×30
$\max. g_1 - 1$	-0.6690	-0.7680	-0.7617
$\max. g_2 - 1$	-0.7033	-0.7435	-0.7363
$\max. g_3 - 1$	-0.3995	-0.3848	-0.3664
$\max. g_4 - 1$	-0.0330	-0.0895	-0.0826
$\max. g_5 - 1$	-0.0469	-0.0774	-0.0678
$\max. g_6 - 1$	-0.2900	-0.3791	-0.2704
$\max. g_7 - 1$	-0.1955	-0.2317	-0.2389
$\max. g_8 - 1$	-0.3146	-0.1783	-0.1700
$\max. g_9 - 1$	-0.2347	-0.4728	-0.4740
$\max. g_{10} - 1$	-0.3014	-0.3106	-0.2953
$\max. g_{11} - 1$	-0.2711	-0.2833	-0.2792
$\max. g_{12} - 1$	-0.2852	-0.2671	-0.2698
$C(s)$ [kg]	7006	7299	7131

6 Conclusions

The RDO optimization problem for steel frames with complex nonlinear constraint functions, discrete design variables, and interval uncertainty in external loads and material properties is a challenging problem on which most of the existing methods are not applicable since they require either special expressions for the constraint functions, e.g., convex functions in the random parameters, or continuous design variables. The random search method proposed in this study is able to handle this problem by directly addressing its global optimal and anti-optimal solutions. The method consists of three key

ingredients: RBF models to approximate black-box functions of the structural responses, an adaptive optimization strategy coupled with the DCA to solve the anti-optimization problem, and an iterative optimization process to solve the upper-level problem. The global optimal solution is approximated by the best solution obtained after a finite number of iterations of the iterative optimization process.

Two test problems of a two-bar truss and a welded beam, and two design examples of steel frames are used for assessing the performance of the proposed method. The obtained solutions by the proposed method are compared with the solutions by other methods in the literature, or verified by the MCS. It is shown that the exact optimal or good feasible solution can be achieved after about ten iterations of the optimization algorithm.

Although the size of the design variables in the two design examples is not very large, using the RBF models as approximations of the structural responses to solve the anti-optimization problem still offers advantages in handling multiple semi-infinite constraint functions and many random parameters, which is encountered when the RDO problem is formulated for a steel frame. These advantages arise from the following two reasons. First, the RBF models provide accurate approximation of structural responses, which is further decomposed as the difference of two convex functions. Therefore, the anti-optimization problem can be efficiently solved using the DCA. Second, the RBF models also enable us to devise an intelligent way using the adaptive optimization strategy to solve the anti-optimization problem in which only promising region of the random parameter space is of interest. This overcomes the drawbacks of using either metaheuristic techniques [40,41] or the scenario approach [23] for solving the anti-optimization problem.

Since the proposed method is random in nature, different solutions may be found if one attempts to apply it to solving a problem using different tuning parameters or performing the algorithm several times with the same tuning parameters and different random seeds. Thus, the trade-off between the computational cost and the quantity of the obtained solution should be made. The following four steps can be executed for the application. (1) Select a number of iterations of the algorithm, which may be based on the number of design variables or the number of semi-infinite constraint functions; (2) implement the algorithm several times to obtain the solutions; (3) check whether the obtained solutions

meet the expectation on the objective function value and/or the margin in the constraints of the design; and (4) increase the number of iterations of the algorithm if the obtained solutions do not satisfy the expectation.

The computational cost of the proposed method mainly depends on the number of initial samples and the number of iterations of the adaptive optimization process. This is due to the separation of the exploration of the design variable space from the exploitation of the random variable space for checking the feasibility of the design variables. For improvement, a method that simultaneously addresses the design variable and random parameter spaces via a criterion to balance the exploration and exploitation may be of interest.

The proposed method has been tested against a problem with seven discrete design variables, ten random parameters, and 20 constraints, it may be unsuitable for problems beyond the mentioned thresholds since the method is surrogate-based. Readers may refer to Wang and Shan [53] for a detailed discussion on challenges involved in utilizing surrogate-based models for solving high-dimensional optimization problems. Future studies may focus on this point to either modify the proposed algorithm to be suitable for very high-dimensional problems or improve it when the computational cost required for these problems is unacceptable.

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Appendix A. Radial basis function model

Based on the training data set $\mathcal{D} = \{\mathbf{X}, \mathbf{y}\} = \{\mathbf{x}_i, y_i\}_{i=1}^N$, we seek the mapping $y = g(\mathbf{x}): \mathbb{R}^k \rightarrow \mathbb{R}$, where $g(\mathbf{x})$ is an unknown function. An RBF approximation of $g(\mathbf{x})$, denoted by $\hat{g}(\mathbf{x})$, is defined by a weighted combination of basis functions and a linear polynomial tail as follows [42,46]:

$$\hat{g}(\mathbf{x}) = \sum_{i=1}^N \lambda_i \varphi(\|\mathbf{x} - \mathbf{x}_i\|) + p(\mathbf{x}) \quad (\text{A.1})$$

where $\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_N]^T \in \mathbb{R}^N$ is the coefficient vector to be determined; $\varphi(\cdot): \mathbb{R}^+ \rightarrow \mathbb{R}$ is the radial basis function; and $p(\mathbf{x}) = \mathbf{w}^T \mathbf{x} + a$ is the linear polynomial tail with $\mathbf{w} = [w_1, \dots, w_k]^T \in \mathbb{R}^k$ and $a \in \mathbb{R}$. The cubic RBF, defined by $\varphi(r) = r^3$, where $r = \|\mathbf{x} - \mathbf{x}_i\|$, is used in this study. The unknown parameters $\boldsymbol{\lambda}$, $\mathbf{w}$, and a are determined by solving the following linear system:

$$\begin{bmatrix} \boldsymbol{\Phi} & \mathbf{P} \\ \mathbf{P}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} \quad (\text{A.2})$$

where $\boldsymbol{\Phi} \in \mathbb{R}^{N \times N}$ is the Gram matrix with $\phi_{ij} = \varphi(\|\mathbf{x}_i - \mathbf{x}_j\|)$, $i, j = 1, \dots, N$ and

$$\mathbf{P} = \begin{bmatrix} \mathbf{x}_1^T & 1 \\ \vdots & \vdots \\ \mathbf{x}_N^T & 1 \end{bmatrix} \in \mathbb{R}^{N \times (k+1)}, \mathbf{c} = \begin{bmatrix} \mathbf{w} \\ a \end{bmatrix} \in \mathbb{R}^{k+1} \quad (\text{A.3})$$

The linear system in (A.2) has a unique solution if and only if $\text{rank}(\mathbf{P}) = k + 1$ [42].

Since $\varphi(r)$ is convex in $[0, +\infty]$, the RBF model in (A.1) can be decomposed as the difference of two convex functions as follows:

$$\hat{g}(\mathbf{x}) = l(\mathbf{x}) - h(\mathbf{x}) \quad (\text{A.4})$$

where

$$l(\mathbf{x}) = \sum_{\lambda_i \geq 0} \lambda_i \varphi(\|\mathbf{x} - \mathbf{x}_i\|) + p(\mathbf{x}) \quad (\text{A.5})$$

$$h(\mathbf{x}) = \sum_{\lambda_i < 0} -\lambda_i \varphi(\|\mathbf{x} - \mathbf{x}_i\|) \quad (\text{A.6})$$

The Jacobian of the RBF model in (A.1) can be derived as

$$\frac{\partial \hat{g}(\mathbf{x})}{\partial \mathbf{x}} = \sum_{i=1}^N \lambda_i \frac{\partial \varphi(r_i)}{\partial r_i} \frac{\partial r_i}{\partial \mathbf{x}} + \mathbf{w} = \sum_{i=1}^N \lambda_i \frac{\partial \varphi(r_i)}{\partial r_i} \frac{(\mathbf{x} - \mathbf{x}_i)}{r_i(\mathbf{x})} + \mathbf{w} \quad (\text{A.7})$$

where

$$r_i(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_i\| \quad (\text{A.8})$$

$$\frac{\partial \varphi(r_i)}{\partial r_i} = 3\|\mathbf{x} - \mathbf{x}_i\|^2 \quad (\text{A.9})$$

The Hessian of the RBF model is also derived from the Jacobian in (A.7) as

$$\frac{\partial^2 \hat{g}(\mathbf{x})}{\partial \mathbf{x}^2} = \sum_{i=1}^N \frac{\lambda_i}{r_i(\mathbf{x})} \left\{ \frac{\partial \varphi(r_i)}{\partial r_i} \mathbf{I} + \left[\frac{\partial^2 \varphi(r_i)}{\partial r_i^2} - \frac{\partial \varphi(r_i)}{\partial r_i} \frac{1}{r_i(\mathbf{x})} \right] \frac{(\mathbf{x} - \mathbf{x}_i)(\mathbf{x} - \mathbf{x}_i)^T}{r_i(\mathbf{x})} \right\} \quad (\text{A.10})$$

where $\mathbf{I}$ is a k -by- k unit matrix and

$$\frac{\partial^2 \varphi(r_i)}{\partial r_i^2} = 6\|\mathbf{x} - \mathbf{x}_i\| \quad (\text{A.11})$$

Appendix B. Normalized limit state functions of serviceability and strength requirements for frame designs

In this paper, the design of steel frames complies with certain serviceability and strength requirements in ANSI/AISC 360-16 [52]. The serviceability requirements are related to the maximum inter-story drift, total drift of the frame, and long-term displacements of the beam members, while the strength requirements are associated with the strength of the column and beam members. Let h , L_b , and H denote the height of each story, the span of a beam, and the overall height of the frame. As specified in ANSI/AISC 360-16, the maximum inter-story drift for the story, the long-term displacement of the beam, and the total drift of the frame should be bounded above by $h/300$, $L_b/360$, and $H/400$, respectively, which are used to normalize all the serviceability requirements. Similarly, the strength limit state function for a column or a beam is normalized using the following AISC-LRFD interaction formula

$$g(\mathbf{s}, \mathbf{r}) = \begin{cases} \frac{P_r}{P_c} + \frac{8}{9} \left(\frac{M_{rx}}{M_{cx}} + \frac{M_{ry}}{M_{cy}} \right), & \text{if } \frac{P_r}{P_c} \geq 0.2 \\ \frac{P_r}{2P_c} + \left(\frac{M_{rx}}{M_{cx}} + \frac{M_{ry}}{M_{cy}} \right), & \text{if } \frac{P_r}{P_c} < 0.2 \end{cases} \quad (\text{B.1})$$

where P_r and P_c are the required and available axial strengths of the structural member, respectively; M_{rx} and M_{ry} are respectively the required flexural strengths about the major axis x , and the minor axis

y ; and M_{cx} and M_{cy} denote the available flexural strengths about the major axis x , and the minor axis y , respectively. The required axial and flexural strengths in Eq. (B.1) are obtained from a linear elastic frame analysis under static loads. The available axial and flexural strengths, estimated in accordance with chapters E, F, and H of ANSI/AISC 360-16 [52], are functions of the cross-sectional properties of the steel section, Young's modulus E , the yield stress σ_y , and the ultimate tensile strength σ_u of the steel material, and the effective length of the structural members.

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