

WHITTAKER FUNCTIONS ON GROUPS OF LOW RANK

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§1. We shall discuss two topics in this report. One is the commutation relations among differential operators. The other is concerned with an explicit formula of Whittaker functions on $Sp_2(\mathbf{R})$. Whittaker functions on other groups of low rank are rather known. See [2],[4],[16],[17] for instance.

As usual, we consider an element in the center of the universal enveloping algebra of Lie algebra of Lie groups G as a differential operator on G . Generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \mathbf{R})$ are given in [6] and the way to find generators of that of Lie algebra of classical groups in [6],[3].

Put

$$\begin{aligned} H_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & H_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ X_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, & X_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ X_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & X_4 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Then the generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \mathbf{R})$ in [6] are

$$\begin{aligned} \lambda(L_1) &= H_1 H_1 + H_2 H_2 + 6H_1 \\ &\quad + 2H_2 + 4X_{-1}X_1 + 8X_{-4}X_4 + 4X_{-3}X_3 + 8X_{-2}X_2, \end{aligned}$$

$$\begin{aligned}
\lambda(L_2) = & 16X_{-4}X_{-4}X_4X_4 + 16X_{-4}X_{-3}X_3X_4 \\
& - 32X_{-4}X_{-2}X_2X_4 + 16X_{-4}X_{-2}X_3X_3 \\
& + 16X_{-4}X_{-1}X_1X_4 + 8X_{-4}H_1H_2X_4 \\
& + 8X_{-4}(H_1 - H_2)X_1X_3 - 16X_{-4}X_1X_1X_2 \\
& + 16X_{-3}X_{-3}X_2X_4 + 16X_{-3}X_{-2}X_2X_3 \\
& + 8X_{-3}X_{-1}(H_1 - H_2)X_4 + 4X_{-3}H_2H_2X_3 \\
& + 8X_{-3}(H_1 + H_2)X_1X_2 + 16X_{-2}X_{-2}X_2X_2 \\
& - 16X_{-2}X_{-1}X_{-1}X_4 + 8X_{-2}X_{-1}(H_1 + H_2)X_3 \\
& + 16X_{-2}X_{-1}X_1X_2 - 8X_{-2}H_1H_2X_2 \\
& + 4X_{-1}H_1H_1X_1 + H_1H_1H_2H_2 \\
& - 16X_{-4}H_1X_4 + 32X_{-4}H_2X_4 + 32X_{-4}X_1X_3 \\
& + 32X_{-3}X_{-1}X_4 - 8X_{-3}H_1X_3 + 16X_{-3}X_1X_2 \\
& + 16X_{-2}X_{-1}X_3 - 16X_{-2}(H_1 + H_2)X_2 \\
& + 24X_{-1}H_1X_1 + 2H_1H_1H_2 \\
& + 6H_1H_2H_2 \\
& - 48X_{-4}X_4 - 24X_{-3}X_3 - 48X_{-2}X_2 \\
& + 24X_{-1}X_1 - 2H_1H_1 + 12H_1H_2 \\
& + 6H_2H_2 - 12H_1 + 12H_2.
\end{aligned}$$

In order to describe generators of the center of the universal enveloping algebra of $\mathfrak{sl}(2, \mathbf{R})$, put

$$\begin{aligned}
B_{12} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & B_{13} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
B_{14} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & B_{23} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
B_{24} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & B_{34} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
B_{11} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & B_{22} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
B_{33} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & B_{ij} &= {}^t B_{ji} \quad (i > j),
\end{aligned}$$

and define a symmetrizer S_n on the space of differential operators by

$$S_n(X_1, X_2, \dots, X_n) = \sum_{\sigma \in \mathfrak{S}_n} X_{\sigma(1)} X_{\sigma(2)} \dots X_{\sigma(n)}.$$

Then the generators are

$$\begin{aligned}
\beta_2 &= -\{3S_2(B_{11}, B_{11}) + 4S_2(B_{11}, B_{22}) + 2S_2(B_{11}, B_{33}) \\
&\quad + 8S_2(B_{12}, B_{21}) + 8S_2(B_{13}, B_{31}) + 8S_2(B_{14}, B_{41}) \\
&\quad + 4S_2(B_{22}, B_{22}) + 4S_2(B_{22}, B_{33}) + 8S_2(B_{23}, B_{32}) \\
&\quad + 8S_2(B_{24}, B_{42}) + 3S_2(B_{33}, B_{33}) + 8S_2(B_{34}, B_{43})\}/128,
\end{aligned}$$

$$\begin{aligned}
\beta_3 &= -\{S_3(B_{11}, B_{11}, B_{11}) + 2S_3(B_{11}, B_{11}, B_{22}) \\
&\quad + S_3(B_{11}, B_{11}, B_{33}) + 4S_3(B_{11}, B_{12}, B_{21}) \\
&\quad + 4S_3(B_{11}, B_{13}, B_{31}) + 4S_3(B_{11}, B_{14}, B_{41}) \\
&\quad - 4S_3(B_{11}, B_{23}, B_{32}) - 4S_3(B_{11}, B_{24}, B_{42}) \\
&\quad - S_3(B_{11}, B_{33}, B_{33}) - 4S_3(B_{11}, B_{34}, B_{43}) \\
&\quad + 8S_3(B_{12}, B_{21}, B_{22}) + 4S_3(B_{12}, B_{21}, B_{33}) \\
&\quad + 8S_3(B_{12}, B_{23}, B_{31}) + 8S_3(B_{12}, B_{24}, B_{41}) \\
&\quad + 8S_3(B_{13}, B_{21}, B_{32}) + 4S_3(B_{13}, B_{31}, B_{33}) \\
&\quad + 8S_3(B_{13}, B_{34}, B_{41}) + 8S_3(B_{14}, B_{21}, B_{42}) \\
&\quad + 8S_3(B_{14}, B_{31}, B_{43}) - 4S_3(B_{14}, B_{33}, B_{41})\}
\end{aligned}$$

$$\begin{aligned}
& -2S_3(B_{22}, B_{33}, B_{33}) - 8S_3(B_{22}, B_{34}, B_{43}) \\
& + 4S_3(B_{23}, B_{32}, B_{33}) + 8S_3(B_{23}, B_{34}, B_{42}) \\
& + 8S_3(B_{24}, B_{32}, B_{43}) - 4S_3(B_{24}, B_{33}, B_{42}) \\
& - S_3(B_{33}, B_{33}, B_{33}) - 4S_3(B_{33}, B_{34}, B_{43}) \} / 512,
\end{aligned}$$

$$\begin{aligned}
\beta_4 = & -(3S_4(B_{11}, B_{11}, B_{11}, B_{11}) + 8S_4(B_{11}, B_{11}, B_{11}, B_{22}) \\
& + 4S_4(B_{11}, B_{11}, B_{11}, B_{33}) + 16S_4(B_{11}, B_{11}, B_{12}, B_{21}) \\
& + 16S_4(B_{11}, B_{11}, B_{13}, B_{31}) + 16S_4(B_{11}, B_{11}, B_{14}, B_{41}) \\
& - 8S_4(B_{11}, B_{11}, B_{22}, B_{22}) - 8S_4(B_{11}, B_{11}, B_{22}, B_{33}) \\
& - 48S_4(B_{11}, B_{11}, B_{23}, B_{32}) - 48S_4(B_{11}, B_{11}, B_{24}, B_{42}) \\
& - 14S_4(B_{11}, B_{11}, B_{33}, B_{33}) - 48S_4(B_{11}, B_{11}, B_{34}, B_{43}) \\
& + 64S_4(B_{11}, B_{12}, B_{21}, B_{22}) + 32S_4(B_{11}, B_{12}, B_{21}, B_{33}) \\
& + 64S_4(B_{11}, B_{12}, B_{23}, B_{31}) + 64S_4(B_{11}, B_{12}, B_{24}, B_{41}) \\
& + 64S_4(B_{11}, B_{13}, B_{21}, B_{32}) + 32S_4(B_{11}, B_{13}, B_{31}, B_{33}) \\
& + 64S_4(B_{11}, B_{13}, B_{34}, B_{41}) + 64S_4(B_{11}, B_{14}, B_{21}, B_{42}) \\
& + 64S_4(B_{11}, B_{14}, B_{31}, B_{43}) - 32S_4(B_{11}, B_{14}, B_{33}, B_{41}) \\
& - 32S_4(B_{11}, B_{22}, B_{22}, B_{22}) - 48S_4(B_{11}, B_{22}, B_{22}, B_{33}) \\
& - 128S_4(B_{11}, B_{22}, B_{23}, B_{32}) - 128S_4(B_{11}, B_{22}, B_{24}, B_{42}) \\
& - 8S_4(B_{11}, B_{22}, B_{33}, B_{33}) + 64S_4(B_{11}, B_{22}, B_{34}, B_{43}) \\
& - 160S_4(B_{11}, B_{23}, B_{32}, B_{33}) - 192S_4(B_{11}, B_{23}, B_{34}, B_{42}) \\
& - 192S_4(B_{11}, B_{24}, B_{32}, B_{43}) + 32S_4(B_{11}, B_{24}, B_{33}, B_{42}) \\
& + 4S_4(B_{11}, B_{33}, B_{33}, B_{33}) + 32S_4(B_{11}, B_{33}, B_{34}, B_{43}) \\
& + 64S_4(B_{12}, B_{21}, B_{22}, B_{22}) + 64S_4(B_{12}, B_{21}, B_{22}, B_{33}) \\
& - 48S_4(B_{12}, B_{21}, B_{33}, B_{33}) - 256S_4(B_{12}, B_{21}, B_{34}, B_{43}) \\
& + 128S_4(B_{12}, B_{22}, B_{23}, B_{31}) + 128S_4(B_{12}, B_{22}, B_{24}, B_{41}) \\
& + 192S_4(B_{12}, B_{23}, B_{31}, B_{33}) + 256S_4(B_{12}, B_{23}, B_{34}, B_{41}) \\
& + 256S_4(B_{12}, B_{24}, B_{31}, B_{43}) - 64S_4(B_{12}, B_{24}, B_{33}, B_{41}) \\
& + 128S_4(B_{13}, B_{21}, B_{22}, B_{32}) + 192S_4(B_{13}, B_{21}, B_{32}, B_{33}) \\
& + 256S_4(B_{13}, B_{21}, B_{34}, B_{42}) - 64S_4(B_{13}, B_{22}, B_{22}, B_{31}) \\
& - 128S_4(B_{13}, B_{22}, B_{31}, B_{33}) - 128S_4(B_{13}, B_{22}, B_{34}, B_{41})
\end{aligned}$$

$$\begin{aligned}
& -256S_4(B_{13}, B_{24}, B_{31}, B_{42}) + 256S_4(B_{13}, B_{24}, B_{32}, B_{41}) \\
& -48S_4(B_{13}, B_{31}, B_{33}, B_{33}) - 64S_4(B_{13}, B_{33}, B_{34}, B_{41}) \\
& + 128S_4(B_{14}, B_{21}, B_{22}, B_{42}) + 256S_4(B_{14}, B_{21}, B_{32}, B_{43}) \\
& - 64S_4(B_{14}, B_{21}, B_{33}, B_{42}) - 64S_4(B_{14}, B_{22}, B_{22}, B_{41}) \\
& - 128S_4(B_{14}, B_{22}, B_{31}, B_{43}) + 256S_4(B_{14}, B_{23}, B_{31}, B_{42}) \\
& - 256S_4(B_{14}, B_{23}, B_{32}, B_{41}) - 64S_4(B_{14}, B_{31}, B_{33}, B_{43}) \\
& + 16S_4(B_{14}, B_{33}, B_{33}, B_{41}) - 16S_4(B_{22}, B_{22}, B_{22}, B_{22}) \\
& - 32S_4(B_{22}, B_{22}, B_{22}, B_{33}) - 64S_4(B_{22}, B_{22}, B_{23}, B_{32}) \\
& - 64S_4(B_{22}, B_{22}, B_{24}, B_{42}) - 8S_4(B_{22}, B_{22}, B_{33}, B_{33}) \\
& + 64S_4(B_{22}, B_{22}, B_{34}, B_{43}) - 128S_4(B_{22}, B_{23}, B_{32}, B_{33}) \\
& - 128S_4(B_{22}, B_{23}, B_{34}, B_{42}) - 128S_4(B_{22}, B_{24}, B_{32}, B_{43}) \\
& + 8S_4(B_{22}, B_{33}, B_{33}, B_{33}) + 64S_4(B_{22}, B_{33}, B_{34}, B_{43}) \\
& - 48S_4(B_{23}, B_{32}, B_{33}, B_{33}) - 64S_4(B_{23}, B_{33}, B_{34}, B_{42}) \\
& - 64S_4(B_{24}, B_{32}, B_{33}, B_{43}) + 16S_4(B_{24}, B_{33}, B_{33}, B_{42}) \\
& + 3S_4(B_{33}, B_{33}, B_{33}, B_{33}) + 16S_4(B_{33}, B_{33}, B_{34}, B_{43})/65536.
\end{aligned}$$

§2. We define the Weil representation r_n of $Sp_2(\mathbf{R})$ on $V_n = M_{n,2}(\mathbf{R})$ by putting

$$\begin{aligned}
r_n \begin{pmatrix} E & X \\ 0 & E \end{pmatrix} f \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &= \exp(2\pi i \operatorname{tr}(X^t X_1 X_2)) f \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \\
r_n \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} f \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &= (\det A)^{n/2} f \begin{pmatrix} X_1 A \\ X_2 A \end{pmatrix}, \\
r_n \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} f \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &= \int_V \int_V \exp(2\pi i \operatorname{tr}({}^t Y_1 X_2 + {}^t Y_2 X_1)) \\
&\quad f \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} dY_1 dY_2
\end{aligned}$$

for $f \in \mathcal{S}(V_n \times V_n)$, $X = {}^t X \in M_{2,2}(\mathbf{R})$, $A \in M_{2,2}(\mathbf{R})$ with $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $G_1 = SL(2, \mathbf{R})$, $G_3 = SL(4, \mathbf{R})$. Then we can define representations ρ_2, ρ_3 of $G_1 \times G_1, G_3$ on $\mathcal{S}(V_2 \times V_2), \mathcal{S}(V_3 \times V_3)$ in the following manner. First, we define linear mappings σ_1, σ_3 by

$$\sigma_1(X) = \begin{pmatrix} a & d \\ b & -c \end{pmatrix}$$

for

$$X = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in M_{4,1}(\mathbf{R})$$

and

$$\sigma_3(X) = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & f & -e \\ -b & -f & 0 & d \\ -c & e & -d & 0 \end{pmatrix}$$

for

$$X = \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \in M_{6,1}(\mathbf{R}).$$

Then $(g, h) \in G_1 \times G_1$ acts on $V_2 \times V_2$ by

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}^{(g,h)} = \left(\sigma_1^{-1} \left({}^t g \left(\sigma_1 \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \end{pmatrix} \right) h \right), \sigma_1^{-1} \left({}^t g \left(\sigma_1 \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \end{pmatrix} \right) h \right) \right)$$

for

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{pmatrix} \in M_{4,2}(\mathbf{R}) = V_2 \times V_2,$$

and $g \in G_3$ acts on V_3 by

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}^g = \left(\sigma_3^{-1} \left({}^t g \left(\sigma_3 \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \\ x_{51} \\ x_{61} \end{pmatrix} \right) g \right), \sigma_3^{-1} \left({}^t g \left(\sigma_3 \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \\ x_{52} \\ x_{62} \end{pmatrix} \right) g \right) \right)$$

for

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \\ x_{51} & x_{52} \\ x_{61} & x_{62} \end{pmatrix} \in M_{6,2}(\mathbf{R}) = V_3 \times V_3.$$

Put

$$\rho_2(g)f(X) = f(X^g)$$

for $f \in \mathcal{S}(V_2 \times V_2)$, $g \in G_1 \times G_1$, and put

$$\rho_3(g)f(X) = f(X^g)$$

for $f \in \mathcal{S}(V_3 \times V_3)$, $g \in G_3$. Then, the representations r_2, r_3, ρ_2, ρ_3 induce the representations (differential representations) of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \mathbf{R})$, $\mathfrak{sl}(2, \mathbf{R}) \oplus \mathfrak{sp}(2, \mathbf{R})$, $\mathfrak{sl}(4\mathbf{R})$ which we denote by the same letters r_2, r_3, ρ_2, ρ_3 .

With this notation, we get

THEOREM 1.

$$\begin{aligned} \rho_3(\beta_2) &= -\frac{1}{32}r_3(\lambda(L_1)), \\ \rho_3(\beta_3) &= 0, \\ \rho_3(\beta_4) &= \frac{3}{512}r_3(\lambda(L_2)) + \frac{1}{128}r_3(\lambda(L_1)), \\ r_2(\lambda(L_1)) &= \rho_2(\gamma, 1) + \rho_2(1, \gamma) - 8, \\ r_2(\lambda(L_2)) &= \rho_2(\gamma, 1)\rho_2(1, \gamma) - 2\rho_2(\gamma, 1) - 2\rho_2(1, \gamma) + 16. \end{aligned}$$

§3. By using Theorem 1, we can construct Whittaker functions on $Sp_2(\mathbf{R})$ which are standard Whittaker functions not generalized Whittaker functions in [8]. (See [11].) First, we consider same theta functions $\Theta(g, z_1, z_2)$ as in [8] attached to the Weil representation r_2 and define a lift

$$F(g) = F_{\varphi_1, \varphi_2}(g) = \int_{\Gamma \backslash H} \Theta(g, z_1, z_2) \varphi_1(z_1) \varphi_2(z_2) d_0 z_1 d_0 z_2$$

where φ_1, φ_2 are Mass wave forms on the upper half plane H . Define a character Ψ_0 by

$$\Psi_0 \left(\begin{pmatrix} 1 & n_0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -n_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & n_1 & n_2 \\ 0 & 1 & n_2 & n_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = \exp(2\pi i(n_0 + n_3))$$

as in [10] of the unipotent radical N of a Borel subgroup of $Sp_2(\mathbf{R})$. Considering

$$\int_{N \cap Sp_2(\mathbf{Z}) \backslash N} F(n g) \Psi_0(n) dn,$$

we get a following Whittaker function

$$\begin{aligned} & W_{\nu_1, \nu_2} \left(\begin{pmatrix} 1 & n_0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -n_0 & 1 \end{pmatrix} \begin{pmatrix} y_1 & 0 & n_1/y_1 & n_2/y_2 \\ 0 & y_2 & n_2/y_1 & n_3/y_2 \\ 0 & 0 & 1/y_1 & 0 \\ 0 & 0 & 0 & 1/y_2 \end{pmatrix} k \right) \\ &= \exp(2\pi i(n_0 + n_3)) \int_0^\infty \int_0^\infty v_1^{-1} v_2^{-1} y_1^2 y_2 K_{\nu_1}(2\pi v_1) K_{\nu_2}(2\pi v_2) \\ & \quad \exp(-\pi y_1^2/v_1 v_2 - \pi v_1 v_2/y_2^2 - \pi v_1 y_2^2/v_2 - \pi v_2 y_2^2/v_1) dv_1 dv_2 \end{aligned}$$

for $k \in SO(4) \cap Sp_2(\mathbf{R})$ with the modified Bessel function K_ν . (See [5].) The latter part of Theorem 1 implies

THEOREM 2.

$$\begin{aligned} \lambda(L_1) W_{\nu_1, \nu_2} &= 4(\lambda_1 + \lambda_2 - 2) W_{\nu_1, \nu_2}, \\ \lambda(L_2) W_{\nu_1, \nu_2} &= 8(2\lambda_1 \lambda_2 - \lambda_1 - \lambda_2 + 2) W_{\nu_1, \nu_2} \end{aligned}$$

with $\lambda_1 = \nu_1^2 - 1/4$, $\lambda_2 = \nu_2^2 - 1/4$.

We can calculate Mellin transforms of W_{ν_1, ν_2} and can derive an analogue of Barne's second lemma from them by using [9],[10],[15].

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