

A non-standard proof of the Peano existence theorem in WKL_0

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This note is a sort of supplement to our paper [6], but can be read independently. Except for lacking in the proof of our self-embedding theorem that every countable non-standard model of WKL_0 has a proper initial part isomorphic to itself, our argument here is essentially self-contained. The goal of this note is to carry out the popular non-standard proof of the Peano existence theorem for solutions of ordinary differential equations within WKL_0 .

The usual standard proof of Peano's theorem depends much on the Ascoli lemma, by which one can make a solution of initial value problem from a sequence of piecewise linear approximations. It was Simpson [3] who first proved the theorem within WKL_0 by avoiding the use of the Ascoli lemma. In regard to the program of Reverse Mathematics, he [3] has actually shown that Peano's theorem is provably equivalent to WKL_0 over RCA_0 , while the Ascoli lemma is equivalent to the stronger system ACA_0 . Subsequently, we [2] obtained another WKL_0 proof of Peano's theorem based on a version of Schauder's fixed point theorem. See [4], [5] for more information.

On the other hand, the non-standard proof of Peano's theorem is also known to be free from the Ascoli lemma. Thus, the non-standard proof and the WKL_0 proofs share the same feelings of constructivity (cf. Albeverio et

al. [1, p.31]). In fact, by our self-embedding theorem, a considerable portion of non-standard analysis could be developed in WKL_0 .

To begin with, recall some basic definitions and the self-embedding theorem. The system RCA_0 consists of the axioms of ordered semirings, Σ^0_1 induction and Δ^0_1 comprehension, and WKL_0 is obtained from RCA_0 by adding weak König's lemma: every infinite tree of sequences of 0's and 1's has an infinite path. A structure V of second-order arithmetic is often expressed as a pair (M, S) , where M is its first-order part and S consists of subsets of (the underlying set of) M . For an initial segment I of M , let $V \upharpoonright I = (I, S \upharpoonright I)$ where $S \upharpoonright I = \{X \cap I : X \in S\}$. Now, we have

The Self-Embedding Theorem. Let $V = (M, S)$ be a countable non-standard model of WKL_0 . Then there exists a proper initial part $V \upharpoonright I = (I, S \upharpoonright I)$ of V and an isomorphism $f: V \rightarrow V \upharpoonright I$.

See [6] for a more general statement and its proof.

Fix a countable non-standard model $V = (M, S)$ of WKL_0 , in which we are going to develop analysis. By the above theorem, V has an initial part isomorphic to itself. Since the initial part and V are isomorphic to each other, they may exchange their roles, and so they can be regarded as V and its extension, respectively. Then, let $*V = (*M, *S)$ denote an isomorphic extension of V , which will be used as a non-standard universe.

Following our paper [6], a *real number* in the closed unit interval $[0,1]$ is defined as its binary expansion. Intuitively, a binary function α codes the real $\sum_i \frac{\alpha(i)}{2^{i+1}}$. Then, each real in V is an initial segment of a $*V$ -finite sequence. A set F of pairs of finite binary sequences is said to be (a code for) a *continuous (partial) function* f from $[0, 1]$ to itself if the following conditions hold:

1. if $(s, t) \in F$ and $(s, t') \in F$, then t extends t' or t' extends t ;
2. if $(s, t) \in F$ and s' extends s , then $(s', t) \in F$;
3. if $(s, t) \in F$ and t extends t' , then $(s, t') \in F$.

For a sequence s with length $lh(s)$, we set

$$a_s = \sum_{i < lh(s)} \frac{s(i)}{2^{i+1}}, \quad b_s = a_s + \frac{1}{2^{lh(s)+1}}.$$

Then, $(s, t) \in F$ intuitively means that the image of open interval (a_s, b_s) via f is included in the closed interval $[a_t, b_t]$. Finally, we write $f(\alpha) = \beta$ iff for each M -finite initial segment t of β , there exists an M -finite initial segment s of α such that $(s, t) \in F$.

Suppose that F is a code for a “total” continuous function in V . Let $*F$ be a set of $*V$ such that $F = *F \cap V$. Since “ F is a code for a continuous function” is a Π_1^0 predicate, by overspill, there is a $p \notin M$ such that $*F$ satisfies the above three conditions for all the binary sequences with length $\leq p$. Fix such a p . Let $\text{Seq}(p)$ be the set of binary sequences with length p . We then define the function $*f$ on $\text{Seq}(p)$ by

$$*f(\tilde{s}) = \text{the longest sequence } \tilde{t} \text{ such that } (\tilde{s}, \tilde{t}) \in *F \text{ and } lh(\tilde{t}) \leq p.$$

It is clear from conditions 1 and 2 that this function is well-defined. It is also obvious that for each $\tilde{s} \in \text{Seq}(p)$, the length of $*f(\tilde{s})$ is not in M , since f is total. Again by overspill, there is a $q \notin M$ such that the length of $*f(\tilde{s})$ is $\geq q$ for every $\tilde{s} \in \text{Seq}(p)$. So, by pruning, $*f$ can be seen as a function from $\text{Seq}(p)$ to $\text{Seq}(q)$.

Lemma 1. Let f be a total continuous function in V . And let $*f$ be a function from $\text{Seq}(p)$ to $\text{Seq}(q)$ constructed as above. Then, $f(\tilde{s} \cap M) = *f(\tilde{s}) \cap M$ for each $\tilde{s} \in \text{Seq}(p)$.

Proof. Let $y = f(\tilde{s} \cap M)$. Choose any M -finite initial segment t of y . By the definition of $f(\alpha) = \beta$, there exists an M -finite initial segment s of $\tilde{s}$ such that $(s, t) \in F$. Hence we have $(\tilde{s}, t) \in {}^*F$ by condition 2 of the definition of continuous partial functions. So, t must be an initial segment of ${}^*f(\tilde{s})$ by condition 1. Since t is chosen as an arbitrary initial segment of y , y is also an initial segment of ${}^*f(\tilde{s})$. $\square$

Theorem 2 (WKL₀). Any continuous function f on $[0, 1]$ attains a maximal value.

Proof. If *f is maximal at $\tilde{s} \in \text{Seq}(p)$, f attains a maximal value ${}^*f(\tilde{s}) \cap M$ at $\tilde{s} \cap M$. $\square$

Next, we show the converse to Lemma 1.

Lemma 3. Suppose we are first given a function ${}^*f: \text{Seq}(p) \rightarrow \text{Seq}(q)$ with $p, q \notin M$ such that for all $\tilde{s}, \tilde{t} \in \text{Seq}(p)$,

$$(*) \quad \tilde{s} \cap M = \tilde{t} \cap M \Rightarrow {}^*f(\tilde{s}) \cap M = {}^*f(\tilde{t}) \cap M.$$

Then there exists a continuous function f in V such that $f(\tilde{s} \cap M) = {}^*f(\tilde{s}) \cap M$ for all $\tilde{s} \in \text{Seq}(p)$.

Proof. We first put

$${}^*F = \{(s, t) \in \bigcup_{r \leq p} \text{Seq}(r) \times \bigcup_{r \leq q} \text{Seq}(r) : \forall \tilde{s} \in \text{Seq}(p) (s \subseteq \tilde{s} \rightarrow t \subseteq {}^*f(\tilde{s}))\}.$$

Then it is easy to see that *F satisfies the three conditions of continuous functions with respect to sequences $s \in \bigcup_{r \leq p} \text{Seq}(r)$ and $t \in \bigcup_{r \leq q} \text{Seq}(r)$.

Hence, $F = {}^*F \cap M$ is a code for a continuous (partial) function in V .

To show that F is total and $f(\tilde{s} \cap M) = {}^*f(\tilde{s}) \cap M$, take any real $\alpha \in [0, 1]$. Let $\tilde{s} \in \text{Seq}(p)$ be a sequence extending α , and t be any M -finite initial segment of ${}^*f(\tilde{s})$. By condition (*), for any $s \subseteq \tilde{s}$ such that $s \notin M$, we have $(s, t) \in {}^*F$.

So, by underspill, there is an M -finite $s \subseteq \tilde{s}$ such that $(s, t) \in {}^*F$, hence $(s, t) \in F$. This shows that $f(\alpha)$ is defined and its value is ${}^*f(\tilde{s}) \cap M$. Thus, F is a code for a desired continuous function f . $\square$

Theorem 4 (WKL₀). Any continuous function f on $[0, 1]$ is uniformly continuous, that is, for each $n \in M$, there exists $m \in M$ such that $\forall s \in \text{Seq}(m) \exists t \in \text{Seq}(n) (s, t) \in F$.

Proof. Fix any $n \in M$. As in the proofs of the above lemmas, we can easily see that for each $p \notin M$, $\forall s \in \text{Seq}(p) \exists t \in \text{Seq}(n) (s, t) \in F$. Hence, also by underspill, there exists $m \in M$ such that $\forall s \in \text{Seq}(m) \exists t \in \text{Seq}(n) (s, t) \in F$. $\square$

Theorem 5 (WKL₀). Any continuous function f on $[\alpha, \beta] \subseteq [0, 1]$ is Riemann integrable.

Proof. With the help of Theorem 4, the usual argument using the upper and lower sums works. $\square$

Remark. The Riemann integral of a continuous function f on $[0, 1]$ is given by

$$\begin{aligned} \int_0^1 f(x) dx &= \lim_{n \rightarrow \infty} \sum_{s \in \text{Seq}(n)} \max_{\alpha \supseteq s} f(\alpha) \cdot \frac{1}{2^n} = \lim_{n \rightarrow \infty} \sum_{s \in \text{Seq}(n)} \min_{\alpha \supseteq s} f(\alpha) \cdot \frac{1}{2^n} \\ &= \left(\sum_{\tilde{s} \in \text{Seq}(p)} {}^*f(\tilde{s}) \cdot \frac{1}{2^p} \right) \cap M. \end{aligned}$$

Theorem 6 (WKL₀). Let $f(x, y)$ be a continuous function from $D = [0, 1]^2$ to $[0, 1]$. Then the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(0) = 0$$

has a solution $y(x)$ on the interval $[0, 1]$. (The Peano Existence Theorem)

Proof. Given a continuous function $f(x, y)$, we take *f , $p \notin M$, $q \notin M$ as before so that

$$f = *f \cap V, \quad *f: \text{Seq}(p) \times \text{Seq}(p) \rightarrow \text{Seq}(q).$$

Then define a function $*y: \text{Seq}(p) \rightarrow \text{Seq}(p+q)$ by recursion as follows:

$$\begin{aligned} *y\left(\frac{0}{2^p}\right) &= \frac{0}{2^{p+q}}, \\ *y\left(\frac{i+1}{2^p}\right) &= *y\left(\frac{i}{2^p}\right) + \frac{1}{2^p} \cdot *f\left(\frac{i}{2^p}, *y\left(\frac{i}{2^p}\right) \upharpoonright p\right), \end{aligned}$$

where a fraction form $\frac{i}{2^p}$ denotes the binary sequence in $\text{Seq}(p)$ encoding the real $\frac{i}{2^p}$, and $*y\left(\frac{i}{2^p}\right) \upharpoonright p$ is the initial segment of $*y\left(\frac{i}{2^p}\right)$ with length p .

First, it is easy to see that

$$|*y\left(\frac{i}{2^p}\right) - *y\left(\frac{j}{2^p}\right)| \leq \frac{|i-j|}{2^p},$$

since $|*f(x)| \leq 1$. So, by Lemma 3, there exists a continuous function $y(x)$ in V such that $y\left(\frac{i}{2^p} \cap M\right) = *y\left(\frac{i}{2^p}\right) \cap M$. By the definition of $*y$,

$$*y\left(\frac{k}{2^p}\right) = \sum_{i < k} *f\left(\frac{i}{2^p}, *y\left(\frac{i}{2^p}\right) \upharpoonright p\right) \cdot \frac{1}{2^p}.$$

We also have

$$\int_0^{\frac{k}{2^p} \cap M} f(x, y) dx = \left(\sum_{i < k} *f\left(\frac{i}{2^p}, *y\left(\frac{i}{2^p}\right) \upharpoonright p\right) \cdot \frac{1}{2^p}\right) \cap M,$$

by the remark after Theorem 5. So, letting $\alpha = \frac{k}{2^p} \cap M$, we have

$$y(\alpha) = \int_0^\alpha f(x, y) dx.$$

Thus, $y(x)$ is a solution of the differential equation. □

References

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