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Stability of cubic hypersurfaces of dimension 4

By

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§ 1. Introduction

Hilbert's idea of *null forms* appeared again as the *(semi-)stability* and plays an important role in constructing the moduli space and its compactification in Geometric Invariant Theory of Mumford **[6]**. By virtue of the numerical criterion, one can determine the stable objects explicitly. For example, Hilbert proved the following. (See **[2]** §19 and **[7]** p15.)

Theorem 1.1. *Let S be a cubic surface in the projective space $\mathbf{P}^3$.*

(1) S is stable if and only if it has only rational double points of type A_1.

(2) S is semi-stable if and only if it has only rational double points of type A_1 or A_2.

(3) The moduli of stable ones is compactified by adding one point corresponding to the semi-stable cubic $xyz + w^3 = 0$ with 3 A_2 singularities.

Applying the sane criterion to cubic 3-folds, *i.e.* hypersurfaces of degree 3 in $\mathbf{P}^4$, we can prove the following. (See **[1]** and **[9]**)

Theorem 1.2. *Let X be a cubic 3-fold.*

(1) X is stable if and only if it has only double points of type $A_n : v^2 + w^2 + x^2 + y^{n+1} = 0$ with $n \leq 4$.

(2) A non-stable cubic 3-fold is contained in a closed orbit if and only if it is either stable or its defining equation is projectively equivalent to either

$$\phi_{\alpha,\beta} = vy^2 + w^2z - vxz - \alpha wxy + \beta x^3 \text{ with } (\alpha, \beta) \neq (0,0) \text{ or}$$
$$vwz + x^3 + y^3.$$

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According to the numerical criterion for hypersurfaces in $\mathbf{P}^n$, in order to classify stable ones, it is enough to determine certain finite number of hyperplane sections passing through the center of gravity in an n-dimensional simplex. In Hilbert's case, we can determine these hyperplane sections by intuition. Although it becomes more difficult in the case $n \geq 4$, we prove Theorem 1.2 without an assistant of computer in **[9]**. In Section 3 we prove the following by aid of computer.

Main Theorem 1.3. *A cubic 4-fold X is not stable if and only if it satisfies either*

(1) $\mathrm{Sing}X$ *contains a conic,*

(2) $\mathrm{Sing}X$ *contains a line,*

(3) $\mathrm{Sing}X$ *contains the intersection of two hyperquadrics in a space,*

(4) X *has a double point of rank* ≤ 2,

(5) there exist a double point p of rank 3 and a hyperplane section Y through p with a line L as singular locus such that the point p on L is of rank 1 and any points on L are of rank ≤ 2, or

(6) there exist a double point p of rank 3 such that the singular locus of the tangent cone at p of X is a 2-palne in X.

In Section 4 we give an algorithm to determine the family of hypersurfaces with closed orbits. In Section 5 applying the algorithm, we determine non-stable cubic 4-folds contained in closed orbits, that is, we prove the following.

Main Theorem 1.4. *A non-stable cubic 4-fold is contained in a closed orbit if and only if it is either stable or its defining equation is projectively equivalent to one of the following:*

[C.1] $uq_1(w,x,y,z) + vq_2(w,x,y,z)$ *where* $V(u,v,q_1,q_2)$ *is a smooth curve*;

[C.2] $u(xy + xz + yz + \alpha z^2) + v^2x + w^2y + 2\beta vwz$
where $\alpha \neq 1,\ -\beta^2 \pm 2\beta,\ -5\beta^2 \pm 2\beta\sqrt{4\beta^2+1}$;

[C.3] $uy^2 + v^2z + l_1(w,x)uz + 2l_2(w,x)vy + c(w,x)$ *where* $l_1 \not| c$ *and* $l_2^2 \not| c$;

[C.4] $uvw + c(x,y,z)$ *where* $V(u,v,w,c)$ *is smooth*;

[C.5] $\alpha uy^2 + v^2z + w^2x - uxz + 2vwy \quad (\alpha \neq 0)$;

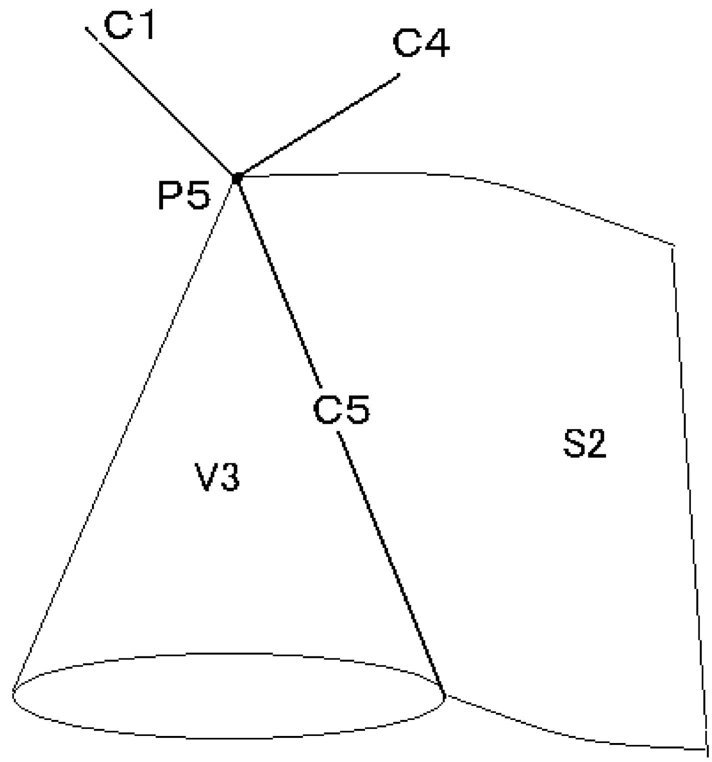
[C.6] $uvw + xyz$. .

We note that the symbols l_i, q_i and c denote a linear, quadratic and cubic homogeneous polynomial respectively. And $V(f_1, \cdots, f_k)$ means $\{f_1 = \cdots = f_k = 0\}$.

For the relation among the above families, we have the following. The proof is given in Section 3.

Proposition 1.5. *The families of [C.1] to [C.6] have dimension 1, 2, 3, 1, 1 and 0 respectively. If we denote them by C_1, S_2, V_3, C_4, C_5 and P_6 respectively, then*

$$C_5 \subseteq \overline{S_2} \cap \overline{V_3} \text{ and } P_6 \in \overline{C_1} \cap \overline{C_4} \cap \overline{C_5}. \text{ (See Figure)}$$



Figure

Remark 1.6. The maximal tori of the stabilizer groups of [C.1] to [C.6] are 1-PS's $\gamma^3 = [2, 2, -1, -1, -1, -1]$, $\gamma^1 = [4, 1, 1, -2, -2, -2]$, $\gamma^5 = [2, 1, 0, 0, -1, -2]$, $[a, b, -a-b, 0, 0, 0]$, $< \gamma^1, \gamma^5 >$ and $[a, b, -a-b, c, d, -c-d]$ respectively.

§ 2. Preparations

In this section we state some criterions playing an important role. A one-parameter subgroup, 1-PS for short, of $\mathrm{SL}(n+1)$ is a homomorphism $\lambda : \mathbf{G}_m \to \mathrm{SL}(n+1)$ of algebraic groups. Such λ can always be diagonalized in a suitable basis:

$$\lambda(t) = \mathrm{diag}\ (t^{r_0}, t^{r_1}, \cdots, t^{r_n}) \text{ and } r_0 \geq r_1 \geq \cdots \geq r_n,\ r_0 + r_1 + \cdots + r_n = 0.$$

It is simply expressed by $\lambda(t) = [r_0, r_1, \cdots, r_n](t)$ or $\lambda = [r_0, r_1, \cdots, r_n]$. Since $[r_0, r_1, \cdots, r_n] \neq [0, 0, \cdots, 0]$, r_0 is positive and r_n is negative.

Theorem 2.1. (**Numerical Criterion**) *A hypersurface of degree d in $\mathbf{P}^n$ defined by a homogeneous polynomial $f(x_0, x_1, \cdots, x_n)$ of degree d is not stable (resp. semi-stable) if and only if there exists an element σ of* $\mathrm{SL}(n+1)$ *and a 1-PS $\lambda(t) =$* $\mathrm{diag}\,(t^{r_0}, t^{r_1}, \cdots, t^{r_n}) \in \mathrm{SL}(n+1)$ *such that* $\lim_{t\to 0} \lambda(t)(\sigma f)$ *exists (resp. exists and is equal to 0). Expressing $\sigma f = \sum a_{ij\cdots k} x_0^i x_1^j \cdots x_n^k$, this is equivalent to the condition*

$$\exists \text{ 1-PS } [r_0, r_1, \cdots, r_n] \text{ s.t. } r_0 i + r_1 j + \cdots + r_n k \geq 0 \ (\text{ resp. } > 0) \text{ if } a_{ij\cdots k} \neq 0.$$

Let $\mathbf{I}$ be the set of exponents of monomials $x_0^i x_1^j \cdots x_n^k$, that is,

$$\mathbf{I} := \{(i, j, \cdots, k) \in \mathbf{Z}^{n+1} \mid i, j, \cdots, k \geq 0 \text{ and } i + j + \cdots + k = d\}.$$

Then the determination of all non-stable (resp. unstable) hypersurfaces is reduced to that of the subsets in $\mathbf{I}$

$$M^{\oplus}(\mathbf{r}) := \{\mathbf{i} \in \mathbf{I} \mid \mathbf{i} \cdot \mathbf{r} \geq 0\} \ \big(\text{ resp. } M^{+}(\mathbf{r}) := \{\mathbf{i} \in \mathbf{I} \mid \mathbf{i} \cdot \mathbf{r} > 0\}\big)$$

for all 1-PS $\mathbf{r} = [r_0, r_1, \cdots, r_n]$. We note that if $\mathbf{i} = (i, j, \cdots, k)$ and $\mathbf{r} = [r_0, r_1, \cdots, r_n]$, then $\mathbf{i} \cdot \mathbf{r} := r_0 i + r_1 j + \cdots + r_n k$. We seek for only maximal ones instead of all such subsets.

The following criterion is useful to show the closeness of the orbit.

Theorem 2.2. (**Luna's Criterion** [**3**] or [**8**] Theorem 6.17) *Suppose that a reductive group G acts on an affine variety X, H is a reductive subgroup of G, and x belongs to the set X^H of fixed points of H. Then the following are equivalent:*

(1) the orbit Gx is closed;

(2) the orbit $N_G(H)x$ over the normalizer is closed;

(3) the orbit $Z_G(H)x$ over the centralizer is closed.

Lemma 2.3. ([**8**] 6.15) *Suppose that T is an algebraic torus acting linearly on a finite-dimensional vector space V and $v \in V$ be a vector. Then the following conditions are equivalent:*

(1) the orbit Tv is closed in V;

(2) 0 is an interior point of the set supp v *in $X(T) \otimes_{\mathbf{Z}} \mathbf{Q}$,*

where $X(T)$ is the group of character of T.

Notation 2.4. We use the following notations without further mention. The symbols l, q and c denote a linear, quadratic and cubic homogeneous polynomial respectively. $f \simeq g$ means that $f = \sigma g$ for some linear transformation σ.

§ 3. Stability for cubic 4-folds

In this section we begin the following Lemma which is obtained by computer calculation.

Lemma 3.1. (1) *For any 1-PS* $\mathbf{r}$, $M^{\oplus}(\mathbf{r}) \subseteq M^{\oplus}(\gamma^i)$ *for some* $1 \le i \le 8$, *where*

$$\begin{array}{lll} \gamma^1 = [4,1,1,-2,-2,-2], & \gamma^2 = [1,1,1,1,-2,-2], & \gamma^3 = [2,2,-1,-1,-1,-1], \\ \gamma^4 = [1,1,0,0,0,-2], & \gamma^5 = [2,1,0,0,-1,-2], & \gamma^6 = [2,2,2,-1,-1,-4], \\ \gamma^7 = [2,0,0,0,-1,-1], & \gamma^8 = [1,0,0,0,0,-1]. & \end{array}$$

(2) *For any 1-PS* $\mathbf{r}$, $M^{+}(\mathbf{r}) \subseteq M^{+}(\lambda^i)$ *for some* $1 \le i \le 10$, *where* $\epsilon = 0.01$ *and*

$$\begin{array}{ll} \lambda^1 = \gamma^1 + [-1,0,-2,1,1,1]\epsilon, & \lambda^2 = \gamma^2 + [0,0,0,-2,1,1]\epsilon, \\ \lambda^3 = \gamma^2 + [2,0,-2,-6,5,1]\epsilon, & \lambda^4 = \gamma^2 + [2,0,-2,-2,1,1]\epsilon, \\ \lambda^5 = \gamma^3 + [8,0,-8,1,-3,2]\epsilon, & \lambda^6 = \gamma^4 + [0,-6,1,1,1,3]\epsilon, \\ \lambda^7 = \gamma^4 + [0,0,7,1,-11,3]\epsilon, & \lambda^8 = \gamma^5 + [-1,-6,1,1,2,3]\epsilon, \\ \lambda^9 = \gamma^5 + [11,0,1,-11,-4,3]\epsilon, & \lambda^{10} = \gamma^8 + [-7,7,7,1,-11,3]\epsilon. \end{array}$$

Lemman 3.2. *modulo* SL(6)*-action, there are the following relations among the maximal subsets:*

(1) $M^{\oplus}(\gamma^7) \subseteq M^{\oplus}(\gamma^4) = M^{\oplus}(\gamma^8)$,

(2) $M^{+}(\lambda^1) \subseteq M^{+}(\lambda^9)$, $M^{+}(\lambda^2) \subseteq M^{+}(\lambda^3) \supseteq M^{+}(\lambda^4)$, $M^{+}(\lambda^8) \subseteq M^{+}(\lambda^{10}) \subseteq M^{+}(\lambda^7)$.

Proof. We take $(u{:}v{:}w{:}x{:}y{:}z)$ as a homogeneous coordinate system of $\mathbf{P}^5$.

(1) Let [A.k] be the ideal generated by monomials of $M^{\oplus}(\gamma^k)$ where $k = 1, 2, \cdots, 8$. Then we have the following list.

$$\begin{array}{ll} [\mathrm{A}.1] & (u,v,w)^3 + (u,v,w)^2(x,y,z) + u(x,y,z)^2; \\ [\mathrm{A}.2] & (u,v,w,x)^3 + (u,v,w,x)^2(y,z); \\ [\mathrm{A}.3] & (u,v)^3 + (u,v)^2(w,x,y,z) + (u,v)(w,x,y,z)^2; \\ [\mathrm{A}.4] & (u,v,w,x,y)^3 + (u,v)^2 z; \\ [\mathrm{A}.5] & (u,v,w,x)^3 + (u,v)(u,v,w,x)y + (uy^2) + (u,v)^2 z + u(w,x)z; \\ [\mathrm{A}.6] & (u,v,w)^3 + (u,v,w)^2(x,y) + (u,v,w)(x,y)^2 + (u,v,w)^2 z; \\ [\mathrm{A}.7] & (u,v,w,x)^3 + u(u,v,w,x)(y,z) + u(y,z)^2; \\ [\mathrm{A}.8] & (u,v,w,x,y)^3 + u(u,v,w,x,y)z. \end{array}$$

Since any polynomials in [A.4] and [A.8] have a double point of rank ≤ 2, we have [A.4] $\simeq$ [A.8]. For any $F \in$ [A.7], we have

$$F = c(u,v,w,x) + ul_1(u,v,w,x)y + ul_2(u,v,w,x)z + uq(y,z)$$

$$\begin{aligned}&\simeq c(u,v,w,x)+ul_1'(u,v,w,x)y+ul_2'(u,v,w,x)z+ul_3(y,z)y\\&=\{c(u,v,w,x)+ul_1'(u,v,w,x)y+a_1uy^2\}+u\{l_2'(u,v,w,x)+a_2y\}z\in[A.8],\end{aligned}$$

where $l_3(y,z)=a_1y+a_2z$. Hence we have $M^{\oplus}(\gamma^7)\subseteq M^{\oplus}(\gamma^8)$ modulo SL(5) action.

(2) Let [B.k] be the ideal generated by monomials of $M^{+}(\lambda^k)$ where $k=1,2,\cdots,10$. Then we have the following list.

[B.1] $(u,v,w)^3+\{(u,v)^2+(uw)\}(x,y,z)+u(x,y,z)^2$;
[B.2] $(u,v,w,x)^3+(u,v,w)^2(y,z)$;
[B.3] $(u,v,w,x)^3+\{(u,v,w)^2+(ux)\}y+\{(u,v)^2+(uw)\}z$;
[B.4] $(u,v,w,x)^3+\{(u,v)^2+u(w,x)\}(y,z)$;
[B.5] $(u,v,w)^3+(u,v,w)^2(x,y)+\{(u,v)^2+(uw)\}z+u(x,y)^2+(vy^2)$;
[B.6] $(u,v,w,x,y)^3+(u^2z)$;
[B.7] $(u,v,w,x)^3+(u,v)y^2+\{(u,v,w)^2+(u,v)x\}y+(u,v)^2z$;
[B.8] $(u,v,w,x)^3+(uy^2)+\{(u,v)^2+u(w,x)\}y+u(u,v,w,x)z$;
[B.9] $(u,v,w)^3+(u,v)(u,v,w)x+\{(u,v)^2+(uw)\}(y,z)+u\{(x,y)^2+(xz)\}+(vx^2)$;
[B.10] $(u,v,w,x)^3+(uy^2)+\{(u,v,w)^2+(ux)\}y+u(u,v,w)z$.

For any $F_k\in[B.k]$, we have

$$\begin{aligned}F_1&=c(u,v,w)+\{q_1(u,v)+a_1uw\}x+\{q_2(u,v)+a_2uw\}y\\&\qquad+\{q_3(u,v)+a_3uw\}z+uq_4(x,y,z)\\&\simeq c(u,v,w)+\{q_1'(u,v)+a_1'uw\}x+\{q_2'(u,v)+a_2'uw\}y\\&\qquad+\{q_3'(u,v)+a_3'uw\}z+u\{q_4'(x,y)+a_4xz\}\in[B.9]\\&\qquad\quad\text{by } q_4(x,y,z)\mapsto q_4'(x,y)+a_4xz,\\F_2&=c(u,v,w)+q_1(u,v,w)y+q_2(u,v,w)z\\&\simeq c'(u,v,w)+q_1'(u,v,w)y+\{q_2'(u,v)+a_2uw\}z\in[B.3]\\&\qquad\text{by } q_2(u,v,w)\mapsto q_2'(u,v)+a_2uw,\\F_4&=c(u,v,w,x)+\{q_1(u,v)+ul_1(w,x)\}y+\{q_2(u,v)+ul_2(w,x)\}z\\&\simeq c'(u,v,w,x)+\{q_1'(u,v)+a_1ux\}y+\{q_2'(u,v)+a_2uw\}z\in[B.3]\qquad(*)\\&\qquad\text{by } (l_1(w,x),l_2(w,x))\mapsto(x,w),\\F_8&=c(u,v,w,x)+auy^2+\{q(u,v)+ul_1(w,x)\}y+ul_2(u,v,w,x)z\\&\simeq c'(u,v,w,x)+a_1uy^2+\{q(u,v)+a_2ux\}y+ul_2'(u,v,w)z\in[B.10]\\&\qquad\text{by } (l_1(w,x),l_2(u,v,w,x))\mapsto(x,l_2'(u,v,w)),\\F_{10}&=c(u,v,w,x)+a_1uy^2+\{q(u,v,w)+a_2ux\}y+ul(u,v,w)z\\&\simeq c'(u,v,w,x)+a_1uy^2+\{q'(u,v,w)+a_2ux\}y+ul'(u,v)z\in[B.7]\\&\qquad\text{by } l(u,v,w)\mapsto l'(u,v).\end{aligned}$$

In $(*)$, if $l_1(w,x)=l_2(w,x)$, then by $(y,z,l_2(w,x))\mapsto(y,z-y,w)$,

$$F_4\simeq c'(u,v,w,x)+q_1'(u,v)y+\{q_2(u,v)+uw\}z\in[B.3].$$

Hence we have $M^+(\lambda^1) \subseteq M^+(\lambda^9)$, $M^+(\lambda^2) \subseteq M^+(\lambda^3)$, $M^+(\lambda^4) \subseteq M^+(\lambda^3)$, $M^+(\lambda^8) \subseteq M^+(\lambda^{10})$ and $M^+(\lambda^{10}) \subseteq M^+(\lambda^7)$ respectively. $\square$

(1) to (6) in Theorem 1.3 are translations of [A.1] to [A.6] into geometric language.

Proof of Theorem 1.3: It is easy to see that (1), $\cdots$, (4) correspond to $M^{\oplus}(\gamma^1), \cdots, M^{\oplus}(\gamma^4)$, respectively.

If a cubic 4-fold X defined by F satisfy (5), then we may assume that p =(0:0:0:0:0:1) is a double point of rank 3 , $Y = X \cap \{u = 0\}$ and $L = \{v = w = x = 0\} \subset Y$. Since Y has a double point or rank 1 at p and Sing$Y$ contains L,

$$F = uq_1(u, v, w, x, y) + c_1(v, w, x) + q_2(v, w, x)y + z\{a_1v^2 + ul_1(u, v, w, x)\}$$

Since any points on L in Y are of rank ≤ 2, we have $q_2(v, w, x) = vl_2(v, w, x)$ and

$$\begin{aligned} F &= uq_1(u, v, w, x, y) + c_1(v, w, x) + vl_2(v, w, x)y + \{a_1v^2 + ul_1(u, v, w, x)\}z \\ &= c_2(u, v, w, x) + ul_3(u, v, w, x)y + a_2uy^2 + vl_2(v, w, x)y + \{a_1v^2 + ul_1(u, v, w, x)\}z. \end{aligned}$$

It is of type [A.5].

If X is a cubic 4-fold defined by a polynomial in [A.6], then p =(0:0:0:0:0:1) is a double point of rank 3 and $S = V(u, v, w)$ is a plane passing through p contained in X. The converse is easy. (6) corresponds to [A.6]. $\square$

§ 4. Family of hypersurfaces with closed orbits

In this section we give an algorithm to determine the family of hypersurfaces with closed orbits, which is essentially depending on **[8]** (**6.13**). And this idea can be traced back to Poincaré (See **[8]** **6.13** Example 1). To state the algorithm we need some preparation.

Notation 4.1. We consider hypersurfaces of degree d in $\mathbf{P}^n$. For 1-PS's $\gamma_1, \cdots, \gamma_m$ of $G = \mathrm{SL}(n+1)$, put

$$H(\gamma_1, \cdots, \gamma_m) = \{\mathbf{i} \in \mathbf{I} \mid \mathbf{i} \cdot \gamma_1 = \cdots = \mathbf{i} \cdot \gamma_m = 0\}.$$

$$< \gamma_1, \cdots, \gamma_m >= (\mathbf{Q}\gamma_1 \oplus \cdots \oplus \mathbf{Q}\gamma_m) \cap \mathbf{Z}^{\oplus m}$$

For a homogeneous polynomial $f = \sum a_{ij\cdots k} x_0^i x_1^j \cdots x_n^k$ of degree d and a subset M of $\mathbf{I}$, if $\{(i, j, \cdots, k) | a_{ij\cdots k} \neq 0\} \subseteq M$, then we denote $f \in M$ for short. And $f \in M$ mod G means that $\sigma f \in M$ for some $\sigma \in G$.

Proposition 4.2. *If a polynomial f is not stable and if its G-orbit is closed, then $\sigma f \in H(\gamma)$ for some 1-PS γ and $\sigma \in G$.*

Proof. By Numerical Criterion 2.1, $\tau f \in M^{\oplus}(\gamma)$ for some $\tau \in G$ and 1-PS γ. On the other hand, we have

$$H(\gamma) \ni \lim_{t\to 0} \gamma(t)\tau f \in \overline{G\cdot f} = G\cdot f.$$

Hence $H(\gamma) \ni \sigma f$ for some $\sigma \in G$. $\square$

Theorem 4.3. *Let f be a polynomial such that $f \in H(\gamma)$. Then $\sigma f \in M^{\oplus}(\lambda)$ for some 1-PS $\lambda \notin <\gamma>$ and $\sigma \in Z_G(\gamma)$ if and only if the orbit $G\cdot f$ is not closed or* rank $(\mathrm{stab}(f)) > 1$.

Proof. Assume $\sigma f \in M^{\oplus}(\lambda)$ for some 1-PS λ and $\sigma \in Z_G(\gamma)$. Then there exists $\lim_{t\to 0}\lambda(t)(\sigma f)$. If it belongs to $G\cdot f$, then rank $(\mathrm{stab}(f)) \geq 2$. Otherwise the orbit $G\cdot f$ is not closed. If rank $(\mathrm{stab}(f)) \geq 2$, then $\sigma f \in H(\lambda)$ as required. If $G\cdot f$ is not closed, the assertion follows from the proposition below. $\square$

Proposition 4.4. *Let $f \in H(\gamma)$ and assume that the orbit $G\cdot f$ is not closed. Then for some 1-PS λ and $\sigma \in Z_G(\gamma)$, the limit $g = \lim_{t\to 0}\lambda(t)\sigma f$ and the orbit $G\cdot g$ is closed.*

Proof. According to Luna's Criterion 2.2, $Z_G(\gamma)\cdot f$ is not closed. By Theorem 4.5, for some 1-dimensional torus T in $Z_G(\gamma)$, there exists $\lim_{t\to 0} T(t)f$ whose orbit over $Z_G(\gamma)$ is closed. Since $T(t) = \sigma^{-1}\lambda(t)\sigma$ for some 1-PS $\lambda(t)$ and elements $\sigma \in Z_G(\gamma)$ by Lemma 4.6, the orbit of $\lim_{t\to 0}\lambda(t)\sigma f$ over $Z_G(\gamma)$ is closed, which is equivalent to the closeness over G by Luna's Criterion 2.2. $\square$

Theorem 4.5. (See **[8]** Theorem 6.9) *Suppose a reductive group G acts on an affine variety X and $x \in X$. Then G contains a 1-dimensional torus T such that the intersection of the variety $\overline{T\cdot x}$ and the (unique) closed orbit in $\overline{G\cdot x}$ is nonempty.*

Lemma 4.6. *Let T be a 1-dimensional torus in $Z_G(\gamma)$, $\gamma(t) = \mathrm{diag}\,(t^{r_0},\cdots,t^{r_n})$. Then there exists $\sigma \in Z_G(\gamma)$ such $T(t) = \sigma\mathrm{diag}\,(t^{s_0},\cdots,t^{s_n})\sigma^{-1}$ for some $s_0,\cdots,s_n$.*

As in the proof of Theorem 4.3, we have the following.

Corollary 4.7. *Assume that $\gamma_1,\cdots,\gamma_m$ are linearly independent 1-PS's. Let f be a polynomial and $f \in H(\gamma_1,\cdots,\gamma_m)$. Then $\sigma f \in M^{\oplus}(\lambda)$ for some 1-PS $\lambda \notin <\gamma_1,\cdots,\gamma_m>$ and $\sigma \in Z_G(\gamma)$ if and only if either the orbit $G\cdot f$ is not closed or* rank $(\mathrm{stab}(f)) > m$.

We state here the method to find the family of hypersurfaces contained in closed orbits. In Step $k = 0,1,\cdots,n$ we determine the subfamily of ones whose stabilizers are of rank k.

Step 0: Take 1-PS's $\gamma_1, \cdots, \gamma_\ell$ such that $M^{\oplus}(\gamma_i)$'s are all the maximal subsets of $\mathbf{I}$. Then f is stable if and only if $f \notin M^{\oplus}(\gamma_i)$ mod G for any i by Numerical Criterion 2.1. If $f \in M^{\oplus}(\gamma_i)$ mod G, then $\overline{f} = \lim_{t\to 0} \gamma_i(t)f$ belong to $H(\gamma_i)$. So non-stable hypersurfaces with close orbits belong $H(\gamma_i)$ for some i.

Step 1: We determine hypersurfaces in $H(\gamma_i)$ with closed orbits whose stabilizers are rank 1. Take 1-PS's $\lambda_1, \cdots, \lambda_m \notin < \gamma_i >$ such that $M^{\oplus}(\lambda_j) \cap H(\gamma_i)$ $(j = 1, \cdots, m)$ are all the maximal subsets of $H(\gamma_i)$. Then $f \in H(\gamma_i)$ belongs to closed orbit and rank (stab(f)) $= 1$ if and only if $f \notin M^{\oplus}(\lambda_j) \cap H(\gamma_i)$ mod G for any j by Theorem 4.3. If $f \in M^{\oplus}(\lambda_j) \cap H(\gamma_i)$ mod G, then $\overline{f} = \lim_{t\to 0} \lambda_j(t) f \in H(\gamma_i, \lambda_j)$. So the other hypersurfaces belong $H(\gamma_i, \lambda_j)$ for some j.

Step $k = 2, \cdots, n$: Repeating similar procedures, we can determine hypersurfaces which belong to closed orbits and stabilizers are of rank k by Corollary 4.7.

§ 5. The Proof of Theorem 1.4 and Proposition 1.5

In this section, from the algorithm in Section 4, we give [C.1] to [C.6] in Theorem 1.4 as defining equations with closed orbits.

Lemma 5.1. *Let X be a cubic 4-folds defined by F. If it is not stable and belongs to a closed orbit, then $F \in H(\gamma^i)$ mod $SL(6)$ for some $i = 3$, 4, 5 or 6.*

Proof. By Lemma 3.1 and 3.2, we have $F \in M^{\oplus}(\gamma^i)$ for some $1 \leq i \leq 6$. Since its orbit is closed,

$$F \simeq \lim_{t\to 0} \gamma^i(t) F \in H(\gamma^i).$$

We note that $H(\gamma^1) \simeq H(\gamma^6)$ and $H(\gamma^2) \simeq H(\gamma^3)$. □

Hence we may assume that $F \in H(\gamma^i)$ for $i = 3,\ 4,\ 5$ or 6. First we consider the case $F \in H(\gamma^4)$.

Lemma 5.2. *If $F \in H(\gamma^4)$ belongs to a closed orbit, then it is of type either [C.4] or [C.6].*

Proof. Since $\gamma^4 = [1, 1, 0, 0, 0, -2]$ and $F \in H(\gamma^4)$, we have

$$F = q(u, v)z + c(w, x, y) \simeq uvz + c(w, x, y).$$

We note that F belongs to a closed orbit if and only if $c(w, x, y)$ belongs to a closed orbit. Hence we have either $c(w, x, y)$ is smooth or $c(w, x, y) \simeq wxy$. Therefore F is of type either [C.4] or [C.6]. □

Second we consider the case $F \in H(\gamma^3)$. To carry out Step 1 in Section 4 we use computer in the following.

Lemma 5.3. *Suppose $F \in H(\gamma^3)$ belongs to a closed orbit. Then its stabilizer is of rank 1 if and only if it is of type [C.1]. If its stabilizer is of rank > 1, then it is of type either [C.4] or [C.6].*

Proof. Since $\gamma^3 = [2, 2, -1, -1, -1, -1]$ and $F \in H(\gamma^3)$, we denote

$$F = uq(w, x, y, z) + vq'(w, x, y, z).$$

By $\mathcal{E}(F)$ we define $\{q(w, x, y, z) = q'(w, x, y, z) = 0\}$ in $\mathbf{P}^4(w{:}x{:}y{:}z)$. We note that

$$\mathcal{E}(F) \text{ is singular} \iff F \simeq u\{q_1(w, x, y) + l(w, x, y)z\} + vq_2(w, x, y). \qquad (*)$$

The computer calculation show that the following 1-PS's

$$\begin{aligned} &\lambda_1 = [0, -2, 1, 1, 1, -1], \qquad &&\lambda_2 = [2, 0, 1, -1, -1, -1], \qquad &&\lambda_3 = [0, 0, 1, 0, 0, -1], \\ &\lambda_4 = [0, -2, 1, 1, 0, 0], &&\lambda_5 = [0, 0, 1, 1, -1, -1] \end{aligned}$$

give all the maximal subsets $M^{\oplus}(\lambda_j) \cap H(\gamma^3)$ of $H(\gamma^3)$, which correspond to the following homogeneous polynomials respectively:

$$\begin{aligned} \phi_1 &= u\{q_1(w, x, y) + l(w, x, y)z\} + vq_2(w, x, y), \\ \phi_2 &= uq(w, x, y, z) + vwl(w, x, y, z), \\ \phi_3 &= u\{q_1(w, x, y) + a_1 wz\} + v\{q_2(w, x, y) + a_2 wz\}, \\ \phi_4 &= uq_1(w, x, y, z) + vq_2(w, x), \\ \phi_5 &= u\{q_1(w, x) + l_1(w, x)y + l_2(w, x)z\} + v\{q_2(w, x) + l_3(w, x)y + l_4(w, x)z\}. \end{aligned}$$

Since $\mathcal{E}(\phi_i)$ is singular for any $1 \leq i \leq 5$, if $\mathcal{E}(F)$ is smooth, then $F \not\simeq \phi_i$ for any i. Hence F belongs to a closed orbit and its stabilizer is of rank 1 and F is of type [C.1].

If $\mathcal{E}(F)$ is singular, then we have $F \simeq \phi_1$ by $(*)$ and

$$\lim_{t \to 0} \lambda_1(t)\phi_1 = ul(w, x, y)z + vq(w, x, y) \simeq uwz + vq(w, x, y).$$

Therefore F is of type either [C.4] or [C.6]. $\square$

Next we consdier the case $F \in H(\gamma^5)$.

Proposition 5.4. *Suppose $F \in H(\gamma^5)$ belongs to a closed orbit. Then its stabilizer is of rank 1 if and only if it is of type [C.3]. If its stabilizer is of rank > 1, then it is either of type [C.5] or [C.6].*

Proof. Since $\gamma^5 = [2, 1, 0, 0, -1, -2]$ and $F \in H(\gamma^5)$, we have

$$F = a_1 uy^2 + a_2 v^2 z + l_1(w, x)uz + 2l_2(w, x)vy + c(w, x).$$

The computer calculation show that the following 1-PS's

$$\lambda_1 = [0, -1, 0, 0, 1, 0], \quad \lambda_2 = [0, 1, 0, 0, -1, 0], \quad \lambda_3 = [0, 1, 0, 0, 1, -2],$$
$$\lambda_4 = [0, -1, 1, -2, 0, 2], \quad \lambda_5 = [0, 1, 2, -1, 0, -2], \quad \lambda_6 = [0, -1, -2, -2, 3, 2]$$

give all the maximal subsets $M^{\oplus}(\lambda_j) \cap H(\gamma^5)$ of $H(\gamma^5)$, which correspond to the following homogeneous polynomials:

$$\begin{aligned}
\phi_1 &= uzl_1(w, x) + vyl_2(w, x) + auy^2 + c(w, x),\\
\phi_2 &= uzl_1(w, x) + vyl_2(w, x) + av^2 z + c(w, x),\\
\phi_3 &= vyl(w, x) + a_1 v^2 z + a_2 uy^2 + c(w, x),\\
\phi_4 &= uzl_1(w, x) + a_1 vwy + a_2 v^2 z + a_3 uy^2 + w^2 l(w, x),\\
\phi_5 &= a_1 uzw + vyl(w, x) + a_2 v^2 z + a_3 uy^2 + wq(w, x),\\
\phi_6 &= uzl_1(w, x) + vyl_2(w, x) + a_1 v^2 z + a_2 uy^2.
\end{aligned}$$

We note that ϕ_2, ϕ_3 and ϕ_6 are the special cases of ϕ_1, ϕ_5 and ϕ_4 respectively.

If $F = a_1 uy^2 + a_2 v^2 z + l_1(w, x)uz + 2l_2(w, x)vy + c(w, x) \notin H(\lambda_j) \bmod G$ for any j then $F \not\simeq \phi_1$, $F \not\simeq \phi_4$ and $F \not\simeq \phi_5$, which mean $a_1, a_2 \neq 0$, $l_2^2 \not| c$ and $l_1 \not| c$ respectively. Therefore we have obtained [C.3].

If $F \in H(\lambda_j) \bmod G$ for some j =1, 4 or 5 then there exists the limit below respectively

$$\begin{aligned}
F_1 &= \lim_{t\to 0} \lambda_1(t) f = uzl_1(w, x) + vyl_2(w, x) \simeq uwz + vxy + c'(w, x) \to uwz + vxy,\\
F_4 &= \lim_{t\to 0} \lambda_4(t) f = a_1 vwy + a_2 uxz + a_3 v^2 z + a_4 uy^2 + a_5 w^2 x \text{ or}\\
F_5 &= \lim_{t\to 0} \lambda_5(t) f = a_1 uwz + a_2 vxy + a_3 v^2 z + a_4 uy^2 + a_5 wx^2.
\end{aligned}$$

Hence they are either of tyep [C.5] or [C.6] by Lemma 5.5. □

Lemma 5.5. *If $F = a_1 uwz + a_2 vxy + a_3 v^2 z + a_4 uy^2 + a_5 wx^2$ belongs to a closed orbit, then either $F \simeq$ [C.5] or [C.6]*

Proof. If $a_1 = 0$, then

$$\lim_{t\to 0}[-1, 1, -1, 1, 1, -1](t)F = 0.$$

Hence we have $a_1 \neq 0$ and $a_2 \neq 0$. If $a_3 = 0$, then

$$\lim_{t\to 0}[1, -2, 1, 0, 0, 0](t)F = a_1 uwz + a_2 vxy,$$

which is of type [C.6]. Hence if $a_3 a_4 a_5 = 0$, then F is of type [C.6]. Otherwise F is of type [C.5]. □

To complete the proof of Theorem 1.4 we consider the case $F \in H(\gamma^5)$.

Lemma 5.6. *Suppose $F \in H(\gamma^6)$ belongs to a closed orbit. If its stabilizer is of rank 1, then it is of type [C.2].*

Proof. Since $\gamma^6 = [2,2,2,-1,-1,-4]$ and $F \in H(\gamma^6)$, we have

$$\begin{aligned} F &= l_1(u,v,w)x^2 + l_2(u,v,w)xy + l_3(u,v,w)y^2 + q(u,v,w)z \\ &\simeq l_1(u,v,w)x^2 + l_2'(u,v)xy + auy^2 + q'(u,v,w)z \\ &\qquad \text{by } (l_3(u,v,w), l_2(u,v,w)) \mapsto (u, l_2'(u,v)). \end{aligned}$$

The computer calculation show that the following 1-PS's

$$\begin{aligned} &\lambda_1 = [0,-2,-2,1,1,2], &&\lambda_2 = [2,2,0,-1,-1,-2], &&\lambda_3 = [0,0,-2,1,1,0], \\ &\lambda_4 = [2,0,0,-1,-1,0], &&\lambda_5 = [0,-1,-2,1,0,2], &&\lambda_6 = [0,0,0,1,-1,0] \end{aligned}$$

give all the maximal subsets $M^{\oplus}(\lambda_j) \cap H(\gamma^6)$ of $H(\gamma^6)$, which correspond to the following homogeneous polynomials:

$$\begin{aligned} \phi_1 &= l_1(u,v,w)x^2 + l_2(u,v,w)xy + l_3(u,v,w)y^2 + ul_4(u,v,w)z, \\ \phi_2 &= l_1(u,v)x^2 + l_2(u,v)xy + l_3(u,v)y^2 + \{q(u,v) + l_4(u,v)w\}z, \\ \phi_3 &= l_1(u,v,w)x^2 + l_2(u,v,w)xy + l_3(u,v,w)y^2 + q(u,v)z \simeq \phi_1, \\ \phi_4 &= uq_1(x,y) + q_2(u,v,w)z, \\ \phi_5 &= l_1(u,v,w)x^2 + l_2(u,v)xy + a_1uy^2 + \{a_2v^2 + ul_3(u,v,w)\}z, \\ \phi_6 &= l_1(u,v,w)x^2 + l_2(u,v,w)xy + q(u,v,w)z. \end{aligned}$$

Since $F \simeq l_1(u,v,w)x^2 + l_2(u,v)xy + auy^2 + q(u,v,w)z \not\simeq \phi_i$ for $i = 1$, 6 and 4, we have rank $q(u,v,w) = 3$, $a \neq 0$ and $\dim \langle l_1(u,v,w), l_2(u,v), au\rangle \geq 2$ respectively.

If $\dim \langle l_1(u,v,w), l_2(u,v), au\rangle = 2$, then we have

$$F \simeq uq_1(x,y) + vq_2(x,y) + q_3(u,v,w)z.$$

Since $F \not\simeq \phi_6$, quadrics $q_1(x,y)$ and $q_2(x,y)$ have no common divisor. Hence

$$F \simeq uy^2 + vx^2 + q_3'(u,v,w)z \text{ by } (u,v) \mapsto (l(u,v), l'(u,v)).$$

Since $F \not\simeq \phi_2$, we have $q_3'(0,0,w) \neq 0$. Hence we have

$$\begin{aligned} q_3'(u,v,w) &= w^2 + l_3(u,v)w + q_4(u,v) \\ &\simeq w^2 + a_1uv + a_2vw + a_3wu \text{ by } w \mapsto w + b_1u + b_2v. \end{aligned}$$

Therefore we have

$$F \simeq uy^2 + vx^2 + (w^2 + a_1uv + a_2vw + a_3wu)z$$

$$\simeq uy^2 + wx^2 + (\alpha v^2 + uv + vw + wu)z$$

where $\alpha \neq 1$ because its rank is equal to 3.

If $\dim \langle l_1(u,v,w), l_2(u,v), au\rangle = 3$, then we have

$$F \simeq ux^2 + vxy + wy^2 + q(u,v,w)z.$$

By Lemma 5.7, we have either

$$F \simeq \begin{cases} ux^2 + vxy + wy^2 + (a_1v^2 + a_2uv + a_3vw + a_4uw)z \text{ or} \\ ux^2 + vxy + wy^2 + \{a_1u^2 + a_2(v^2 - 4uw)\}z. \end{cases}$$

Since $F \not\simeq \phi_5$, the latter case does not hold and we have $a_2a_3 \neq 0$ where

$$F \simeq ux^2 + vxy + wy^2 + (a_1v^2 + a_2uv + a_3vw + a_4uw)z.$$

Since rank $q(u,v,w) = 3$, we also have $a_4 \neq 0$. Therefore we have

$$F \simeq ux^2 + 2\beta vxy + wy^2 + (\alpha v^2 + uv + vw + uw)z$$

where $\alpha \neq 1$. $\square$

Lemma 5.7. *If $F = ux^2 + vxy + wy^2 + q(u,v,w)z$, then either*

$$F \simeq \begin{cases} ux^2 + vxy + wy^2 + (a_1v^2 + a_2uv + a_3vw + a_4uw)z \text{ or} \\ ux^2 + vxy + wy^2 + \{a_1u^2 + a_2(v^2 - 4uw)\}z \end{cases}$$

Proof. If $ad - bc = 1$, then for the linear transformation

$$\sigma : \begin{pmatrix} u' \\ v' \\ w' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a^2 & ac & c^2 & 0 & 0 & 0 \\ 2ab & ad+bc & 2cd & 0 & 0 & 0 \\ b^2 & bd & d^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & c & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \end{pmatrix}$$

we have

$$\sigma^*(ux^2 + vxy + wy^2) = ux^2 + vxy + wy^2 \text{ and } \sigma^*(v^2 - 4uw) = v^2 - 4uw.$$

If $\{q(u,v,w) = v^2 - 4uw = 0\}$ in $\mathbf{P}^2(u{:}v{:}w)$ is not one point, then there exist two points $(d^2 : -2bd : b^2)$ and $(c^2 : -2ac : a^2)$ in $\{q(u,v,w) = 0\}$. Since

$$\sigma(d^2 : -2bd : b^2) = (1 : 0 : 0) \text{ and } \sigma(c^2 : -2ac : a^2) = (0 : 0 : 1),$$

we have $(\sigma^* q)(1:0:0) = (\sigma^* q)(0:0:1) = 0$, hence we have

$$\sigma F = ux^2 + vxy + wy^2 + (a_1 v^2 + a_2 uv + a_3 vw + a_4 uw)z.$$

If $\{\ q(u,v,w) = v^2 - 4uw = 0\}$ in $\mathbf{P}^2(u{:}v{:}w)$ is one point, then we have easily

$$q(u,v,w) = a_1(u+v+w)^2 + a_2(v^2 - 4uw).$$

If $a = c = d = 1$ and $b = 0$, then we have $\sigma^* q = a_1 u^2 + a_2(v^2 - 4uw)$. $\square$

Proposition 5.8. *Suppose $F \in H(\gamma^6)$ belongs to a closed orbit. Then its stabilizer is of rank 1 if and only if*

$$F \simeq ux^2 + 2\beta vxy + wy^2 + (\alpha v^2 + uv + vw + uw)z$$

where $\alpha \neq 1,\ -\beta^2 \pm 2\beta,\ -5\beta^2 \pm 2\beta\sqrt{4\beta^2+1}$

Proof. If $\alpha \neq 1$, then we have $F \not\simeq \phi_i$ for $i = 1,\ 2,\ 3,\ 4$ and 6. Since $F \not\simeq \phi_5$, it follows from Lemma 5.9. $\square$

Lemma 5.9. *Let $F = uy^2 + 2\beta vxy + wx^2 + (\alpha v^2 + uv + vw + uw)z$. Then $\sigma F = \phi_5$ for some $\sigma \in Z_G(\gamma^6)$ if and only if*

$$\alpha = -\beta^2 \pm 2\beta,\ -5\beta^2 \pm 2\beta\sqrt{4\beta^2+1}. \qquad (*)$$

Proof. Let $F = C(u,v,w,x,y) + Q(u,v,w)z$. Then $\sigma F = \phi_5$ for some $\sigma \in Z_G(\gamma^6)$ is equivalent to that if $Q|_{u=l(v,w)} = l_1(v,w)^2$ for some $l(v,w)$ then there exists $l_2(x,y)$ such that

$$F|_{u=l(v,w)} \in (l_1(v,w), l_2(x,y))^2,$$

which means that

$$\{(\alpha+\beta^2)^2 - 4\beta^2(1-\alpha-\beta^2)\}^2 - 16\beta^4(\alpha+\beta^2)^2 = 0.$$

Solving it, we have easily $(*)$. $\square$

Remark 5.10. From Lemma 5.9, if $(*)$ holds then we have $F \in H(\gamma^1, \gamma^5)$, hence F is of type [C.5].

At last we prove Proposition 1.5.

Proof of Proposition 1.5: Since the family of elliptic curves is parameterized by j-invariant, the first statement is trivial.

In case [C.1] if $V(u, v, q_1, q_2)$ is singular, then we may assume that the defining equation

$$F = u\{q_1(w,x,y) + wz\} + q_2(w,x,y) \text{ and}$$
$$\lim_{t\to 0}[-1,-2,2,1,1,-1](t)F = uwz + vq(x,y) \simeq uwz + vxy,$$

which means that $P_6 \in \overline{C_1}$.

In case [C.2] if $\alpha = 1$, then we have

$$F = u(x+z)(y+z) + v^2x + w^2y + \beta vwz \simeq uxy + v^2x + w^2y + zq(v,w) =: F_0 \text{ and}$$
$$\lim_{t\to 0}[0,1,1,0,0,-2](t)F_0 = uxy + zq(v,w) \simeq uxy + vwz,$$

which means that $P_6 \in \overline{S_2}$. The other case refer to Remark 5.10.

In case [C.3] if $l_1|c$, then we may assume that

$$F = uy^2 + v^2z + uwz + l(w,x)vy + wq(w,x) \text{ and}$$
$$\lim_{t\to 0}[0,1,2,-1,0,-2](t)F = uwz + avxy + v^2z + uy^2 + bwx^2.$$

If $l_2^2|c$, then we may assume that

$$F = uy^2 + v^2z + uzl_1(w,x) + vxy + x^2l_2(w,x) \text{ and}$$
$$\lim_{t\to 0}[0,-1,-2,1,0,2](t)F = uy^2 + v^2z + auwz + vxy + bwx^2.$$

Hence we have $C_5 \subseteq \overline{V_3}$.

In case [C.4] if $V(u, v, w, c)$ is singular, then we may assume that

$$F = uvw + c(x,y) + zq(x,y) \text{ and}$$
$$\lim_{t\to 0}[0,0,0,1,1,-2](t)F = uvw + xyz,$$

which means that $P_6 \in \overline{C_4}$.

In case [C.5] if $\alpha = 0$, then we have

$$\lim_{t\to 0}[-2,1,1,0,0,0](t)F \simeq uvw + xyz,$$

which means that $P_6 \in \overline{C_5}$. $\square$

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